ANALYTICAL METHODS FOR STATISTICAL ANALYSIS OF THE CORRECTION OF COUPLING DUE TO ERRORS

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Abstract

We study an analytical method to derive the strengths of the dipolar and skew quadrupole correctors. Analytical expressions to evaluate the effectiveness of the corrections are derived as well. The transport along the machine of the magnet errors and misalignments are considered at first order. A perturbative approach is used to take into account the effect of a non zero central trajectory in the multipoles. The coupling correction is obtained by minimizing the cross-talk central trajectory matrix.

INTRODUCTION

Traditionally the statistical analysis of the impact of the magnet errors and misalignments on the optics design of a machine are done by tracking and Monte Carlo methods [1]. During the preliminary optics design phase, a faster technique can be useful to evaluate the order of magnitude and the effectiveness of the correction system. We derive the analytical expression of the central trajectory at any point as the summation of the transfer along the machine of the kicks due to the magnet errors and misalignments. We treat all the errors in the approximation of thin lenses. First we assume we do not have any coupling terms. The strength of the dipolar correctors as a function of magnet errors and misalignments is derived and the expression for their statistical treatment is provided. The central trajectory after the dipolar correction is used to derive the analytical expression of the trajectory in presence of coupling terms. The coupling correction is obtained by minimizing the cross-talk central trajectory matrix [2]. The effectiveness of the correction is evaluated by the statistical treatment of the beam emittances in presence of coupling errors. In the following we start with the description of the orbit correction and then we treat the coupling correction.

ORBIT CORRECTION

The transverse trajectories x and y at the BPM position i can be expressed as a linear combination of dipolar kicks:

$$z_i = \sum_e c^z_{ie}\theta^z_e + \sum_c c^z_{ic}C^z_c + \delta z_i, \quad z = x, y \qquad (1)$$

where, θ^z_e are the integrated dipolar kicks in the two transverse planes due to magnet errors and misalignments at the location e. C^z_c are the strengths of the dipolar correctors at the location c in the two transverse planes. The coefficients c^z_{im} are the first order terms of transport from the location m of the errors or of the correctors to the location i of the observation (at the BPM). For a circular machine they are [3]:

$$c^z_{im} = \sqrt{\beta_{z,i}\beta_{z,m}}\frac{\cos(\pi\mu_z - |\phi_{z,i} - \phi_{z,m}|)}{2\sin(\pi\mu_z)}$$

where β_z,ϕ_z and μ_z are the betatron Twiss function, the betatron phase advance and the tune of the unperturbed structure. Finally, δz_i are the BPM misalignments in the two transverse planes. Each dipolar corrector strength is obtained by solving the system in Eq. (1) with respect to the correctors in the two planes. The system can be solved by using the Singular Value Decomposition (SVD) or by minimizing the sum of the square of the central trajectories on all the BPM $\sum_i(z_i - z_{\text{target},i})^2$ (LMS). The dipolar corrector strengths can be then written in the general form:

$$C^z_c = \sum_e F^z_{c,e}\theta^z_e + \sum_i D^z_{c,i}(Z_{\text{target},i} - \delta z_i) \qquad (2)$$

where the matrices F^z and D^z, depend on the transfer coefficients of the errors, correctors and on the chosen method of minimization. θ^z_e are the errors and Z_{target} are the desired trajectories values at the BPM, commonly they are null.

Statistical Treatment

We apply statistical rules on the errors and misalignments to obtain statistical prediction on the needed dipolar corrector strengths and on the residual central trajectory after correction. We assume that:

- the errors and the misalignment have a mean value equal to zero;
- the errors are uncorrelated.

The mean value of the corrector strength is then:

$$\langle C^z_c \rangle = \sum_i D^z_{c,i} Z_{\text{target},i}$$

and the variance value of the corrector strength is:

$$\left\langle (C^z_c - < C^z_c >)^2 \right\rangle = \sum_e (F^z_{c,e}\sigma^z_e)^2 + \sum_i (D^z_{c,i}\sigma_{\delta z,i})^2$$

where, σ^z_e are the estimated RMS for each of the error or misalignment distribution considered and $\sigma_{\delta z}$ are the assumed BPM errors. The residual central trajectory is expressed by replacing the corrector strengths in Eq. (1) with the expression in Eq. (2). The effectiveness of the correction is then evaluated by the calculation of the mean and the RMS values of the residual central trajectory:

$$\langle z_i \rangle = \sum_c \sum_{i'} c^z_{ic} D^z_{c,i'} Z_{\text{target},i'}$$

