Analysis of Emittance Growth in a Gridless Spectral Poisson Solver for Fully Symplectic Multiparticle Tracking

ICAP 2018

Key West, Florida, USA

Oct. 24, 2018

Chad Mitchell*

Lawrence Berkeley National Laboratory

*Work done in collaboration with Ji Qiang

Outline

- *Introduction to a symplectic spectral space charge algorithm*
- *Probabilistic model of computed field error*
- *Analysis of emittance growth on a single step*
- *Numerical emittance growth in a FODO channel*
- *Conclusions*

Introduction and Motivation

- Interest has grown in variational (Lagrangian) or "multi-symplectic" (Hamiltonian) algorithms that preserve the geometric properties of the *collective* self-consistent equations of motion for plasmas[1] or beams[2].

- Do such algorithms exhibit a non-physical increase in phase space volume due to the presence of numerical errors? If the physical system possesses one or more dynamical invariants, does the numerical system possess "nearby" invariants?

- Models of numerical emittance growth often treat this effect as a form of collisional Coulomb scattering. Grid heating (for PIC algorithms) significantly complicates this picture.

- Symplectic gridless spectral solvers[2] are sufficiently simple that perhaps numerical noise and its contribution to emittance growth can be understood in more complete detail.

[1] B. Shadwick et al, Physics of Plasmas 21, 055708 (2014), S. Webb, Plasma Phys. Control. Fusion 58, 034007 (2016),
[2] J. Qiang, Phys. Rev. AB 20, 014203 (2017), *previous talks by* Thomas Planche and Paul Jung.

Numerical Hamiltonian of a coasting beam with space charge + external focusing (using particles and modes)

Assume that the collective Hamiltonian of the N_p-particle system is given as the sum of a contribution due to external fields and a contribution due to space charge:

$$H = \sum_{j=1}^{N_p} H_{\text{ext}}(\vec{r}_j, \vec{p}_j, s) - \frac{n}{N_p}\frac{1}{2}\sum_{j=1}^{N_p}\sum_{k=1}^{N_p}\sum_{l=1}^{N_l}\frac{1}{\lambda_l} e_l(\vec{r}_j) e_l(\vec{r}_k)\,.$$

All quantities are computed in the laboratory frame. Each numerical step in the path length coordinate *s* is obtained by applying a second-order operator splitting to *H*.

Ω	bounded domain (1-2D)
e_l, λ_l	*l*th mode and eigenvalue
N_p	number of particles
N_l	number of modes
n	space charge intensity

Eigenmodes of the Laplacian

$$\nabla^2 e_l = \lambda_l e_l \quad e_l|_{\partial\Omega} = 0 \quad (\lambda_l < 0)$$

Symplectic map for a single step:

$$\mathcal{M}(\tau) = \mathcal{M}_{\text{ext}}(\tau/2)\mathcal{M}_{SC}(\tau)\mathcal{M}_{\text{ext}}(\tau/2) + O(\tau^3)$$

J. Qiang, Phys. Rev. ST Accel. Beams 20, 014203 (2017).

Numerical Hamiltonian of a coasting beam with space charge + external focusing (using particles and modes)

Assume that the collective Hamiltonian of the N_p-particle system is given as the sum of a contribution due to external fields and a contribution due to space charge:

$$H = \sum_{j=1}^{N_p} H_{\mathrm{ext}}(\vec{r}_j, \vec{p}_j, s) - \frac{n}{N_p}\frac{1}{2}\sum_{j=1}^{N_p}\sum_{k=1}^{N_p}\sum_{l=1}^{N_l}\frac{1}{\lambda_l}e_l(\vec{r}_j)e_l(\vec{r}_k)\,.$$

Thus, each particle moves in response to the *smooth* space charge potential and force:

$$U(\vec{r}) = -\frac{n}{N_p}\sum_{l=1}^{N_l}\sum_{j=1}^{N_p}\frac{1}{\lambda_l}e_l(\vec{r})e_l(\vec{r}_j) \qquad \vec{F}(\vec{r}) = \frac{n}{N_p}\sum_{l=1}^{N_l}\sum_{j=1}^{N_p}\frac{1}{\lambda_l}e_l(\vec{r}_j)\nabla e_l(\vec{r})$$

where $\nabla^2 U = -\rho, \quad U|_{\partial\Omega} = 0, \quad \rho = \frac{n}{N_p}\sum_{l=1}^{N_l}\sum_{j=1}^{N_p}e_l(\vec{r}_j)\,.$

J. Qiang, Phys. Rev. ST Accel. Beams 20, 014203 (2017).

Probabilistic model of computed field error

Statistical properties of the system of particles

Suppose we sample the *smooth* beam phase space density *P* using N_p macroparticles. The macroparticle coordinates $\{(\vec{r}_j,\vec{p}_j):j=1,2,\ldots,N_p\}$ are treated as i.i.d. random variables described by the probability density *P* on the single-particle phase space.

More precisely, the full beam is (initially) described by the joint probability density:

$$P_N(\vec{r}_1,\vec{p}_1,\ldots,\vec{r}_{N_p},\vec{p}_{N_p}) = P(\vec{r}_1,\vec{p}_1)P(\vec{r}_2,\vec{p}_2)\ldots P(\vec{r}_{N_p},\vec{p}_{N_p})$$

Given a function *a* on the single-particle phase space, we denote its *beam average*:

$$\langle a\rangle = \frac{1}{N_p}\sum_{j=1}^{N_p} a(\vec{r}_j,\vec{p}_j) \qquad \Delta a = a-\langle a\rangle \;.$$

Given functions *F* and *G* defined on the N_p-particle phase space (depending on all particle coordinates within the beam), we define statistics with respect to P_N:

$$\mathrm{E}[F] = \int F dP_N, \qquad \mathrm{Cov}[F,G] = \mathrm{E}[FG]-\mathrm{E}[F]\,\mathrm{E}[G] \;.$$

Statistical properties of the density and computed field

We may now evaluate the statistical properties of the various modes of the (spatial) beam density. Here $\delta\rho = \rho - \rho_{\mathrm{exact}}$. It follows that the first and second moments of the mode coefficients of $\delta\rho$ are given by:

$$\mathrm{E}[\delta\rho^l] = 0, \qquad \mathrm{Cov}[\delta\rho^l, \delta\rho^m] = \frac{n^2}{N_p}\mathrm{Cov}[e_l, e_m].$$

