FEL SIMULATION USING LIE METHOD

Advances in FEL Simulation

Kilean Hwang

Lawrence Berkeley National Laboratory

Oct 23, 2018

Outline

1. Introduction
2. Generalize WPA
   - How?
   - Review : Perturbative Lie Map
   - Hamiltonian
   - Generator
   - Effective Hamiltonian
3. Improve Shot Noise Modeling
   - Review of shot-noise modeling methods
   - Improved shot-noise modeling methods
   - IMPACT code suit and example
4. Conclusion

Outline

Introduction

- FEL design optimization often involve multiple times simulation including start-to-end.
  - Wiggler-Period-Averaging (WPA) : highly efficient
- Generalize WPA perturbatively using Lie map method
  - Leading order : conventional WPA
  - Next order corrections : coupling between betatron and wiggling motion, field envelope gradients,...
- Improve shot-noise model especailly suited for WPA
  - Further improvement : smoother numerical discretization

Introduction

- FEL design optimization often involve multiple times simulation including start-to-end.
  - Wiggler-Period-Averaging (WPA) : highly efficient
- Generalize WPA perturbatively using Lie map method
  - Leading order : conventional WPA
  - Next order corrections : coupling between betatron and wiggling motion, field envelope gradients,...
- Improve shot-noise model especailly suited for WPA
  - Further improvement : smoother numerical discretization

Introduction

- FEL design optimization often involve multiple times simulation including start-to-end.
  - Wiggler-Period-Averaging (WPA) : highly efficient
- Generalize WPA perturbatively using Lie map method
  - Leading order : conventional WPA
  - Next order corrections : coupling between betatron and wiggling motion, field envelope gradients,...
- Improve shot-noise model especailly suited for WPA
  - Further improvement : smoother numerical discretization

Introduction

- FEL design optimization often involve multiple times simulation including start-to-end.
  - Wiggler-Period-Averaging (WPA) : highly efficient
- Generalize WPA perturbatively using Lie map method
  - Leading order : conventional WPA
  - Next order corrections : coupling between betatron and wiggling motion, field envelope gradients,...
- Improve shot-noise model especailly suited for WPA
  - Further improvement : smoother numerical discretization

Introduction

- FEL design optimization often involve multiple times simulation including start-to-end.
  - Wiggler-Period-Averaging (WPA) : highly efficient
- Generalize WPA perturbatively using Lie map method
  - Leading order : conventional WPA
  - Next order corrections : coupling between betatron and wiggling motion, field envelope gradients,...
- Improve shot-noise model especailly suited for WPA
  - Further improvement : smoother numerical discretization

Introduction

- FEL design optimization often involve multiple times simulation including start-to-end.
  - Wiggler-Period-Averaging (WPA) : highly efficient
- Generalize WPA perturbatively using Lie map method
  - Leading order : conventional WPA
  - Next order corrections : coupling between betatron and wiggling motion, field envelope gradients,...
- Improve shot-noise model especailly suited for WPA
  - Further improvement : smoother numerical discretization

Introduction

- FEL design optimization often involve multiple times simulation including start-to-end.
  - Wiggler-Period-Averaging (WPA) : highly efficient
- Generalize WPA perturbatively using Lie map method
  - Leading order : conventional WPA
  - Next order corrections : coupling between betatron and wiggling motion, field envelope gradients,...
- Improve shot-noise model especailly suited for WPA
  - Further improvement : smoother numerical discretization

Outline

1. Introduction
2. Generalize WPA
   - How?
   - Review : Perturbative Lie Map
   - Hamiltonian
   - Generator
   - Effective Hamiltonian
3. Improve Shot Noise Modeling
   - Review of shot-noise modeling methods
   - Improved shot-noise modeling methods
   - IMPACT code suit and example
4. Conclusion

How to generalize WPA from simple averaging?

- In general, a *perturbation* map is built in order of *small parameters*.
- *However*, If we build a *map over a undulator period*,
  - the *wiggling motion average out*
  - leaves small coupling effects between the fast wiggling and slow motions like betatron motion.
- This idea allow us to generalize WPA with perturbative Lie Map method

How to generalize WPA from simple averaging?

