NONLINEAR ACCELERATOR LATTICE WITH TRANSVERSE MOTION INTEGRABLE IN NORMALIZED POLAR COORDINATES

T. Zolkin*, The University of Chicago, Department of Physics, Chicago, IL 60637-1434, USA
Y. Kharkov, I. Morozov†, BINP SB RAS, Novosibirsk 630090, Russia
S. Nagaitsev‡, FNAL, Batavia, IL 60510-5011, USA

Abstract

Several families of nonlinear accelerator lattices with integrable transverse motion were suggested recently [1]. One of the requirements for the existence of two analytic invariants is a special longitudinal coordinate dependence of fields. This paper presents the particle motion analysis when a problem becomes integrable in the normalized polar coordinates. This case is distinguished from the others: it yields an exact analytical solution and has a uniform longitudinal coordinate dependence of the fields (since the corresponding nonlinear potential is invariant under the transformation from the Cartesian to the normalized coordinates). A number of interesting features are revealed: while the frequency of radial oscillations is independent of the amplitude, the spread of angular frequencies in a beam is absolute. A corresponding spread of frequencies of oscillations in the Cartesian coordinates is evaluated via the simulation of transverse Schottky noise.

INTRODUCTION

Modern design of cyclic particle accelerators is based on the transverse linear focusing optics. Despite the simplicity of this concept and the fact that the transverse motion is *integrable*, such a system has a serious disadvantage - it is unstable to the external perturbations. These perturbations are inevitable and vary from the "simple" misalignment of the magnetic optical elements and the fringe fields to the beam-beam interaction effects. Moreover, the perturbations are often introduced deliberately into the system as in an insertion of sextupoles for the chromaticity correction. As a result, the complicated net of resonances appears and does not let to operate a beam with a large spread in a space of frequencies. However, the large spread is desirable for the Landau damping, because it reduces the number of resonant particles.

Three families of 2D integrable nonlinear accelerator lattices were suggested recently [1]. They have a potential to keep the beam with a large spread of frequencies by the design due to the nonlinearity, and to minimize the regions of a chaotic motion due to the integrability at the same time. Here we consider a lattice with the motion integrable in the normalized polar coordinates. A qualitative description of the particles motion, analytical expressions for the angular and radial frequencies of oscillations along with the Schottky noise simulation are presented.

* zolkin@uchicago.edu
† imorozov@fnal.gov
‡ nsergei@fnal.gov

2D INTEGRABLE LATTICES

Consider a Hamiltonian of the 2D linear accelerator lattice with an additional non-linear potential:

$$H(p_x,p_y,x,y;s) = \sum_{q=x,y}\left(\frac{p_q^2}{2} + g_q(s)\frac{q^2}{2}\right) + V(x,y,s).$$

The choice of a betatron phase advance, $\psi(s)$, as a new independent variable, followed by a canonical transformation to the new normalized phase-space coordinates, moves the time dependence into the nonlinear term:

$$\mathcal{H}(\eta_q,\mathcal{P}_q;\psi) = \sum_{q=x,y}\left(\frac{\mathcal{P}_q^2+\eta_q^2}{2}\right) + \underbrace{\beta(s(\psi))V(x(\eta_x,\psi),y(\eta_y,\psi),\psi)}_{\stackrel{\text{def}}{\equiv} U(\eta_x,\eta_y)}.$$

This choice of time is possible in the case of equal linear focusing $g_{x,y}(s) = g(s)$, and the transformation $(q,p) \to (\eta_q,\mathcal{P}_q)$ is given via a generating function

$$F_2(x,y,\mathcal{P}_x,\mathcal{P}_y,\psi) = \sum_{q=x,y}\left(\frac{q\mathcal{P}_q}{\sqrt{\beta}} + \frac{q^2\beta'}{4\beta^2}\right),$$

where $' \stackrel{\text{def}}{\equiv} \frac{d}{d\psi}$ represents a derivative with respect to new "time", $\beta(\psi)$ is a beta-function and subscript q is omitted as long as it does not cause ambiguity. One can see that at least one integral of motion, the Hamiltonian by itself, can be ensured by a special "time"-dependence of the nonlinear potential (which compensate the β-function dependence on ψ).