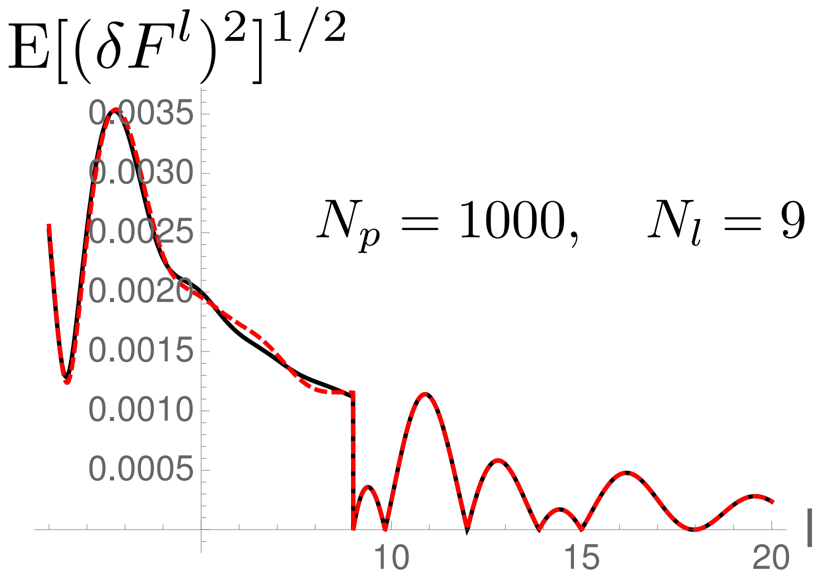
This allows us to evaluate the statistical moments of the error in the various modes of the computed field. Here $\delta\vec{F} = \vec{F} - \vec{F}_{\mathrm{exact}}$. The second moments are given by:

$$\mathrm{E}[\delta F^l \delta F^m] = \frac{1}{N_p}\frac{n^2}{\sqrt{\lambda_l\lambda_m}}\mathrm{Cov}[e_l, e_m] \quad (l, m \leq N_l)$$ (modes below cutoff)

$$\mathrm{E}[\delta F^l \delta F^m] = \frac{n^2}{\sqrt{\lambda_l\lambda_m}}\mathrm{E}[e_l]\,\mathrm{E}[e_m] \quad (l, m > N_l)$$ (modes above cutoff)

1D Example: Errors in the Spectral and Spatial Domains for a parabolic beam distribution

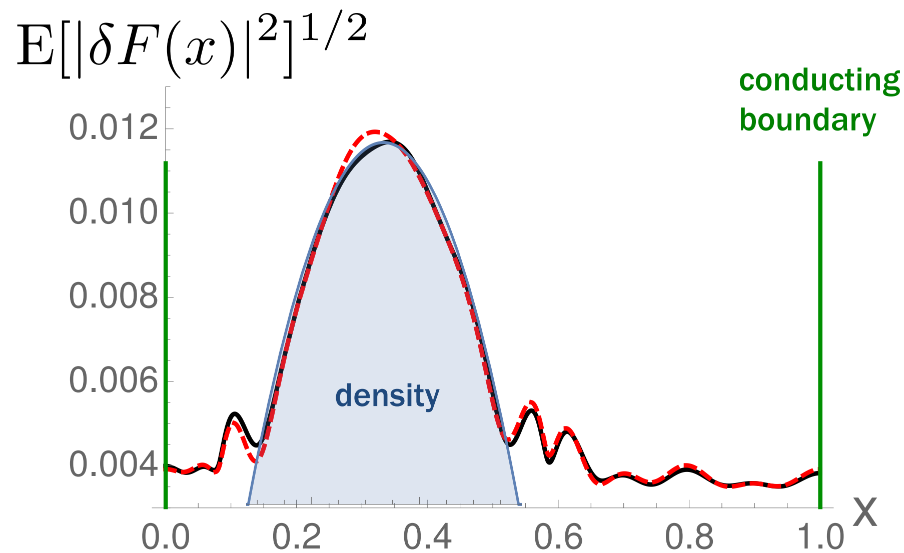
RMS error vs. mode number



particle noise only

truncation error only

RMS error vs. position



- Absolute error is largest in the beam core.
- Gibbs ringing near the edges of the beam.

— Analytical prediction of the rms error in the computed field

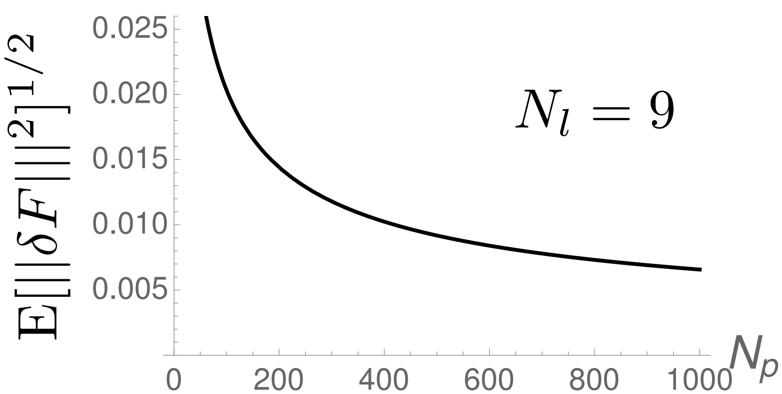
- - - Statistically computed rms field error using 200 random seeds

Expected L^2 norm of the field error and its minimization

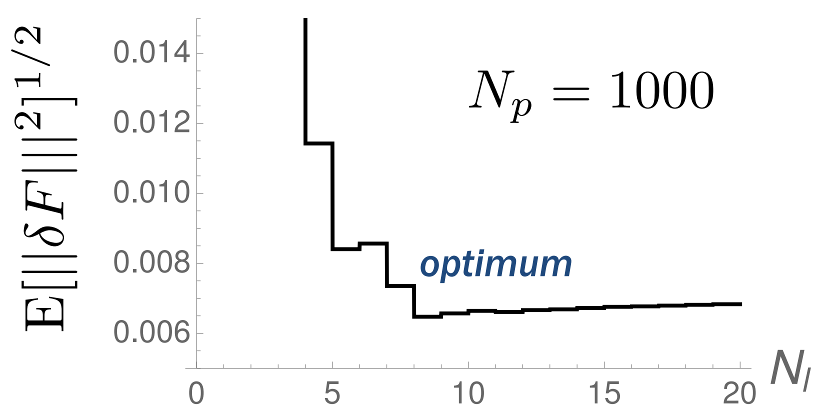
The mean-squared value of the L^2 norm of the error over the domain Ω is given by:

$$\mathrm{E}[||\delta\vec{F}||^2] = \underbrace{\frac{1}{N_p}\sum_{l\in S}\frac{n^2}{\lambda_l}\mathrm{Var}[e_l]}_{\text{particle noise}} + \underbrace{\sum_{l\notin S}\frac{n^2}{\lambda_l}\mathrm{E}[e_l]^2}_{\text{truncation error}}$$

RMS error vs. number of particles



RMS error vs. number of modes



Expected L^2 norm of the field error and its minimization

The mean-squared value of the L^2 norm of the error over the domain Ω is given by:

$$\mathrm{E}[||\delta\vec{F}||^2] = \underbrace{\frac{1}{N_p}\sum_{l\in S}\frac{n^2}{\lambda_l}\mathrm{Var}[e_l]}_{\text{particle noise}} + \underbrace{\sum_{l\notin S}\frac{n^2}{\lambda_l}\mathrm{E}[e_l]^2}_{\text{truncation error}}$$

- Here S denotes the set of indices for all numerically computed modes.
- Every mode contribution is nonnegative, and the L^2 error is globally optimized when we enforce the condition that $l \in S$ if and only if:

$$\frac{\mathrm{E}[(\delta F^l)^2]}{(F^l_{\mathrm{exact}})^2} = \frac{\mathrm{Var}[\delta\rho^l]}{(\rho^l_{\mathrm{exact}})^2} = \frac{1}{N_p}\frac{\mathrm{Var}[e_l]}{\mathrm{E}[e_l]^2} \leq 1$$

- A tighter condition on the variance of computed modes helps with emittance growth[1].