- In general, a *perturbation* map is built in order of *small parameters*.
- *However*, If we build a *map over a undulator period*,
  - the *wiggling motion average out*
  - leaves small coupling effects between the fast wiggling and slow motions like betatron motion.
- This idea allow us to generalize WPA with perturbative Lie Map method

How to generalize WPA from simple averaging?

- In general, a *perturbation* map is built in order of *small parameters*.
- *However*, If we build a *map over a undulator period*,
  - the *wiggling motion average out*
  - leaves small coupling effects between the fast wiggling and slow motions like betatron motion.
- This idea allow us to generalize WPA with perturbative Lie Map method

How to generalize WPA from simple averaging?

- In general, a *perturbation* map is built in order of *small parameters*.
- *However*, If we build a *map over a undulator period*,
  - the *wiggling motion average out*
  - leaves small coupling effects between the fast wiggling and slow motions like betatron motion.
- This idea allow us to generalize WPA with perturbative Lie Map method

How to generalize WPA from simple averaging?

- In general, a *perturbation* map is built in order of *small parameters*.
- *However*, If we build a *map over a undulator period*,
    - the *wiggling motion average out*
    - leaves small coupling effects between the fast wiggling and slow motions like betatron motion.
- This idea allow us to generalize WPA with perturbative Lie Map method

Perturbative Lie Map

- Split the Hamiltonian $H = S + F(z) + V(z)$
  - $S = \oint H\, dz/\lambda_u$ represent slow motion
  - $V(z)$ is the radiation field potential
  - $F(z)$ is the rest : the fast wiggling motion
- Accordingly, Lie map from $z = 0$ to $z = \lambda_u$ is fatored as

$$\begin{aligned}
\mathscr{H}(\lambda_u) &= \mathscr{S}(\lambda_u)\mathscr{F}(\lambda_u)\mathscr{V}(\lambda_u) \\
\mathscr{S}(\lambda_u) &= e^{\mathscr{G}_S(\lambda_u)} \\
\mathscr{F}(\lambda_u) &= e^{\mathscr{G}_F(\lambda_u)} \\
\mathscr{V}(\lambda_u) &= e^{\mathscr{G}_V(\lambda_u)}
\end{aligned}$$

Perturbative Lie Map

- Split the Hamiltonian $H = S + F(z) + V(z)$
  - $S = \oint H\, dz/\lambda_u$ represent slow motion
  - $V(z)$ is the radiation field potential
  - $F(z)$ is the rest : the fast wiggling motion
- Accordingly, Lie map from $z = 0$ to $z = \lambda_u$ is fatored as

$$
\begin{aligned}
\mathscr{H}(\lambda_u) &= \mathscr{S}(\lambda_u)\,\mathscr{F}(\lambda_u)\,\mathscr{V}(\lambda_u) \\
\mathscr{S}(\lambda_u) &= e^{\mathscr{G}_S(\lambda_u)} \\
\mathscr{F}(\lambda_u) &= e^{\mathscr{G}_F(\lambda_u)} \\
\mathscr{V}(\lambda_u) &= e^{\mathscr{G}_V(\lambda_u)}
\end{aligned}
$$

Perturbative Lie Map

- Split the Hamiltonian $H = S + F(z) + V(z)$
  - $S = \oint H\, dz/\lambda_u$ represent slow motion
  - $V(z)$ is the radiation field potential
  - $F(z)$ is the rest : the fast wiggling motion
- Accordingly, Lie map from $z = 0$ to $z = \lambda_u$ is fatored as

$$
\begin{aligned}
\mathscr{H}(\lambda_u) &= \mathscr{S}(\lambda_u)\,\mathscr{F}(\lambda_u)\,\mathscr{V}(\lambda_u) \\
\mathscr{S}(\lambda_u) &= e^{\mathscr{G}_S(\lambda_u)} \\
\mathscr{F}(\lambda_u) &= e^{\mathscr{G}_F(\lambda_u)} \\
\mathscr{V}(\lambda_u) &= e^{\mathscr{G}_V(\lambda_u)}
\end{aligned}
$$