A presence of a second integral of motion can be guaranteed by the choice of new generalized coordinates where variables can be separated. Additional constraint on a potential $U(\eta_x,\eta_y)$ to satisfy the Laplace equation essentially reduces the number of possible choices. In the normalized polar coordinates, $r = \sqrt{\eta_x^2+\eta_y^2}$ and $\theta = \arctan(\eta_y/\eta_x)$, only the following two satisfy it:

$$U(r,\theta) = d\ln r + \frac{A\sin(2\theta+\theta_0)}{r^2},$$

with d, A and θ_0 are being arbitrary constants.

We will consider a special case with $A \neq 0, d = 0$ (see Fig. 1). Initially "time"-independent potential in this form conserves this property under the transformation to the new normalized coordinates. This interesting property gives an opportunity to create exactly integrable nonlinear lens of this type.

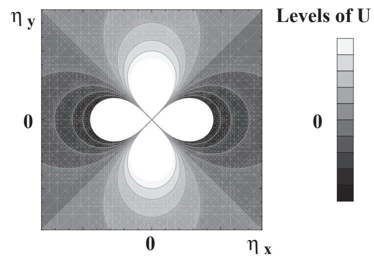

Figure 1: Schematic plot of the non-linear potential in normalized coordinates ($d = 0$, $\theta_0 = -\pi/2$), showing lines of constant potential $U(\eta_x, \eta_y)$. Note that U has a singularity at the origin: $\lim_{\eta_x \to \pm 0} U(\eta_x, 0) = -\infty$ and $\lim_{\eta_y \to \pm 0} U(0, \eta_y) = \infty$.

CLASSIFICATION OF THE MOTIONS

In the normalized polar coordinates the Hamiltonian of motion becomes:

$$\mathcal{H}(r, \theta, p_r, p_\theta; \psi) = \frac{p_r^2}{2} + \frac{1}{r^2}\frac{p_\theta^2}{2} + \frac{r^2}{2} + \frac{A\sin(2\theta + \theta_0)}{r^2}.$$

The variables separation gives the Hamilton's principal function and determines integrals of motion, E and W:

$$\begin{cases} S = -E\tau + \int p_\theta d\theta + \int p_r dr \\ p_\theta = \sqrt{2}\sqrt{W - \underbrace{A\sin(2\theta + \theta_0)}_{U_\theta^{\text{eff}}}} \\ p_r = \sqrt{2}\sqrt{E - \underbrace{\left(\frac{r^2}{2} + \frac{W}{r^2}\right)}_{U_r^{\text{eff}}}} \end{cases}.$$

Effective potential energies along with phase curves are given by graphs in Fig. 2 for the angular and the radial motions respectively.

The particle motion may be classified according to the value of W (see Fig. 3):

- *Falling to the center* ($-A \le W < 0$). The angular motion is bounded between $\pm\theta_+$[1], while the radial motion is only bounded from above, $r < r_+$, and a particle falls to the origin;
- *Oscillation* ($0 < W < A$). The motion is bounded for both degrees of freedom and particle executes oscillations ($r_- < r < r_+, |\theta| < \theta_+$). The trajectory in the normalized coordinates resembles the Lissajous curve performed in the polar coordinates;
- *Rotation* ($W > A$). The motion is unbounded in θ and bounded in $r_- < r < r_+$. As a result a particle rounds the singularity.