The mean values of the residual central trajectory after the correction depend on the wanted central trajectory at the BPM.

$$\left\langle (z_i - \langle z_i \rangle)^2 \right\rangle = \sum_e ((c^z_{ie} + \sum_c c^z_{ic} F^z_{c,e})\sigma^z_e)^2 + \sum_{i'} ((\delta_{i,i'} + \sum_c c^z_{c,i'} D^z_{c,i'})\sigma_{\delta z,i'})^2$$

The variance values of the residual central trajectory after the correction depend linearly on the variance of the errors.

COUPLING CORRECTION

In the case of coupling we add to the central trajectory the perturbative terms. We limit our coupling evaluation to the tilt of quadrupoles, and the misalignments of the sextupole and the residual orbit in the sextupoles. Given the integrated strength of the quadrupoles K_q and the tilt error of the quadrupole ϕ_q, the dipolar kicks induced by the quadrupole tilt along x and y are $K_q\phi_q(y_q + \delta y_q)$ and $K_q\phi_q(x_q + \delta x_q)$, respectively. Where y_q, x_q are the residual central trajectories in the two transverse planes at the quadrupole location and $\delta y_q, \delta x_q$ are the quadrupole misalignment. Similarly $-H_h((x_h + \delta x_h)^2 - (y_h + \delta y_h)^2)$ and $2H_h(x_h + \delta x_h)(y_h + \delta y_h)$ are the dipolar kicks induced by the sextupole in the x and y planes, respectively. Where H_h is the sextupole integrated strength at the position h, y_h, x_h are the extensions of the residual central trajectories in the two transverse planes at the sextupole location, and δx_h, δy_h are the sextupole misalignments. We call N_t the skew quadrupole corrector integrated strength at the position t. $N_t y_t$ and $N_t x_t$ are the dipolar kicks induced by the skew quadrupole correctors along x and y, respectively. Thus the central trajectory in the two planes writes:

$$\begin{aligned} x_i &= \sum_e c^x_{ie}\theta^x_e + \sum_c c^x_{ic} C^x_c + \delta x_i + \sum_t c^x_{it} N_t y_t \\ &\quad + \sum_q c^x_{ie} K_q \phi_q (y_q + \delta y_q) \\ &\quad - \sum_h c^x_{ie} H_h ((x_h + \delta x_h)^2 - (y_h + \delta y_h)^2) \\ y_i &= \sum_e c^y_{ie}\theta^y_e + \sum_c c^y_{ic} C^y_c + \delta y_i + \sum_t c^y_{it} N_t x_t \\ &\quad + \sum_q c^y_{iq} K_q \phi_q (x_q + \delta x_q) \\ &\quad + 2\sum_h c^y_{ih} H_h (x_h + \delta x_h)(y_h + \delta y_h) \end{aligned} \tag{3}$$

We assume that the central trajectories (x_n, y_n) at the quadrupoles, at the skew correctors and at the sextupoles locations, are given by the first approximation of the dipolar correction in Eq. (1). In order to correct the coupling we want to minimize the cross talk matrix [2], which means that the horizontal dipolar correctors have no effect in the vertical plane at the BPM location, and viceversa. Thus the skew quadrupole correctors can be calculated by solving (SVD) or minimizing (LMS) a system of the type:

$$\begin{cases} \frac{\partial x_i}{\partial C^y_c} = 0 \\ \frac{\partial y_i}{\partial C^x_c} = 0 \end{cases}$$

Which leads to the following expression for the skew quadrupole correctors:

$$\begin{aligned} \vec{N}_t &= M \cdot \begin{pmatrix} G_1(\theta^y_e) \\ G_2(\phi_q) \\ G_3(\delta y_h) \\ G_4(y_{\text{target},i} - \delta y_i) \end{pmatrix} \\ &= M \cdot \vec{N}_e \end{aligned} \tag{4}$$

As in the orbit correction the matrix M and the functions G_n depend on the transfer coefficients of the errors and of the correctors, and on the solving method. In the following we will use $\vec{N}_e$ to express the amplitude of the skew errors. The same statistical rules used for the dipolar correctors can be applied to the skew correctors.