Perturbative Lie Map

- Split the Hamiltonian $H = S + F(z) + V(z)$
  - $S = \oint H\,dz/\lambda_u$ represent slow motion
  - $V(z)$ is the radiation field potential
  - $F(z)$ is the rest : the fast wiggling motion
- Accordingly, Lie map from $z = 0$ to $z = \lambda_u$ is fatored as

$$
\begin{aligned}
\mathscr{H}(\lambda_u) &= \mathscr{S}(\lambda_u)\,\mathscr{F}(\lambda_u)\,\mathscr{V}(\lambda_u) \\
\mathscr{S}(\lambda_u) &= e^{\mathscr{G}_S(\lambda_u)} \\
\mathscr{F}(\lambda_u) &= e^{\mathscr{G}_F(\lambda_u)} \\
\mathscr{V}(\lambda_u) &= e^{\mathscr{G}_V(\lambda_u)}
\end{aligned}
$$

Perturbative Lie Map

- Split the Hamiltonian $H = S + F(z) + V(z)$
  - $S = \oint H\,dz/\lambda_u$ represent slow motion
  - $V(z)$ is the radiation field potential
  - $F(z)$ is the rest : the fast wiggling motion
- Accordingly, Lie map from $z = 0$ to $z = \lambda_u$ is fatored as

$$\begin{aligned}
\mathscr{H}(\lambda_u) &= \mathscr{S}(\lambda_u)\,\mathscr{F}(\lambda_u)\,\mathscr{V}(\lambda_u) \\
\mathscr{S}(\lambda_u) &= e^{\mathscr{G}_S(\lambda_u)} \\
\mathscr{F}(\lambda_u) &= e^{\mathscr{G}_F(\lambda_u)} \\
\mathscr{V}(\lambda_u) &= e^{\mathscr{G}_V(\lambda_u)}
\end{aligned}$$

Magnus Series

- Slow

$$\mathscr{G}_S(z) = -z : S :$$

- Fast

$$\begin{aligned}\mathscr{G}_F(z) &= -\int_0^z dz_1 : F^{\text{int}}(z_1) : \\ &+\frac{1}{2!}\int_0^z dz_1 \int_0^{z_1} dz_2 : \left[F^{\text{int}}(z_2), F^{\text{int}}(z_1)\right] : \\ &-\frac{1}{3!}\int_0^z dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 : \left[F^{\text{int}}(z_3), \left[F^{\text{int}}(z_2), F^{\text{int}}(z_1)\right]\right] \\ &+\left[\left[F^{\text{int}}(z_3), F^{\text{int}}(z_2)\right], F^{\text{int}}(z_1)\right] :\end{aligned}$$

where $F^{\text{int}}(z_i) \equiv \mathscr{S}(z_i)F(z_i)$

- Field Potential

$$\mathscr{G}_V(z) = -\int_0^z dz : \mathscr{S}(z_i)\mathscr{F}(z_i)V(z_i) :$$

Magnus Series

- Slow

$$\mathscr{G}_S(z) = -z :S:$$

- Fast

$$\begin{aligned}\mathscr{G}_F(z) &= -\int_0^z dz_1 :F^{\text{int}}(z_1): \\ &+\frac{1}{2!}\int_0^z dz_1 \int_0^{z_1} dz_2 :\left[F^{\text{int}}(z_2), F^{\text{int}}(z_1)\right]: \\ &-\frac{1}{3!}\int_0^z dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 :\left[F^{\text{int}}(z_3), \left[F^{\text{int}}(z_2), F^{\text{int}}(z_1)\right]\right] \\ &+\left[\left[F^{\text{int}}(z_3), F^{\text{int}}(z_2)\right], F^{\text{int}}(z_1)\right]:\end{aligned}$$

where $F^{\text{int}}(z_i) \equiv \mathscr{S}(z_i)F(z_i)$

- Field Potential

$$\mathscr{G}_V(z) = -\int_0^z dz :\mathscr{S}(z_i)\mathscr{F}(z_i)V(z_i):$$

Magnus Series

- Slow

$$\mathscr{G}_S(z) = -z :S:$$

- Fast

$$\begin{aligned}\mathscr{G}_F(z) = & -\int_0^z dz_1 :F^{\text{int}}(z_1): \\ & +\frac{1}{2!}\int_0^z dz_1 \int_0^{z_1} dz_2 :\left[F^{\text{int}}(z_2), F^{\text{int}}(z_1)\right]: \\ & -\frac{1}{3!}\int_0^z dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 :\left[F^{\text{int}}(z_3), \left[F^{\text{int}}(z_2), F^{\text{int}}(z_1)\right]\right] \\ & +\left[\left[F^{\text{int}}(z_3), F^{\text{int}}(z_2)\right], F^{\text{int}}(z_1)\right]:\end{aligned}$$