[1] J. Qiang, "Long-term simulation of space charge fields," submitted NIMA (2018).

Analysis of emittance growth on a single step

Change in RMS emittance after a single space charge step

A single space charge kick of step size τ of the form $(x,p) \rightarrow (x, p + \tau F(x))$ induces a change of RMS emittance given exactly by:

$$\epsilon^2 - \epsilon_0^2 = 2\tau A + \tau^2 B$$ where

Change in RMS emittance after a single space charge step

A single space charge kick of step size τ of the form $(x,p) \rightarrow (x, p+\tau F(x))$ induces a change of RMS emittance given exactly by:

$$\epsilon^2 - \epsilon_0^2 = 2\tau A + \tau^2 B$$ where

$$A = \langle \Delta x^2 \rangle \langle \Delta p \Delta F \rangle - \langle \Delta x \Delta p \rangle \langle \Delta x \Delta F \rangle = \langle \Delta x^2 \rangle \langle \Delta p_u \Delta F_u \rangle$$

measures the size of nonlinear correlations between *p* and *F*

variable sign

Here F_u and p_u denote *F* and *p* after subtracting linear correlations with *x*.

Change in RMS emittance after a single space charge step

A single space charge kick of step size τ of the form $(x, p) \to (x, p + \tau F(x))$ induces a change of RMS emittance given exactly by:

$$\epsilon^2 - \epsilon_0^2 = 2\tau A + \tau^2 B$$ where

$$A = \langle \Delta x^2 \rangle \langle \Delta p \Delta F \rangle - \langle \Delta x \Delta p \rangle \langle \Delta x \Delta F \rangle = \langle \Delta x^2 \rangle \langle \Delta p_u \Delta F_u \rangle$$

measures the size of nonlinear correlations between *p* and *F*

variable sign

$$B = \langle \Delta x^2 \rangle \langle \Delta F^2 \rangle - \langle \Delta x \Delta F \rangle^2 = \langle \Delta x^2 \rangle \langle \Delta F_u^2 \rangle$$

measures the size of the nonlinear part of *F*

always nonnegative

Here F_u and p_u denote *F* and *p* after subtracting linear correlations with *x*.

Statistical properties of emittance change after a single space charge step (1)

Our probabilistic model gives the statistics of *A* and *B* as sums over spectral modes:

$$\mathrm{E}[A] = \sum_{l=1}^{N_l} \frac{n}{\lambda_l} A^l \quad , \qquad \mathrm{Var}[A] = \sum_{l,m=1}^{N_l} \frac{n^2}{\lambda_l \lambda_m} A^{lm} \quad ,$$

$$\mathrm{E}[B] = \sum_{l,m=1}^{N_l} \frac{n^2}{\lambda_l \lambda_m} B^{lm} \quad , \qquad \mathrm{Var}[B] = \sum_{l,m,l',m'=1}^{N_l} \frac{n^4}{\lambda_l \lambda_m \lambda_{l'} \lambda_{m'}} B^{lml'm'}$$

In the smooth beam limit $N_p \to \infty$ we have nonzero emittance change given by*:

$$A^l = \mathrm{Var}[x]\,\mathrm{Cov}[p, e_l']\,\mathrm{E}[e_l] \quad , \qquad A^{lm} = 0$$

$$B^{lm} = \mathrm{Var}[x]\,\mathrm{Cov}[e_l', e_m']\,\mathrm{E}[e_l]\,\mathrm{E}[e_m] \quad , \qquad B^{lml'm'} = 0$$

*after removing linear correlations of *p* and e_l with *x*

Statistical properties of emittance change after a single space charge step (2)

When we include corrections through order $1/N_p$, we introduce the effects of particle noise. Term A is simple when p and x have no nonlinear correlation:

$$\mathrm{E}[A] = 0 \ , \qquad \mathrm{Var}[A] = \frac{1}{N_p} \mathrm{Var}[x]\,\mathrm{Var}[p]\,\mathrm{E}[B] \ .$$

Term B is quite complicated, but can be determined via computer algebra. For example:

$$B^{lm} = \lim_{N_p \to \infty} B^{lm} + \frac{1}{2N_p}(T^{lm} + T^{ml}) \ ,$$

$$T^{l,m} =$$

$$\mathrm{Var}[x]\,\mathrm{Cov}[e'_l, e'_m]\,\mathrm{Cov}[e_l, e_m] - 3\,\mathrm{Var}[x]\,\mathrm{Cov}[e'_l, e'_m]\,\mathrm{E}[e_l]\,\mathrm{E}[e_m]$$

$$+ 2\,\mathrm{Cov}[x^2, e_l]\,\mathrm{Cov}[e'_l, e'_m]\,\mathrm{E}[e_m] + 2\,\mathrm{Var}[x]\,\mathrm{Cov}[e'_l e'_m, e_l]\,\mathrm{E}[e_m]$$

Statistical properties of emittance change after a single space charge step (2)

When we include corrections through order $1/N_p$, we introduce the effects of particle noise. Term A is simple when p and x have no nonlinear correlation:

$$\mathrm{E}[A] = 0 \ , \qquad \mathrm{Var}[A] = \frac{1}{N_p} \mathrm{Var}[x]\,\mathrm{Var}[p]\,\mathrm{E}[B] \ .$$

Term B is quite complicated, but can be determined via computer algebra. For example:

$$B^{lm} = \lim_{N_p \to \infty} B^{lm} + \frac{1}{2N_p}(T^{lm} + T^{ml}) \ ,$$

$$T^{l,m} =$$

$$\mathrm{Var}[x]\,\mathrm{Cov}[e'_l, e'_m]\,\mathrm{Cov}[e_l, e_m] - 3\,\mathrm{Var}[x]\,\mathrm{Cov}[e'_l, e'_m]\,\mathrm{E}[e_l]\,\mathrm{E}[e_m]$$
$$+ 2\,\mathrm{Cov}[x^2, e_l]\,\mathrm{Cov}[e'_l, e'_m]\,\mathrm{E}[e_m] + 2\,\mathrm{Var}[x]\,\mathrm{Cov}[e'_l e'_m, e_l]\,\mathrm{E}[e_m]$$

This result is consistent with that of Kesting[1] if we keep only the first term.