[1]
$$r_\pm = \left(E \pm \sqrt{E^2 - 2W}\right)^{1/2},$$
$$\theta_+ = \arccos(-W/A)$$

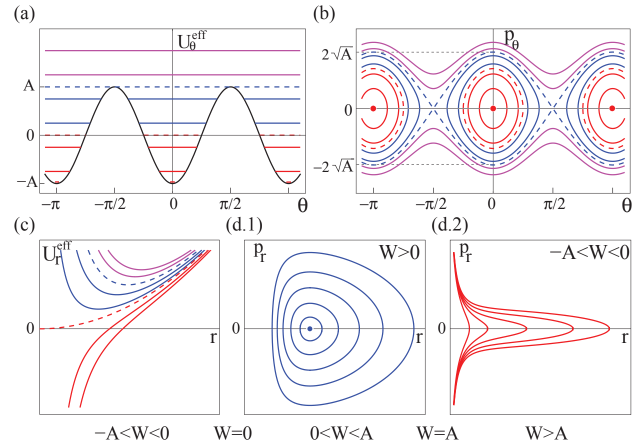

Figure 2: (a) Angular effective potential with level sets of angular invariant, W; (b) Phase curves of the angular motion; (c) Radial effective potential for different values of W; (d.1,2) Phase curves of the radial motion for the case of $W > 0$ and $-A < W < 0$ respectively.

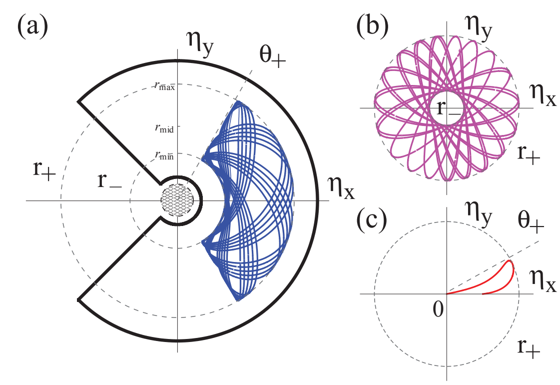

Figure 3: Example of particle trajectories for different values of W: (a) oscillations, (b) rotation and (c) falling to the center. The trajectories in Cartesian coordinates are given by the "time"-dependent scaling $x, y = \eta_{x,y}\sqrt{\beta(\psi)}$.

As one can see, only the particles which perform an oscillations are suitable for the usage since they do not round the singularity or do not fall to the center. The vacuum pipe can be realized as it shown in a Fig. 3 (a) by the black thick line; such a shape allows to place a nonlinear lens at the origin.

The Hamiltonian can be re-expressed in terms of canonical action-angle variables and for particles with oscillations is given by

$$\mathcal{H}(\Psi_r, \Psi_\theta, J_r, J_\theta; \psi) = 2J_r + \sqrt{2W(J_\theta)}, \qquad (1)$$

where W and J_θ are related as

$$J_\theta = \frac{4\sqrt{A}}{\pi}\left[\mathcal{E}(\kappa) - (1 - \kappa^2)\mathcal{K}(\kappa)\right],$$

$\mathcal{K}(\kappa)$ and $\mathcal{E}(\kappa)$ are the complete elliptic integrals of the first and second kinds respectively, and $\kappa = \sqrt{(A + W)/2A}$.

FREQUENCY SPREAD

Normalized Polar Coordinates

The frequencies of oscillations in the normalized polar coordinates can be calculated using Eq. (1) as $\omega_{r,\theta} = \partial\mathcal{H}/\partial J_{r,\theta}$, which gives:

$$\omega_r = 2,$$
$$\omega_\theta = \frac{\partial\sqrt{2W}}{\partial W}\frac{\partial W}{\partial J_\theta} = \frac{1}{\sqrt{2W}}\frac{\pi\sqrt{A}}{\mathcal{K}(\kappa)}.$$

Thus the radial oscillations are linear with respect to ψ while the angular one isn't.

