

RELATIONS BETWEEN PROJECTED EMITTANCES AND EIGENEMITTANCES

V.Balandin*, W.Decking, N.Golubeva
DESY, Hamburg, Germany

Abstract

We give necessary and sufficient conditions that two sets of positive real numbers must satisfy in order to be realizable as eigenemittances and projected emittances of a beam matrix. The information provided by these conditions sets limits on what one can achieve when designing a beam line to perform advanced emittance manipulations.

INTRODUCTION

Projected emittances are quantities which are used to characterize transverse and longitudinal beam dimensions in the laboratory coordinate system and are invariants under linear uncoupled (with respect to the laboratory coordinate system) symplectic transport. Eigenemittances are quantities which give beam dimensions in the coordinate frame in which the beam matrix is uncoupled between degrees of freedom and are invariants under arbitrary (possibly coupled) linear symplectic transformations. If the beam matrix is uncoupled already in the laboratory frame, then the set of projected emittances coincides with the set of eigenemittances, and if the beam matrix has correlations between different degrees of freedom, then these two sets are different. This fact, though looking simple, has interesting applications in accelerator physics and gives the theoretical basis for the round-to-flat transformation of angular momentum dominated beams invented by Derbenev [1]. In his scheme the beam with equal transverse projected emittances (round beam) but with nonequal eigenemittances is first produced in an axial magnetic field. Then the correlations in the beam matrix are removed by a downstream set of skew quadrupoles and projected emittances become equal to the eigenemittances, which means that the beam transverse dimensions become different from each other.

This work and further development of the advanced emittance manipulation techniques (see, for example [2, 3, 4] and references therein) naturally raise the following question: what are the relations between projected emittances and eigenemittances? As concerning already known results, in general situation they are limited to the so-called classical uncertainty principle, which states that none of projected emittances can be smaller than the minimal eigenemittance (see, for example, [5]). Besides that, in the specific two degrees of freedom case, a number of useful results can be found in [6].

The purpose of this article is to give the necessary and sufficient conditions which two sets of positive real numbers must satisfy in two and three degrees of freedom cases in order to be realizable as eigenemittances and projected emittances of a beam matrix.

* vladimir.balandin@desy.de

BEAM MATRIX AND EMITTANCES

Let us consider a collection of points in $2n$ -dimensional phase space (a particle beam) and let, for each particle,

$$z = (q_1, p_1, \dots, q_n, p_n)^T \quad (1)$$

be a vector of canonical coordinates q_m and momenta p_m . Then, as usual, the beam (covariance) matrix is defined as

$$\Sigma = \langle (z - \langle z \rangle) \cdot (z - \langle z \rangle)^T \rangle, \quad (2)$$

where the brackets $\langle \cdot \rangle$ denote an average over a distribution of the particles in the beam. By definition, the beam matrix Σ is symmetric positive semidefinite and in the following we will restrict our considerations to the situation when this matrix is nondegenerated and therefore positive definite. For simplification of notations and without loss of generality, we will also assume that the beam has vanishing first-order moments, i.e. $\langle z \rangle = 0$.

Let s be the independent variable and let $T = T(\tau)$ be the nondegenerated matrix which propagates particle coordinates from the state $s = 0$ to the state $s = \tau$, i.e let

$$z(\tau) = T z(0). \quad (3)$$

Then from (2) and (3) it follows that the matrix Σ evolves between these two states according to the congruence

$$\Sigma(\tau) = T \Sigma(0) T^T. \quad (4)$$

Let us write the $2n \times 2n$ matrix Σ in block-matrix form

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{n1} & \Sigma_{n2} & \cdots & \Sigma_{nn} \end{pmatrix}, \quad (5)$$

where the entries Σ_{mk} are 2×2 matrices. Because Σ is symmetric, the blocks satisfy the relations $\Sigma_{mk} = \Sigma_{km}^T$ for all $m, k = 1, \dots, n$. One says that the beam matrix Σ is uncoupled if all its 2×2 blocks Σ_{mk} with $m \neq k$ are equal to zero, and one says that the m -th degree of freedom in the beam matrix Σ is decoupled from the others if $\Sigma_{mk} = \Sigma_{km} = 0$ for all $k \neq m$.