where $F^{\text{int}}(z_i) \equiv \mathscr{S}(z_i)F(z_i)$

- Field Potential

$$\mathscr{G}_V(z) = -\int_0^z dz :\mathscr{S}(z_i)\mathscr{F}(z_i)V(z_i):$$

Next leading order

	Leading order (WPA)	Next leading order
$\mathscr{G}_S$	integand S is linear	S includes higher order terms
$\mathscr{G}_F$	integrand is F	integrand is $\mathscr{S}F$
$\mathscr{G}_V$	integrand is $\mathscr{F}V$	integrand $\mathscr{S}\mathscr{F}V$

Hamiltonian

Example

Start from

$$H(\boldsymbol{x}, \boldsymbol{p}, ct, -\gamma; z) = -\sqrt{\gamma^2 - 1 - (p_x - a_x)^2 - (p_y - a_y)^2}$$

where

$$a_x = K \cosh(k_x x)\cosh(k_y y)\cos(k_u z) + a_r$$
$$a_y = K \frac{k_x}{k_y} \sinh(k_x x)\sinh(k_y y)\cos(k_u z)$$

and we write the vector potential of the radiation field as

$$a_r = \Re \sum_{h \geq 1} K_h(\boldsymbol{x}, t; z)\, e^{ihk_r(z-ct)}$$

Hamiltonian

Example

Start from

$$H(\boldsymbol{x}, \boldsymbol{p}, ct, -\gamma; z) = -\sqrt{\gamma^2 - 1 - (p_x - a_x)^2 - (p_y - a_y)^2}$$

where

$$\begin{aligned} a_x &= K \cosh(k_x x) \cosh(k_y y) \cos(k_u z) + a_r \\ a_y &= K \frac{k_x}{k_y} \sinh(k_x x) \sinh(k_y y) \cos(k_u z) \end{aligned}$$

and we write the vector potential of the radiation field as

$$a_r = \Re \sum_{h \geq 1} K_h(\boldsymbol{x}, t; z) e^{ihk_r(z-ct)}$$

Hamiltonian

Example

Start from

$$H(\boldsymbol{x}, \boldsymbol{p}, ct, -\gamma; z) = -\sqrt{\gamma^2 - 1 - (p_x - a_x)^2 - (p_y - a_y)^2}$$

where

$$\begin{aligned} a_x &= K \cosh(k_x x) \cosh(k_y y) \cos(k_u z) + a_r \\ a_y &= K \frac{k_x}{k_y} \sinh(k_x x) \sinh(k_y y) \cos(k_u z) \end{aligned}$$

and we write the vector potential of the radiation field as

$$a_r = \Re \sum_{h \geq 1} K_h(\boldsymbol{x}, t; z) \, e^{ihk_r(z - ct)}$$

Co-moving frame / FEL phase

- Canonical transformation using

$$G_2(ct,\eta) = [k_r(z-ct)+k_u z]\,\eta$$

- New Hamiltonian

$$H = (k_u+k_r)\eta - \sqrt{k_r^2\eta^2 - 1 - (p_x-a_x)^2 - (p_y-a_y)^2}$$

- New longitudinal conjugate pair

$$\theta \equiv k_r(z-ct)+k_u z, \qquad \eta \equiv \gamma/k_r$$

  - $\theta' \simeq 0$ on resonance (in undulator), $\theta' \simeq k_u$ in drift

Co-moving frame / FEL phase

- Canonical transformation using

$$G_2(ct, \eta) = [k_r(z - ct) + k_u z]\,\eta$$

- New Hamiltonian

$$H = (k_u + k_r)\,\eta - \sqrt{k_r^2\eta^2 - 1 - (p_x - a_x)^2 - (p_y - a_y)^2}$$