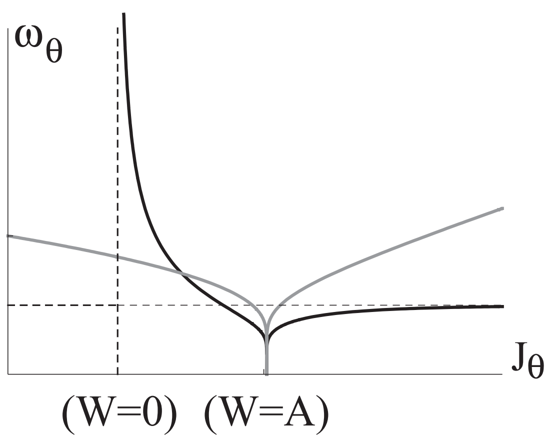

Figure 4: Angular frequency dependence of the amplitude J_θ (black curve) compared to one of the simple pendulum (gray curve).

The ω_θ dependence of J_θ is equivalent to one of the simple pendulum[2] divided over $\sqrt{2W}$ (see Fig. 4). As can be seen,

- the angular frequency is not determined for $-A \leq W < 0$ since particles fall to the center;
- $\omega_\theta \in (0;\infty)$ for $0 < W < A$, which corresponds to the absolute frequency spread;
- for the particles with large amplitudes, J_θ, the frequency stabilization is observed $\lim_{W\to\infty} \omega_\theta = 1$.

Cartesian Coordinates

In order to study the spread of frequencies in the Cartesian coordinates the Schottky noise simulation has been performed (see Fig. 5). It is revealed that all possible fractional parts of frequencies are presented in a Fourier spectra.

So despite the fact that considered potential provides nonlinear motion only for the angular degree of freedom, both frequency spreads are absolute in the Cartesian coordinates. On the other hand, additional studies of the stability of this system with respect to the external perturbations and inclusion of the longitudinal motion into consideration should be done to answer the question about feasibility of such an accelerator.

[2]The Hamiltonian of such a pendulum is

$$H_{osc} = \frac{p_{osc}^2}{2} + A\sin(2\theta + \theta_0)$$

and the correspondence is given up to a scale factor of 2 with respect to θ.

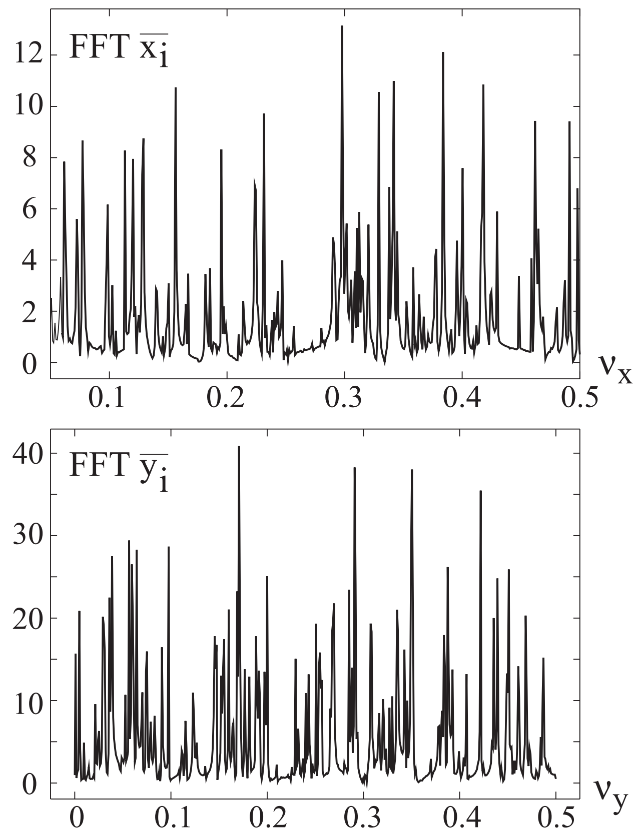

Figure 5: FFT of the turn-by-turn Schottky noise simulation in Cartesian coordinates x, y.

REFERENCES

[1] V. Danilov and S. Nagaitsev, Phys. Rev. ST Accel. Beams 13 084002 (2010).