If, similar to the matrix Σ , we will partition the matrix T into submatrices T_{mk} , then one can rewrite the transport equation (4) in the form of a system involving only 2×2 submatrices of the matrices Σ and T

$$\Sigma_{mk}(\tau) = \sum_{l,p=1}^n T_{ml} \Sigma_{lp}(0) T_{kp}^T, \quad m, k = 1, \dots, n. \quad (6)$$

In analogy with the matrix Σ , one says that the transport matrix T is uncoupled if all its blocks T_{mk} with $m \neq k$ are equal to zero, and one says that the m -th degree of freedom in the transport matrix T is decoupled from the others if $T_{mk} = T_{km} = 0$ for all $k \neq m$.

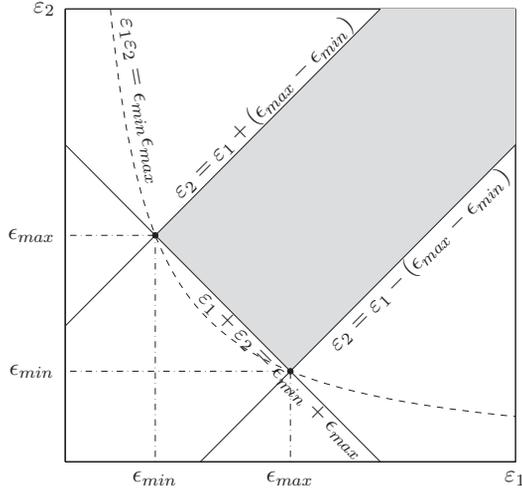


Figure 1: Shaded area shows all possible values of projected emittances ε_1 and ε_2 of a 4×4 beam matrix Σ with fixed eigenemittances ε_{min} and ε_{max} . If $\varepsilon_{min} = \varepsilon_{max}$, then the shaded half-strip turns into a ray (half-line).

In the following we will assume that the beam transport matrix T is symplectic, which is equivalent to say that it satisfies the relations

$$T J_{2n} T^\top = T^\top J_{2n} T = J_{2n} \quad (7)$$

where

$$J_{2n} = \text{diag} \left(\underbrace{\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)}_n \right) \quad (8)$$

is the $2n \times 2n$ symplectic unit matrix.

Using partitioning into 2×2 submatrices the two (equivalent) conditions for the matrix T to be symplectic (7) can be rewritten in the form of the following set of equations:

$$\sum_{l=1}^n T_{ml} J_2 T_{kl}^\top = \sum_{l=1}^n T_{lm}^\top J_2 T_{lk} = \delta_{mk} J_2, \quad (9)$$

where $m, k = 1, \dots, n$ and δ_{mk} is Kronecker's delta.

Because for an arbitrary 2×2 matrix X

$$X J_2 X^\top = X^\top J_2 X = \det(X) \cdot J_2, \quad (10)$$

the equations (9) give us the following important identities

$$\sum_{l=1}^n \det(T_{ml}) = \sum_{l=1}^n \det(T_{lm}) = 1, \quad (11)$$

which are valid for all $m = 1, \dots, n$.

Projected emittances ε_m are the rms phase space areas covered by projections of the particle beam onto each coordinate plane (q_m, p_m)

$$\varepsilon_m = \det^{1/2}(\Sigma_{mm}) = \sqrt{\langle q_m^2 \rangle \langle p_m^2 \rangle - \langle q_m p_m \rangle^2}. \quad (12)$$

Let us assume that in the matrix T the m -th degree of freedom is decoupled from the others. Then from equations (6) one obtains that

$$\Sigma_{mm}(\tau) = T_{mm} \Sigma_{mm}(0) T_{mm}^\top, \quad (13)$$

and because due to (11) the submatrix T_{mm} has unit determinant, we see that the projected emittance ε_m is conserved during the beam transport independently if the m -th degree