- New longitudinal conjugate pair

$$\theta \equiv k_r(z - ct) + k_u z, \qquad \eta \equiv \gamma / k_r$$

  - $\theta' \simeq 0$ on resonance (in undulator), $\theta' \simeq k_u$ in drift

Co-moving frame / FEL phase

- Canonical transformation using

$$G_2(ct,\eta) = [k_r(z-ct) + k_u z]\,\eta$$

- New Hamiltonian

$$H = (k_u + k_r)\,\eta - \sqrt{k_r^2\eta^2 - 1 - (p_x - a_x)^2 - (p_y - a_y)^2}$$

- New longitudinal conjugate pair

$$\theta \equiv k_r(z-ct) + k_u z, \qquad \eta \equiv \gamma/k_r$$

- $\theta' \simeq 0$ on resonance (in undulator), $\theta' \simeq k_u$ in drift

Co-moving frame / FEL phase

- Canonical transformation using

$$G_2(ct,\eta) = [k_r(z-ct)+k_u z]\,\eta$$

- New Hamiltonian

$$H = (k_u+k_r)\,\eta - \sqrt{k_r^2\eta^2 - 1 - (p_x-a_x)^2 - (p_y-a_y)^2}$$

- New longitudinal conjugate pair

$$\theta \equiv k_r(z-ct)+k_u z, \qquad \eta \equiv \gamma/k_r$$

  - $\theta' \simeq 0$ on resonance (in undulator), $\theta' \simeq k_u$ in drift

Split Hamiltonians

Propagated Field Model

- Slow Hamiltonian

$$S \equiv \frac{k_u}{k_s}\gamma + \frac{1}{2\gamma}\left[1 + p_x^2 + p_y^2 + \frac{K^2}{2}\left(1 + k_x^2 x^2 + k_y^2 y^2\right)\right] + \frac{K^2}{4\gamma}\left[\frac{1}{3}\left(k_x^4 x^4 + k_y^4 y^4\right) + k_x^2 k_u^2 x^2 y^2\right] + \frac{1}{(2\gamma)^3}\left(1 + K^2 + \frac{3}{8}K^4\right) + O\left(\frac{q_\perp^6}{\gamma}, \frac{q_\perp^2}{\gamma^3}, \frac{1}{\gamma^5}\right)$$

- Fast Hamiltonian

$$F \equiv \frac{K_{\text{eff}}^2}{4\gamma}\cos\left(2k_u z\right) + \frac{K_{\text{eff}}}{\gamma} p_x \cos\left(k_u z\right) + O\left(\frac{q_\perp^3}{\gamma}, \frac{1}{\gamma^3}\right)$$

where $K_{\text{eff}} \equiv K\left(1 + k_x^2 \frac{x^2}{2} + k_y^2 \frac{y^2}{2}\right)$

- Field Potential

$$V \equiv -\Re\sum_h \left[\frac{K_{eff}}{\gamma}\cos\left(k_u z\right) + \frac{p_x}{\gamma}\right] K_h e^{ih(\theta - k_u z)} + O\left(\frac{K_h q_\perp^2}{\gamma}, \frac{K_h}{\gamma^3}, \frac{K_h^2}{\gamma}\right)$$

Split Hamiltonians

Propagated Field Model

- Slow Hamiltonian

$$S \equiv \frac{k_u}{k_s}\gamma + \frac{1}{2\gamma}\left[1 + p_x^2 + p_y^2 + \frac{K^2}{2}\left(1 + k_x^2 x^2 + k_y^2 y^2\right)\right] + \frac{K^2}{4\gamma}\left[\frac{1}{3}\left(k_x^4 x^4 + k_y^4 y^4\right) + k_x^2 k_u^2 x^2 y^2\right] + \frac{1}{(2\gamma)^3}\left(1 + K^2 + \frac{3}{8}K^4\right) + O\left(\frac{q_\perp^6}{\gamma}, \frac{q_\perp^2}{\gamma^3}, \frac{1}{\gamma^5}\right)$$

- Fast Hamiltonian

$$F \equiv \frac{K_{\text{eff}}^2}{4\gamma}\cos(2k_u z) + \frac{K_{\text{eff}}}{\gamma} p_x \cos(k_u z) + O\left(\frac{q_\perp^3}{\gamma}, \frac{1}{\gamma^3}\right)$$

where $K_{\text{eff}} \equiv K\left(1 + k_x^2 \frac{x^2}{2} + k_y^2 \frac{y^2}{2}\right)$