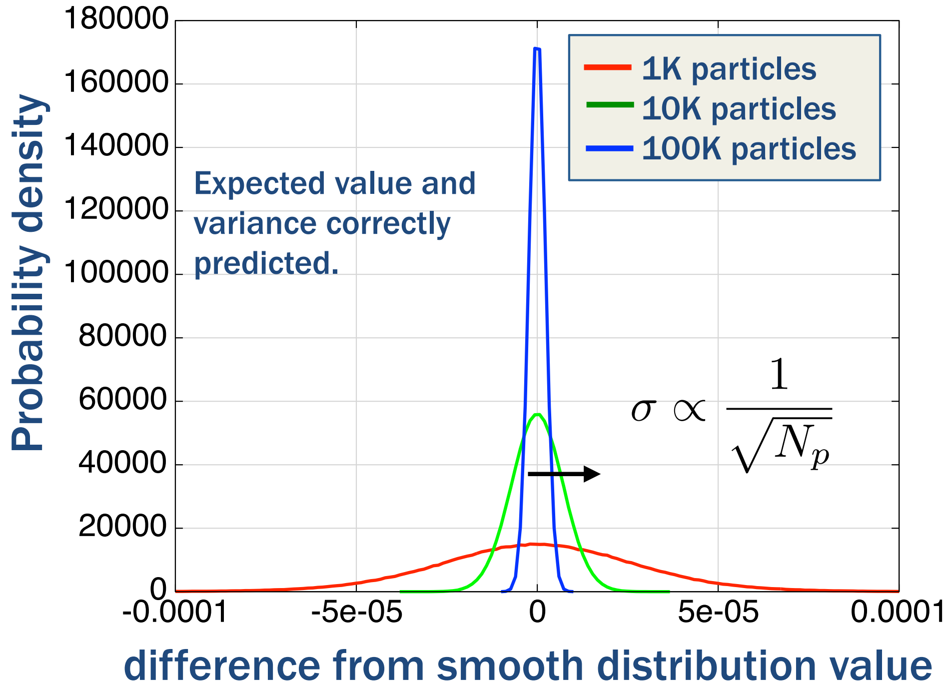
[1] F. Kesting and G. Franchetti, PRAB **18**, 114201 (2015).

Statistical properties of excess emittance growth on a single numerical step (uniform beam w/ *x-p* correlation)

1D uniform beam using 15 spectral modes, using 1M random seeds

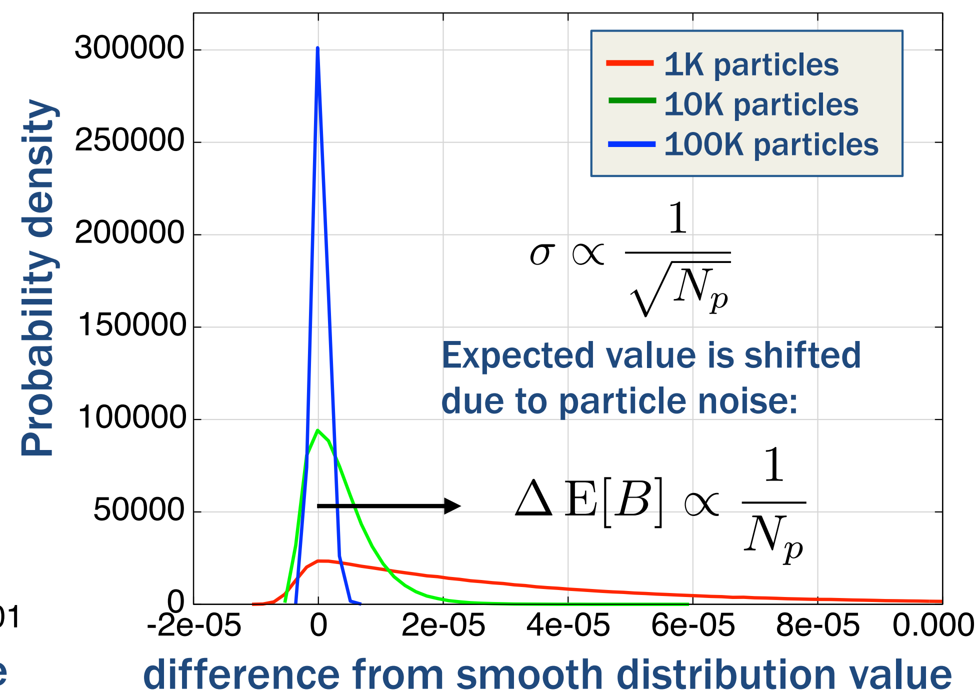
term A

$$\langle \Delta x^2 \rangle \langle \Delta p \Delta F \rangle - \langle \Delta x \Delta p \rangle \langle \Delta x \Delta F \rangle$$



term B

$$\langle \Delta x^2 \rangle \langle \Delta F^2 \rangle - \langle \Delta x \Delta F \rangle^2$$



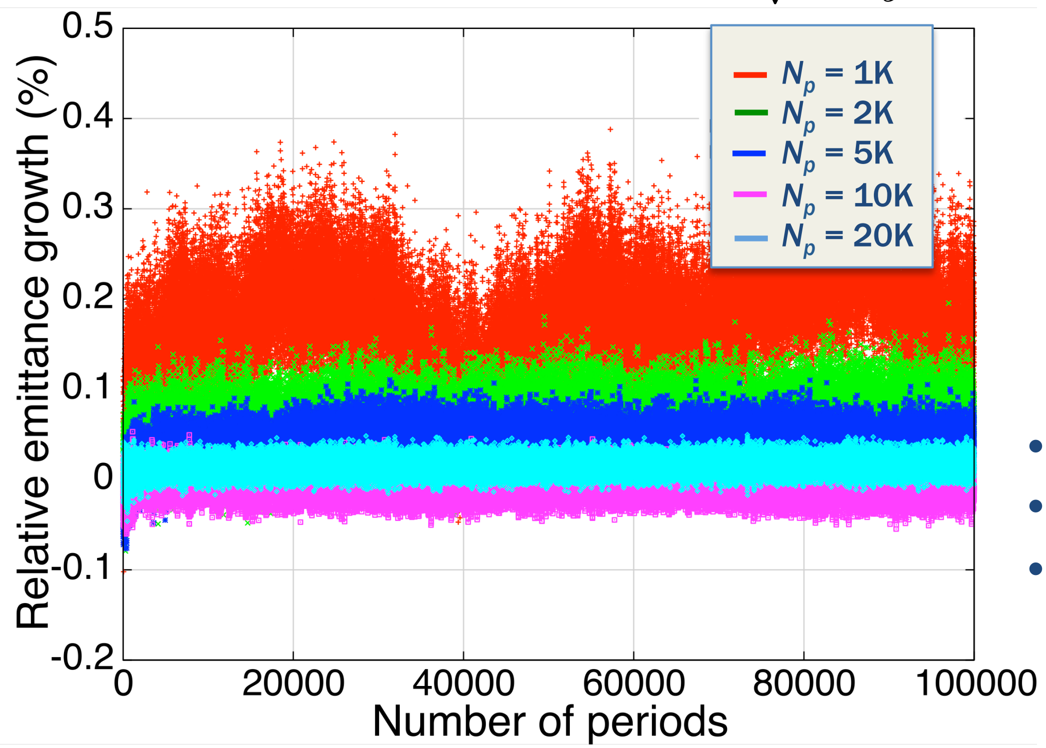
Numerical emittance growth in a FODO channel