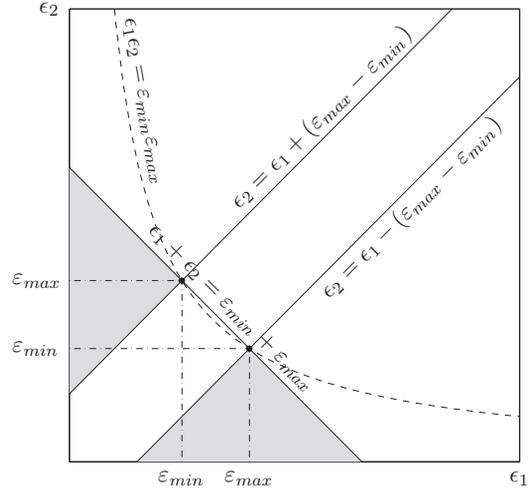


Figure 2: Shaded area (with exception of axes $\varepsilon_1 = 0$ and $\varepsilon_2 = 0$) shows all possible values of eigenemittances ε_1 and ε_2 of a 4×4 beam matrix Σ with fixed projected emittances ε_{min} and ε_{max} . If $\varepsilon_{min} = \varepsilon_{max}$, then two shaded triangles merge into one.

of freedom in the matrix $\Sigma(0)$ is decoupled from the others or not.

Using symplecticity of the transport matrix T the congruence (4) can be transformed into the following equivalent form

$$(\Sigma J_{2n})(\tau) = T \cdot (\Sigma J_{2n})(0) \cdot T^{-1}. \quad (14)$$

From this form of the equation (4) we see that the eigenvalues of the matrix ΣJ_{2n} are invariants, because (14) is a similarity transformation. The matrix ΣJ_{2n} is nondegenerated and is similar to the skew symmetric matrix $\Sigma^{1/2} J_{2n} \Sigma^{1/2}$

$$\Sigma J_{2n} = \Sigma^{1/2} \cdot (\Sigma^{1/2} J_{2n} \Sigma^{1/2}) \cdot \Sigma^{-1/2}, \quad (15)$$

which means that its spectrum is of the form

$$\pm i\varepsilon_1, \dots, \pm i\varepsilon_n, \quad (16)$$

where all $\varepsilon_m > 0$ and i is the imaginary unit. The quantities ε_m are called eigenemittances and generalize the property of the projected emittances to be invariants of uncoupled beam transport to the fully coupled case [7].

The other approach to the concept of eigenemittances is the way pointed out by Williamson's theorem (see, for example, references in [7]). This theorem tells us that one can diagonalize any positive definite symmetric matrix Σ by congruence using a symplectic matrix M

$$M \Sigma M^\top = D, \quad (17)$$

and that the diagonal matrix D has the very simple form

$$D = \text{diag}(\Lambda, \Lambda), \quad \Lambda = \text{diag}(\varepsilon_1, \dots, \varepsilon_n) > 0, \quad (18)$$

where the diagonal elements ε_m are the moduli of the eigenvalues of the matrix ΣJ_{2n} . The matrix M in (17) is not unique, but the diagonal entries of the Williamson's normal form D (eigenemittances) are unique up to a re-ordering.

It is clear that not only eigenemittances themselves, but also an arbitrary function of them is an invariant. In particular, in the following we will make use of invariants

$$I_{2m} = (-1)^m \text{tr} [(\Sigma J_{2n})^{2m}] / 2 = \varepsilon_1^{2m} + \dots + \varepsilon_n^{2m}. \quad (19)$$

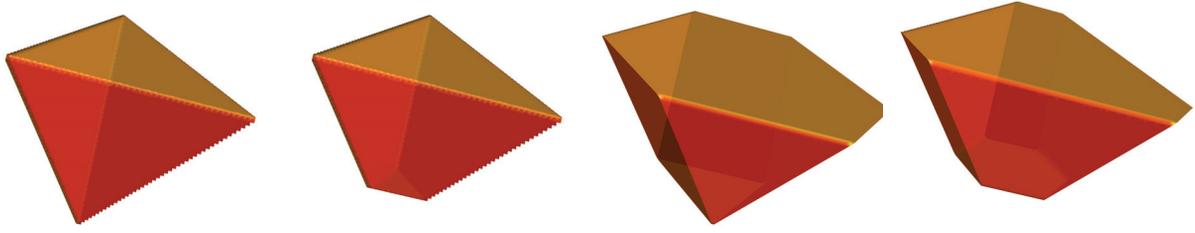


Figure 3: Schematic drawing of area allowed for the projected emittances of a 6×6 beam matrix Σ with fixed eigenemittances. There are four geometrically distinguishable situations which are shown from left to right and correspond to the relations $\epsilon_{min} = \epsilon_{mid} = \epsilon_{max}$, $\epsilon_{min} < \epsilon_{mid} = \epsilon_{max}$, $\epsilon_{min} = \epsilon_{mid} < \epsilon_{max}$, and $\epsilon_{min} < \epsilon_{mid} < \epsilon_{max}$, respectively.