- Field Potential

$$V \equiv -\Re \sum_h \left[\frac{K_{eff}}{\gamma}\cos(k_u z) + \frac{p_x}{\gamma}\right] K_h e^{ih(\theta - k_u z)} + O\left(\frac{K_h q_\perp^2}{\gamma}, \frac{K_h}{\gamma^3}, \frac{K_h^2}{\gamma}\right)$$

Split Hamiltonians

Propagated Field Model

- Slow Hamiltonian

$$S \equiv \frac{k_u}{k_s}\gamma + \frac{1}{2\gamma}\left[1 + p_x^2 + p_y^2 + \frac{K^2}{2}\left(1 + k_x^2 x^2 + k_y^2 y^2\right)\right] + \frac{K^2}{4\gamma}\left[\frac{1}{3}\left(k_x^4 x^4 + k_y^4 y^4\right) + k_x^2 k_u^2 x^2 y^2\right] + \frac{1}{(2\gamma)^3}\left(1 + K^2 + \frac{3}{8}K^4\right) + O\left(\frac{q_\perp^6}{\gamma}, \frac{q_\perp^2}{\gamma^3}, \frac{1}{\gamma^5}\right)$$

- Fast Hamiltonian

$$F \equiv \frac{K_{\text{eff}}^2}{4\gamma}\cos\left(2k_u z\right) + \frac{K_{\text{eff}}}{\gamma} p_x \cos\left(k_u z\right) + O\left(\frac{q_\perp^3}{\gamma}, \frac{1}{\gamma^3}\right)$$

where $K_{\text{eff}} \equiv K\left(1 + k_x^2 \frac{x^2}{2} + k_y^2 \frac{y^2}{2}\right)$

- Field Potential

$$V \equiv -\Re \sum_h \left[\frac{K_{eff}}{\gamma}\cos\left(k_u z\right) + \frac{p_x}{\gamma}\right] K_h e^{ih(\theta - k_u z)} + O\left(\frac{K_h q_\perp^2}{\gamma}, \frac{K_h}{\gamma^3}, \frac{K_h^2}{\gamma}\right)$$

Slow / Fast Map

Magnus Series

- Slow Map

$$\mathscr{G}_S(\lambda_u) = -\lambda_u S$$

- Fast Map

$$\mathscr{G}_F(\lambda_u) = -\lambda_u \frac{K^4 k_x^2}{16 k_u^2 \gamma^3}$$

- coupling b/w slow betatron and fast wiggling motion
- small coupling due to large frequency ratio b/w slow and fast motion
  - Can become relevant when strong quads present on top of undulator

Slow / Fast Map

Magnus Series

- Slow Map

$$\mathscr{G}_S(\lambda_u) = -\lambda_u S$$

- Fast Map

$$\mathscr{G}_F(\lambda_u) = -\lambda_u \frac{K^4 k_x^2}{16 k_u^2 \gamma^3}$$

- coupling b/w slow betatron and fast wiggling motion
- small coupling due to large frequency ratio b/w slow and fast motion
  - Can become relavent when strong quads present on top of undulator

Slow / Fast Map

Magnus Series

- Slow Map

$$\mathscr{G}_S(\lambda_u) = -\lambda_u S$$

- Fast Map

$$\mathscr{G}_F(\lambda_u) = -\lambda_u \frac{K^4 k_x^2}{16 k_u^2 \gamma^3}$$

  - coupling b/w slow betatron and fast wiggling motion
  - small coupling due to large frequency ratio b/w slow and fast motion
    - Can become relavent when strong quads present on top of undulator

Slow / Fast Map

Magnus Series

- Slow Map

$$\mathscr{G}_S(\lambda_u) = -\lambda_u S$$

- Fast Map

$$\mathscr{G}_F(\lambda_u) = -\lambda_u \frac{K^4 k_x^2}{16 k_u^2 \gamma^3}$$

  - coupling b/w slow betatron and fast wiggling motion
  - small coupling due to large frequency ratio b/w slow and fast motion
    - Can become relavent when strong quads present on top of undulator

Slow / Fast Map

Magnus Series

- Slow Map

$$\mathscr{G}_S(\lambda_u) = -\lambda_u S$$

- Fast Map

$$\mathscr{G}_F(\lambda_u) = -\lambda_u \frac{K^4 k_x^2}{16 k_u^2 \gamma^3}$$

  - coupling b/w slow betatron and fast wiggling motion
  - small coupling due to large frequency ratio b/w slow and fast motion
    - Can become relavent when strong quads present on top of undulator