Matched KV Beam in a FODO Channel

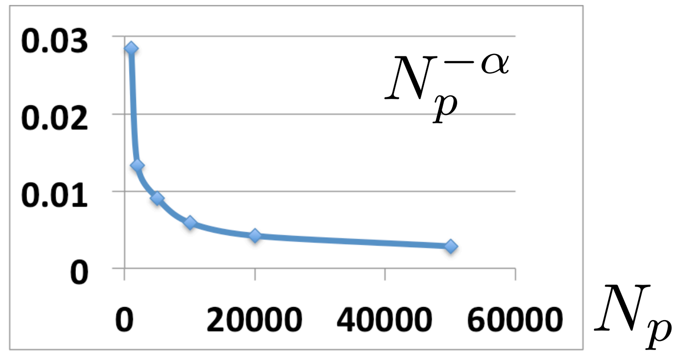
1 GeV proton beam, 100 A current
Zero current phase advance: 87°
Depressed phase advance: 74°

Initial rms emittance: 1 μm
2D domain: [0,6.5] × [0,6.5] mm
Number of modes: 15 × 15

Evolution of 4D emittance $\sqrt{\epsilon_x \epsilon_y}$



Emittance fluctuation (rms) vs. N_p



- Emittance is well-preserved.
- Fluctuations scale w/power $\alpha = 0.57$
- Based on model of a single step:

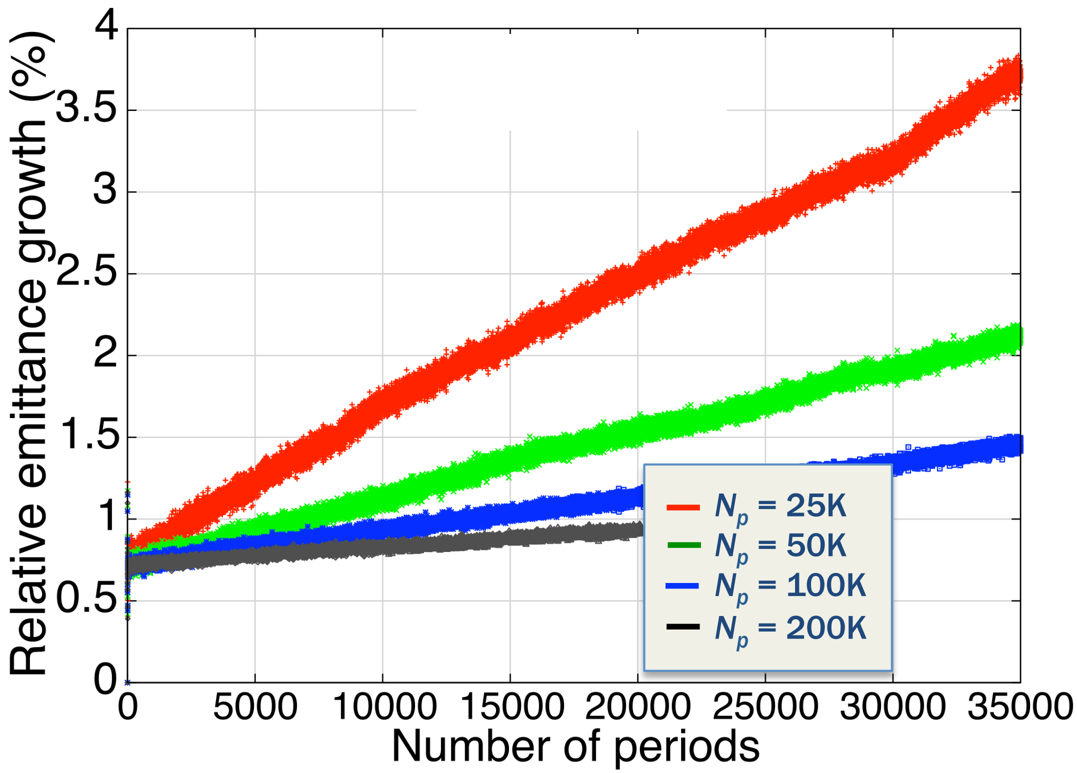
$$\sigma_{\Delta\epsilon} \propto \mathrm{Var}[A]^{1/2} \propto \frac{n}{\sqrt{N_p}}$$

Matched Gaussian Beam in a FODO Channel

1 GeV proton beam, 100 A current
Zero current phase advance: 87°
Depressed phase advance: 74°

Initial rms emittance: 1 μm
2D domain: [0,6.5] × [0,6.5] mm
Number of modes: 32 × 32

Evolution of 4D emittance $\sqrt{\epsilon_x \epsilon_y}$



- **Emittance growth rate** $N_p^{-\beta}$
- **Emittance fluctuations** $N_p^{-\alpha}$

$$\beta = 0.996, \quad \alpha = 0.58$$

- **Based on model of a single step:**

$$\mathrm{E}\left[\frac{d\epsilon}{ds}\right] \propto \mathrm{E}[B] \propto \frac{n^2}{N_p}$$

- **Driven by collisional heat exchange between degrees of freedom[1]:**

$$\frac{dS}{dt} = \frac{1}{2} k_B \beta_f \frac{(T_x - T_y)^2}{T_x T_y}$$

[1] J. Struckmeier, Phys. Rev. E **54**, 830 (1996).

Conclusions

- *The properties of "symplecticity" and "collisionlessness" in particle-based space charge tracking codes are distinct*.

- Symplecticity (in the N_p-particle sense) eliminates non-Hamiltonian artifacts from the numerical integrator, but does *not* imply that the system of macroparticles is collisionless. Additional techniques (particle shapes, noise filtering) can be used.

- This symplectic spectral algorithm is simple enough that probabilistic models of the numerical field error and emittance growth on a numerical step can be applied.

- Two emittance driving terms: *A* (drives fluctuations), *B* (nonnegative, drives growth).