CHARACTERIZATION OF UNCOUPLED BEAM MATRIX AND LOWER BOUNDS FOR PROJECTED EMITTANCES

In this section we summarize what it is possible to say about beam matrix and its emittances in the arbitrary degrees of freedom case. Note that not all presented relations are new. For example, the inequality (21) is the well known classical uncertainty principle.

Proposition 1 *Let Σ be a $2n \times 2n$ beam matrix and let positive real numbers $\epsilon_1, \dots, \epsilon_n$ and $\varepsilon_1, \dots, \varepsilon_n$ be its eigenemittances and projected emittances, respectively. Then, the following statements are equivalent:*

- The beam matrix Σ is uncoupled.
- The set of projected emittances coincides with the set of eigenemittances.
- The product of projected emittances is equal to the product of eigenemittances

$$\varepsilon_1 \varepsilon_2 \dots \varepsilon_n = \epsilon_1 \epsilon_2 \dots \epsilon_n. \quad (20)$$

Proposition 2 *Let us denote by ϵ_{min} and ε_{min} the minimums of the quantities $\epsilon_1, \dots, \epsilon_n$ and $\varepsilon_1, \dots, \varepsilon_n$ respectively. Then, the following statements hold:*

$$\varepsilon_{min} \geq \epsilon_{min}, \quad (21)$$

$$\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_n \geq \epsilon_1 + \epsilon_2 + \dots + \epsilon_n, \quad (22)$$

$$\varepsilon_1 \varepsilon_2 \dots \varepsilon_n \geq \epsilon_1 \epsilon_2 \dots \epsilon_n. \quad (23)$$

TWO DEGREES OF FREEDOM

In the two degrees of freedom case the eigenemittances can be calculated according to the explicit formula

$$\epsilon_{1,2} = \sqrt{\frac{I_2}{2} \pm \sqrt{\left(\frac{I_2}{2}\right)^2 - \det(\Sigma)}}, \quad (24)$$

and the exact relations between them and projected emittances are given by the following proposition:

Proposition 3 *The positive real numbers ϵ_1 , ϵ_2 and ε_1 , ε_2 can be realized as eigenemittances and projected emittances of a 4×4 beam matrix Σ if and only if the following two inequalities hold:*

$$\begin{cases} \varepsilon_1 + \varepsilon_2 \geq \epsilon_1 + \epsilon_2 \\ |\varepsilon_1 - \varepsilon_2| \leq |\epsilon_1 - \epsilon_2| \end{cases} \quad (25)$$

The geometrical interpretation of the inequalities (25) can be seen in Fig.1 and Fig.2.

THREE DEGREES OF FREEDOM

In the three degrees of freedom case the eigenemittances can be found as positive roots of the bicubic equation

$$\epsilon^6 - I_2 \cdot \epsilon^4 + (1/2)(I_2^2 - I_4) \cdot \epsilon^2 - \det(\Sigma) = 0, \quad (26)$$

and the exact relations between them and projected emittances are given by the following proposition:

Proposition 4 *The positive real numbers ε_1 , ε_2 , ε_3 and $\epsilon_{min} \leq \epsilon_{mid} \leq \epsilon_{max}$ can be realized as projected emittances and eigenemittances of a 6×6 beam matrix Σ if and only if the following inequalities hold:*

$$\begin{cases} \varepsilon_1 + \varepsilon_2 - \varepsilon_3 \geq \epsilon_{min} + \epsilon_{mid} - \epsilon_{max} \\ \varepsilon_1 - \varepsilon_2 + \varepsilon_3 \geq \epsilon_{min} + \epsilon_{mid} - \epsilon_{max} \\ -\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \geq \epsilon_{min} + \epsilon_{mid} - \epsilon_{max} \\ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \geq \epsilon_{min} + \epsilon_{mid} + \epsilon_{max} \\ \varepsilon_{min} \geq \epsilon_{min} \end{cases} \quad (27)$$

The geometrical interpretation of these inequalities for the case when eigenemittances are fixed can be seen in Fig.3.

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