Propagated Field Model

Split Hamiltonians

- Need field modeling in order to evaluate the integration

$$\mathscr{G}_V(z) = -\int_0^z dz : \mathscr{S}(z_i)\mathscr{F}(z_i)V(z_i):$$

- assumning slowly varying, we model the propagated field enverlope by

$$K_h^{\text{int}} \equiv \mathscr{S}\mathscr{F}K_h \simeq \mathscr{F}K_h \equiv \mathbb{K}_h + \frac{K_{\text{eff}}}{k_u\gamma}\sin(k_u z)\frac{\partial}{\partial x}\mathbb{K}_h + z\partial_z\mathbb{K}_h$$

  - $\mathbb{K}_h = \frac{1}{z}\int_0^z K_h dz$: averaged field envelope
  - $\partial_z\mathbb{K}_h$: first order longitudinal variation

- where we used

$$\mathscr{F}(z)x = \frac{K_{eff}}{k_u\gamma}\sin(k_u z)$$

Propagated Field Model

- Need field modeling in order to evaluate the integration

$$\mathscr{G}_V(z) = -\int_0^z dz : \mathscr{S}(z_i)\mathscr{F}(z_i)V(z_i) :$$

- assumning slowly varying, we model the propagated field enverlope by

$$K_h^{\text{int}} \equiv \mathscr{S}\mathscr{F}K_h \simeq \mathscr{F}K_h \equiv \mathbb{K}_h + \frac{K_{\text{eff}}}{k_u\gamma}\sin(k_u z)\frac{\partial}{\partial x}\mathbb{K}_h + z\partial_z\mathbb{K}_h$$

  - $\mathbb{K}_h = \frac{1}{z}\int_0^z K_h dz$: averaged field envelope
  - $\partial_z\mathbb{K}_h$: first order longitudinal variation
- where we used

$$\mathscr{F}(z)x = \frac{K_{eff}}{k_u\gamma}\sin(k_u z)$$

Propagated Field Model

Split Hamiltonians

- Need field modeling in order to evaluate the integration

$$\mathscr{G}_V(z) = -\int_0^z dz : \mathscr{S}(z_i)\mathscr{F}(z_i)V(z_i) :$$

- assumning slowly varying, we model the propagated field enverlope by

$$K_h^{\text{int}} \equiv \mathscr{S}\mathscr{F}K_h \simeq \mathscr{F}K_h \equiv \mathbb{K}_h + \frac{K_{\text{eff}}}{k_u\gamma}\sin(k_u z)\frac{\partial}{\partial x}\mathbb{K}_h + z\partial_z\mathbb{K}_h$$

  - $\mathbb{K}_h = \frac{1}{z}\int_0^z K_h dz$: averaged field envelope
  - $\partial_z\mathbb{K}_h$: first order longitudinal variation

- where we used

$$\mathscr{F}(z)x = \frac{K_{eff}}{k_u\gamma}\sin(k_u z)$$

Propagated Field Model

- Need field modeling in order to evaulate the integration

$$\mathscr{G}_V(z) = -\int_0^z dz : \mathscr{S}(z_i)\mathscr{F}(z_i)V(z_i):$$

- assumning slowly varying, we model the propagated field enverlope by

$$K_h^{\rm int} \equiv \mathscr{S}\mathscr{F}K_h \simeq \mathscr{F}K_h \equiv \mathbb{K}_h + \frac{K_{\rm eff}}{k_u\gamma}\sin(k_u z)\frac{\partial}{\partial x}\mathbb{K}_h + z\partial_z\mathbb{K}_h$$

  - $\mathbb{K}_h = \frac{1}{z}\int_0^z K_h dz$: averaged field envelope
  - $\partial_z\mathbb{K}_h$: first order longitudinal variation