Coupling Evaluation

An approach of the coupling is given in [4, 5, 6]. It consists in developing the motion on a basis of eigen vectors of the transfer matrix. The motion invariants of the particle are explicitly given as a function of its initial positions. It is then possible to make a statistical calculation of the mean motion invariant and then to evaluate the effect of the errors. The method developed in [4, 5, 6] uses for the calculation the beam matched to a structure without any error. Therefore, the growth of the motion invariant can come from beta-beating, as the beam is not matched to a structure with errors, and not necessarily from coupling alone. We consider here a beam adapted to the perturbed structure and we propose to characterize the coupling by studying the ratio of the projected emittance over the intrinsic one.

The matrix of an error e can be written under the shape $R_e = \mathbf{1} + N_e \delta R_e$ and

$$\delta R_e = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Let be R the one-turn transfer matrix around the closed orbit. The matrix R is symplectic, which implies that its eigen values are $\lambda_1 = e^{\imath\mu_1}$, $\lambda_2 = e^{-\imath\mu_1}$, $\lambda_3 = e^{\imath\mu_2}$, $\lambda_4 = e^{-\imath\mu_2}$. The reals μ_1 and μ_2 are the tunes of the structure. We shall note μ_x and μ_y the tunes of the structure without any error. Let be V_1 and V_3 two eigen vectors for R and respectively for the eigen values λ_1 and λ_3. It is straightforward that $V_2 = \overline{V_1}$ and $V_4 = \overline{V_3}$ are two eigen vectors for R and respectively for the eigen values $\overline{\lambda_1}$ and $\overline{\lambda_3}$. An expression of these eigen vectors is given in [6].

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The matrix R can be then written:

$$R = M_V \cdot \begin{pmatrix} e^{\imath\mu_1} & 0 & 0 & 0 \\ 0 & e^{-\imath\mu_1} & 0 & 0 \\ 0 & 0 & e^{\imath\mu_2} & 0 \\ 0 & 0 & 0 & e^{-\imath\mu_2} \end{pmatrix} \cdot M_V^{-1}$$

with

$$M_V = \begin{pmatrix} V_1 & V_2 & V_3 & V_4 \end{pmatrix}$$

Let be $J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $J_4 = \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix}$. We have the property:

$$M_V^* \cdot J_4 \cdot M_V = \begin{pmatrix} (V_1^* \cdot J_4 \cdot V_2) J_2 & 0 \\ 0 & (V_3^* \cdot J_4 \cdot V_4) J_2 \end{pmatrix}$$

Let be $W_1 = \frac{V_1}{(V_1^* \cdot J_4 \cdot V_2)^{1/2}}$ and $W_3 = \frac{V_3}{(V_3^* \cdot J_4 \cdot V_4)^{1/2}}$.

By this way, the matrix $M_W = \begin{pmatrix} W_1 & \overline{W_1} & W_3 & \overline{W_3} \end{pmatrix}$ is symplectic. Consider now a beam perfectly matched with the structure with the intrinsic emittance ϵ_1 and ϵ_2. The matched beam matrix Σ is then:

$$\Sigma = M_W \cdot \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_1 & 0 & 0 \\ 0 & 0 & \epsilon_2 & 0 \\ 0 & 0 & 0 & \epsilon_2 \end{pmatrix} \cdot M_W^* = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{xy}^T & \Sigma_{yy} \end{pmatrix}$$

By definition, the projected emittances ϵ_x and ϵ_y are $\epsilon_x = \sqrt{\det \Sigma_{xx}}$ and $\epsilon_y = \sqrt{\det \Sigma_{yy}}$. After calculation, we find by using the expression of the eigen vectors given in [6] and by keeping only the error terms of order 2:

$$\epsilon_x = \epsilon_1 + \frac{\epsilon_1 + \epsilon_2}{16 \sin^2 \mu_+} |A_+|^2 + \frac{\epsilon_2 - \epsilon_1}{16 \sin^2 \mu_-} |A_-|^2 \quad (5)$$

$$\epsilon_y = \epsilon_2 + \frac{\epsilon_1 + \epsilon_2}{16 \sin^2 \mu_+} |A_+|^2 - \frac{\epsilon_2 - \epsilon_1}{16 \sin^2 \mu_-} |A_-|^2 \quad (6)$$