- A first-principles treatment of emittance growth due numerical collisions *with dynamics* would take the complete approach:

Numerical N_p-particle Hamiltonian → BBGKY hierarchy → kinetic equation (Vlasov-Fokker-Planck-like) → moment equations (*a la* Struckmeier)

Backup material

Spectral approach to the Poisson equation on bounded domains

Let Ω be a bounded, open domain in $\mathbb{R}^d$. Consider the Poisson eq. in the form:

$$\nabla^2 U = -\rho \qquad U|_{\partial\Omega} = 0 \ .$$

There exists an orthonormal basis $\{e_l : l = 1, 2, \ldots\}$ for the Hilbert space of square-integrable functions on Ω such that each e_l is a smooth eigenfunction of the Laplace operator:

$$\nabla^2 e_l = \lambda_l e_l \qquad e_l|_{\partial\Omega} = 0 \qquad (\lambda_l < 0) \ .$$

We denote the coefficient of mode *l* of any square-integrable function f on Ω as f^l .

The following vector-valued functions can be extended to an orthonormal basis:

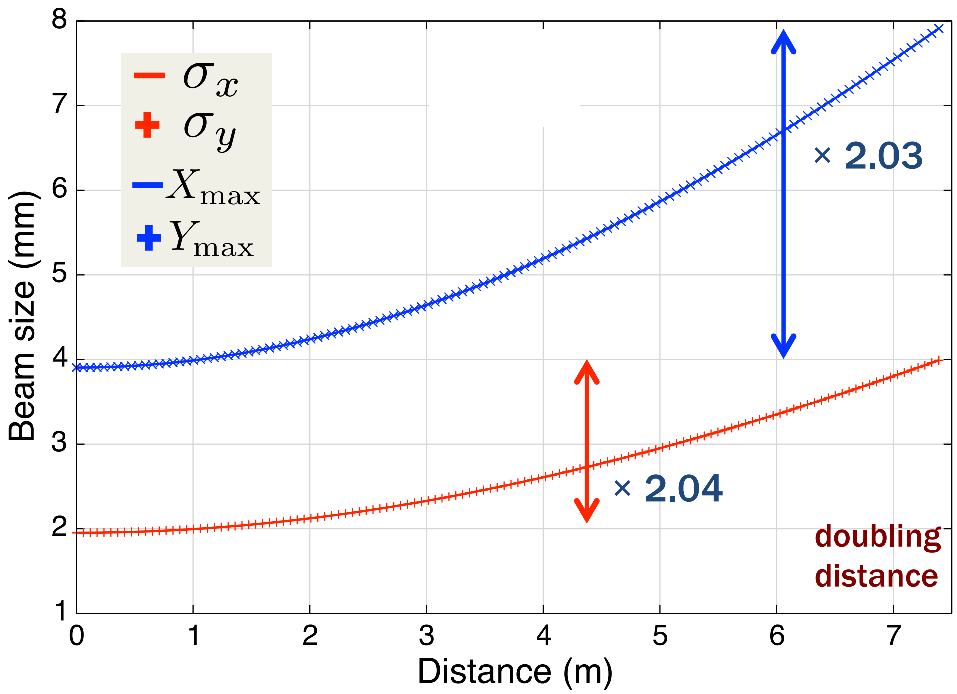
$$\vec{e}_l = \frac{1}{\sqrt{-\lambda_l}} \nabla e_l \qquad (l = 1, 2, \ldots) \ .$$

The modes of U and $\vec{F} = -\nabla U$ satisfy: $U^l = -\rho^l / \lambda_l$, $F^l = -\sqrt{-\lambda_l} U^l$.

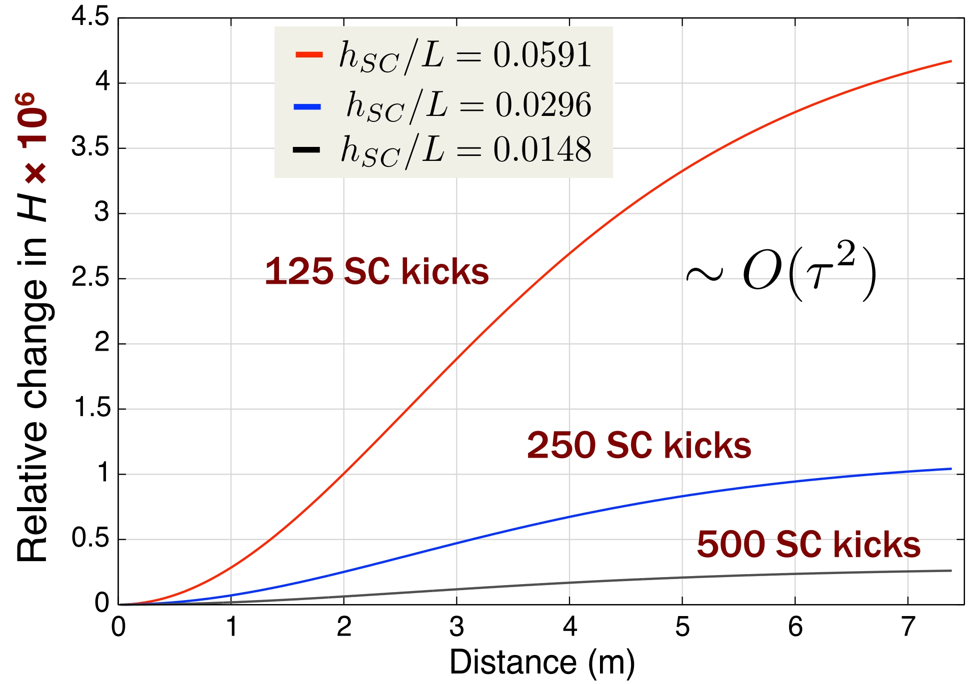
M. Einsiedler and T. Ward, *Functional Analysis, Spectral Theory, and Applications* (2017), Theorem 6.56.

Benchmark: Expansion in a drift space of a cold uniform cylinder beam with 2D transverse space charge

Beam size evolution



Preservation of the N-particle Hamiltonian



KE* = 2.5 MeV *p
***R_0* = 3.905 mm**
***I* = 4.113 mA**
a = b = 5 cm

2D rectangular domain

$$\Omega = (0, a) \times (0, b)$$

Similar behavior for the beam emittance evolution.

BERKELEY LAB U.S. DEPARTMENT OF ENERGY | Office of Science ACCELERATOR TECHNOLOGY & APPLIED PHYSICS DIVISION ATAP

Systematic removal of correlations with *x*

Note that term A and term B are each invariant under any transformation of the form:

$$x \to x + c, \quad p \to p + ax + b, \quad F \to F + gx + h$$

for any constants a, b, c, g, and h. It follows that we can replace *x*, *p*, and e_l using

$$x = \mathrm{E}[x] + x_u \qquad p = \mathrm{E}[p] + \frac{\mathrm{Cov}[x,p]}{\mathrm{Var}[x]}(x - \mathrm{E}[x]) + p_u$$

$$e'_l = \mathrm{E}[e'_l] + \frac{\mathrm{Cov}[x,e'_l]}{\mathrm{Var}[x]}(x - \mathrm{E}[x]) + e'_{l,u}$$

The final result is then made significantly simpler, since we may assume w.l.o.g. that:

$$\mathrm{E}[x] = 0, \quad \mathrm{E}[p] = 0, \quad \mathrm{E}[e_l] = 0, \quad \mathrm{Cov}[x,p] = 0, \quad \mathrm{Cov}[x,e'_l] = 0$$

provided we replace *x*, *p*, and e_l with their uncorrelated values.

Statistical analysis of emittance growth during two numerical steps (numerical tests)

step 1

1) Randomly generate a beam consisting of particle data (x,p).
2) Take ½ step in the external fields (here, a drift).
3) Compute space charge force F(x) at all particle locations using the 1-D symplectic spectral algorithm.
4) Compute the statistical quantities appearing on Slide 2 (averaging over the beam). → term A, term B (kick 1)
5) Take 1 full step in the space charge fields.
6) Take ½ step in the external fields (here, a drift).

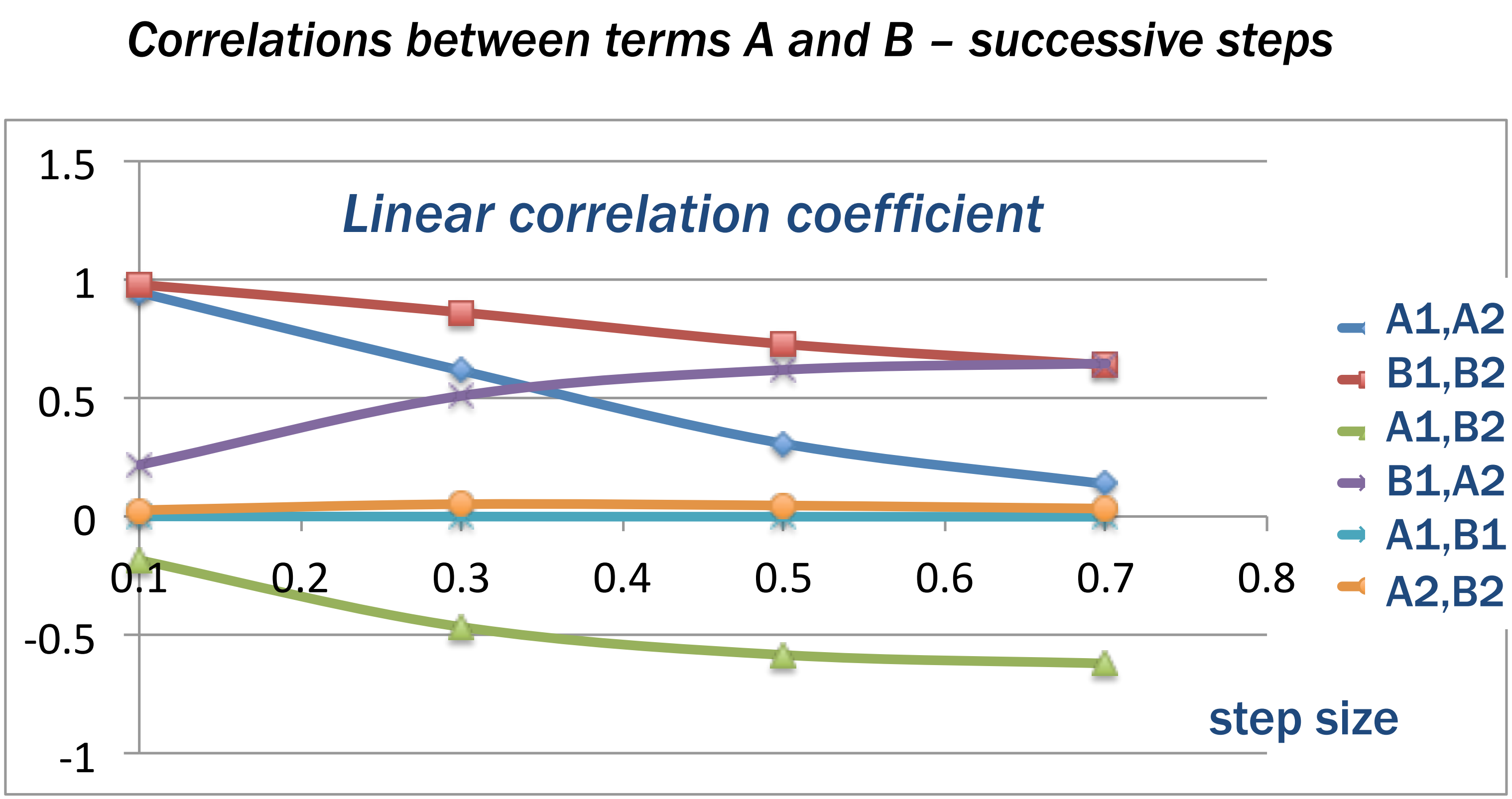
step 2

7) Take ½ step in the external fields (here, a drift).
8) Compute space charge force F(x) at all particle locations using the 1-D symplectic spectral algorithm.
9) Compute the statistical quantities appearing on Slide 2 (averaging over the beam). → term A, term B (kick 2)
10) Take 1 full step in the space charge fields.
11) Take ½ step in the external fields (here, a drift).
12) Repeat 1)-5) for N_{seed} distinct random seeds.
13) Compute statistical moments of quantities computed in 4) and 9) (averaging over random seeds).

Each step:

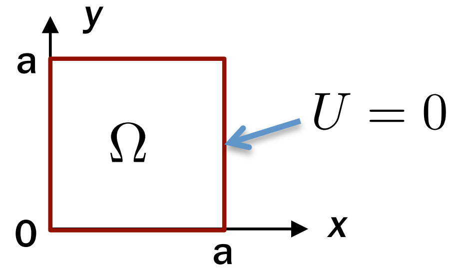
$$\mathcal{M}(\tau) = \mathcal{M}_{ext}(\tau/2)\mathcal{M}_{SC}(\tau)\mathcal{M}_{ext}(\tau/2) + O(\tau^3)$$

Statistical correlations between two successive steps for the Gaussian beam numerical example



Choosing the Optimal Number of Modes (to Minimize Norm of the Field Error) – 2D Example

Domain: $\Omega = (0,a) \times (0,a)$



Orthonormal basis eigenmodes:

$$e_{lm} = \frac{2}{a} \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) \qquad \nabla^2 e_{lm} = \lambda_{lm} e_{lm}, \quad e_{lm}|_{\partial\Omega} = 0$$

Eigenvalues: $$\lambda_{lm} = -\left(\frac{l\pi}{a}\right)^2 - \left(\frac{m\pi}{a}\right)^2 \qquad (l, m = 1, 2, \ldots)$$