$$\epsilon_x \epsilon_y = \epsilon_1 \epsilon_2 + \frac{(\epsilon_1 + \epsilon_2)^2}{16 \sin^2 \mu_+} |A_+|^2 + \frac{(\epsilon_2 - \epsilon_1)^2}{16 \sin^2 \mu_-} |A_-|^2 \quad (7)$$

where:

$$\mu_\pm = \frac{\mu_x \pm \mu_y}{2}$$

$$A_\pm = \sum_e N_e \sqrt{\beta_{x,e}\beta_{y,e}} e^{\imath(\phi_{x,e} \pm \phi_{y,e})} + \sum_t N_t \sqrt{\beta_{x,t}\beta_{y,t}} e^{\imath(\phi_{x,t} \pm \phi_{y,t})} \quad (8)$$

The Eq. (7) implies that the product of the projected emittances is always greater than the product of the intrinsic emittances, which is expected according to Rivkin's inequality. Moreover, the Eq. (5) and (6) show that $|A_+|^2$ gives the amplitude of the excitation due to the sum resonance $\mu_x + \mu_y$ whereas $|A_-|^2$ is linked to the difference resonance $\mu_x - \mu_y$. First of all, if we consider the structure without any skew quadrupole corrector (N_t), as the errors N_e are uncorrelated, we have:

$$\langle |A_+|^2 \rangle = \langle |A_-|^2 \rangle = \sum_e \langle N_e^2 \rangle \beta_{x,e} \beta_{y,e}$$

The mean projected emittances before correction are then:

$$\langle \epsilon_x \rangle = \epsilon_1 + \left[\frac{\epsilon_1 + \epsilon_2}{16 \sin^2 \mu_+} + \frac{\epsilon_2 - \epsilon_1}{16 \sin^2 \mu_-} \right] \sum_e \langle N_e^2 \rangle \beta_{x,e} \beta_{y,e}$$

$$\langle \epsilon_y \rangle = \epsilon_2 + \left[\frac{\epsilon_1 + \epsilon_2}{16 \sin^2 \mu_+} - \frac{\epsilon_2 - \epsilon_1}{16 \sin^2 \mu_-} \right] \sum_e \langle N_e^2 \rangle \beta_{x,e} \beta_{y,e}$$

Introducing the correction given by Eq. (4) in Eq. (8), we have finally:

$$\langle |A_\pm|^2 \rangle = \sum_e \langle N_e^2 \rangle \Big\{ \beta_{x,e}\beta_{y,e} + \sum_t \Big[M_{te}^2 \beta_{x,t}\beta_{y,t} + 2M_{te} \beta_{x,t}^{1/2} \beta_{y,t}^{1/2} \big[\beta_{x,e}^{1/2} \beta_{y,e}^{1/2} \cos(\psi_{x,te} \pm \psi_{y,te}) + \sum_{t'>t} M_{t'e} \beta_{x,t'}^{1/2} \beta_{y,t'}^{1/2} \cos(\psi_{x,tt'} \pm \psi_{y,tt'}) \big] \Big] \Big\} \quad (9)$$

with:

$$\psi_{x,ab} = \phi_{x,a} - \phi_{x,b}$$
$$\psi_{y,ab} = \phi_{y,a} - \phi_{y,b}$$

The statistics on the projected emittance is then directly deduced by putting the expression of $\langle |A_\pm|^2 \rangle$ given in the Eq. (9) in the Eq. (5) and (6).

REFERENCES

[1] P. Nghiem *et al.*, Nucl. Instr. and Meth. A 480, 339 (2002).

[2] J. Safranek, S. Krinsky, " Plans to Increase Source Brightness of NSLS X-Ray Ring", APAC 1993.

[3] G. Leleux, P. Nghiem, " Défauts Dipolaires et Quadrupolaires dans une Ligne de Transport ", Rapport Interne LNS/GT/91-02.

[4] A. Loulergue, J. Payet, " Correction of the betatron coupling by local orbit cross talk ", PAC 1999.

[5] A. Loulergue, J. Payet, " Correction of vertical dispersion and betatron coupling for SOLEIL storage ring", PAC 1999.

[6] A. Loulergue, J. Payet, " Traitement analytique et statistique du couplage bêtatron", DAPNIA/SEA 99/02.