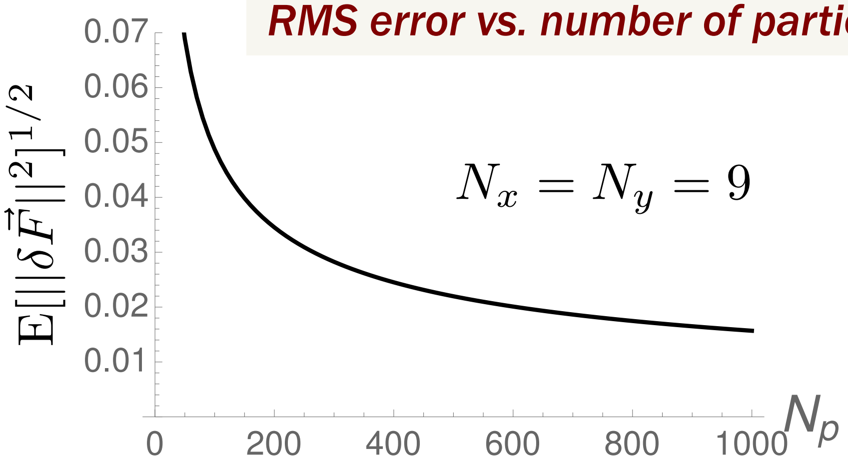
Each 2D mode is a tensor product of 1D modes. For simplicity, we truncate the mode sum such that the max horizontal 1D mode index = the max vertical 1D mode index.

Density:

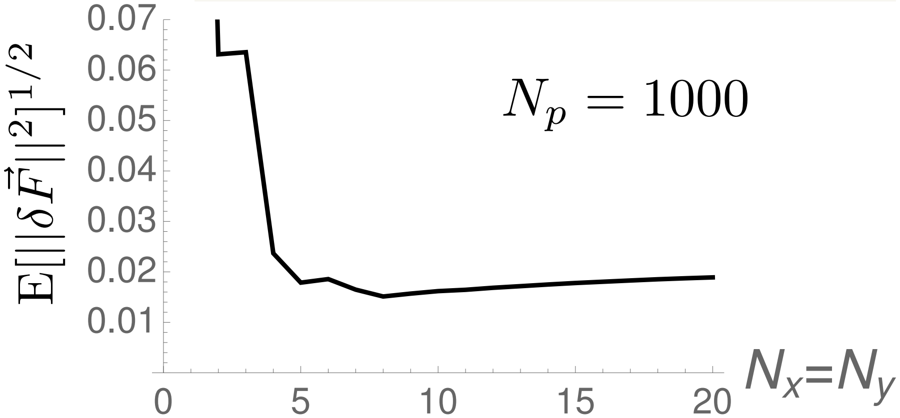
$$P(x,y) = \frac{9}{16h^2}\left(1 - \frac{(x-d)^2}{h^2}\right)\left(1 - \frac{(y-d)^2}{h^2}\right) \qquad |x-d| \leq h, \quad |y-d| \leq h$$

Choosing the Optimal Number of Modes (to Minimize Norm of the Field Error) – 2D Example

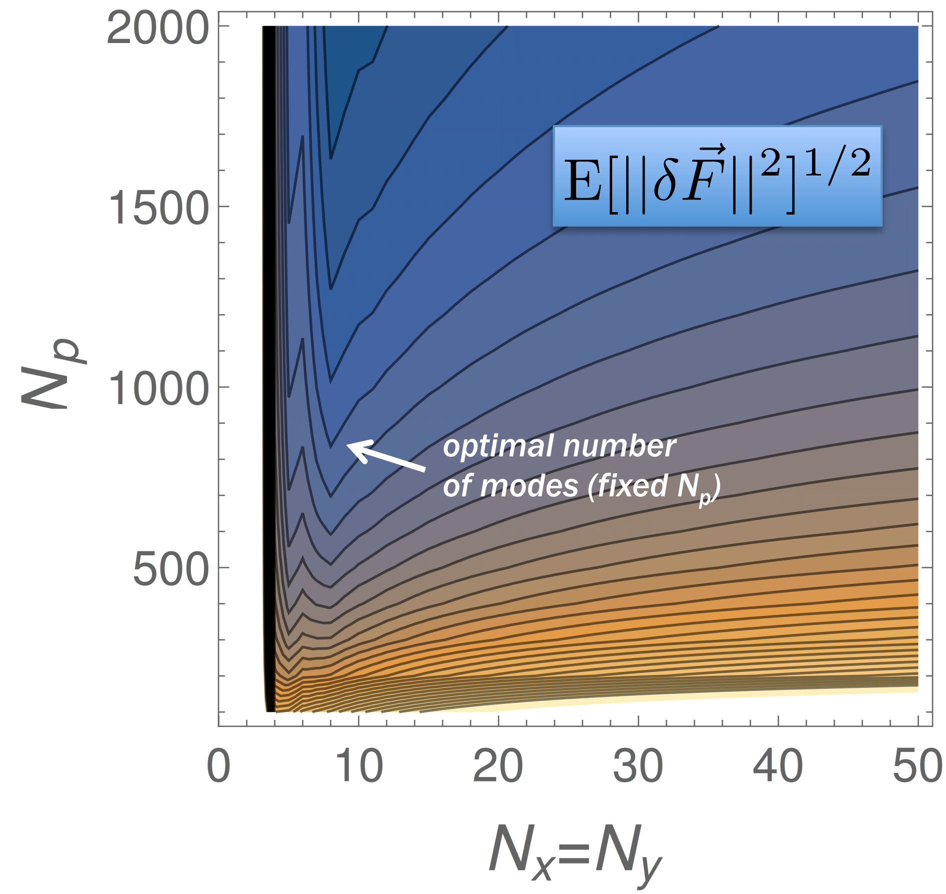
RMS error vs. number of particles



RMS error vs. number of modes



Contours of RMS error



Probabilistic model of particle noise (identities)

If a_j (j=1,..,N), b_k (k=1,..,M) are single-particle dynamical variables, some work gives:

$$\mathrm{E}\left[\prod_{j=1}^{N}\langle a_j\rangle\right] = \prod_{j=1}^{N}\mathrm{E}[a_j] + \frac{1}{N_p}\sum_{\substack{j,k=1\\ j<k}}^{N}\mathrm{Cov}[a_j,a_k]\prod_{\substack{n\neq j\\ n\neq k}}^{N}\mathrm{E}[a_n] + O\left(\frac{1}{N_p^2}\right)$$

$$\mathrm{Cov}\left[\prod_{j=1}^{N}\langle a_j\rangle,\prod_{k=1}^{M}\langle b_k\rangle\right] = \frac{1}{N_p}\sum_{j=1}^{N}\sum_{k=1}^{M}\prod_{r\neq j}^{N}\mathrm{E}[a_r]\prod_{s\neq k}^{M}\mathrm{E}[b_s]\,\mathrm{Cov}[a_j,b_k] + O\left(\frac{1}{N_p^2}\right)$$

Using the linearity of E and Cov, these results allow us to determine the statistics of any quantity that is given as a *polynomial* when expressed using beam-based averages on the single-particle phase space.

This covers all cases of interest here. Higher-order terms in $1/N_p$ are neglected.