- where we used

$$\mathscr{F}(z)x = \frac{K_{eff}}{k_u\gamma}\sin(k_u z)$$

Propagated Field Model

- Need field modeling in order to evaulate the integration

$$\mathscr{G}_V(z) = -\int_0^z dz : \mathscr{S}(z_i)\mathscr{F}(z_i)V(z_i):$$

- assumning slowly varying, we model the propagated field enverlope by

$$K_h^{\mathrm{int}} \equiv \mathscr{S}\mathscr{F}K_h \simeq \mathscr{F}K_h \equiv \mathbb{K}_h + \frac{K_{\mathrm{eff}}}{k_u\gamma}\sin(k_u z)\frac{\partial}{\partial x}\mathbb{K}_h + z\partial_z\mathbb{K}_h$$

  - $\mathbb{K}_h = \frac{1}{z}\int_0^z K_h dz$: averaged field envelope
  - $\partial_z\mathbb{K}_h$: first order longitudinal variation

- where we used

$$\mathscr{F}(z)x = \frac{K_{eff}}{k_u\gamma}\sin(k_u z)$$

Propagated Potential

- Similarily the FEL phase becomes

$$\theta^{\text{int}} \equiv \mathscr{S}\mathscr{F}\theta = \theta + \dot{\theta} z - \xi \sin(2k_u z) - \zeta \sin(k_u z)$$

where

$$\begin{aligned} \dot{\theta} &\equiv k_u - \frac{k_r}{2\gamma^2}\left[1 + p_x^2 + p_y^2 + \frac{K_{\text{eff}}^2}{2}\right] \\ \xi &\equiv \frac{k_r K_{\text{eff}}^2}{8 k_u \gamma^2} \\ \zeta &\equiv \frac{k_r K}{k_u \gamma^2} p_x \end{aligned}$$

- Exactly on resonance $\dot{\theta} \to 0$. The inclusion of it encompasses small off-resonant effects.

Propagated Potential

- Similarily the FEL phase becomes

$$\theta^{\text{int}} \equiv \mathscr{S}\mathscr{F}\theta = \theta + \dot{\theta}z - \xi\sin(2k_u z) - \zeta\sin(k_u z)$$

where

$$\begin{aligned}
\dot{\theta} &\equiv k_u - \frac{k_r}{2\gamma^2}\left[1 + p_x^2 + p_y^2 + \frac{K_{\text{eff}}^2}{2}\right] \\
\xi &\equiv \frac{k_r K_{\text{eff}}^2}{8k_u\gamma^2} \\
\zeta &\equiv \frac{k_r K}{k_u\gamma^2}p_x
\end{aligned}$$

- Exactly on resonance $\dot{\theta} \to 0$. The inclusion of it encompasses small off-resonant effects.

Field Potential Map

Split Hamiltonians

- We can write the propagated field potential by

$$V^{\text{int}} \equiv \mathscr{S}\mathscr{F}V = -\Re\sum_h \left[\frac{K_{eff}}{\gamma}\cos\left(k_u z\right) + \frac{p_x}{\gamma}\right] K_h^{\text{int}} e^{ih\left(\theta^{\text{int}} - k_u z\right)}$$

- Therefore, finally, the generator of the field potential reads,

$$\begin{aligned}\mathscr{G}_V &= -\int_0^{\lambda_u} V^{\text{int}} dz \\ &= \lambda_u \Re \sum_h \frac{e^{ih\theta}}{\gamma}\left[K_{\text{eff}}\int_C^h + p_x\int_1^h + K\int_{zC}^h \partial_z + \frac{K_{\text{eff}}^2}{k_u\gamma}\int_{SC}^h \partial_x\right]\mathbb{K}_h\end{aligned}$$

where $C \equiv \cos\left(k_u z\right)$, $SC \equiv \sin\left(k_u z\right)\cos\left(k_u z\right)$, and the inegration parameter is, for example,

$$\int_C^h = \frac{e^{-ih\theta}}{\lambda_u}\int_0^{\lambda_u}\cos\left(k_u z\right) e^{ih\left(\theta^{\text{int}} - k_u z\right)} dz$$

explicitly....

Field Potential Map

Split Hamiltonians

- We can write the propagated field potential by

$$V^{\text{int}} \equiv \mathscr{S}\mathscr{F}V = -\Re\sum_h \left[\frac{K_{eff}}{\gamma}\cos\left(k_u z\right) + \frac{p_x}{\gamma}\right] K_h^{\text{int}} e^{ih\left(\theta^{\text{int}} - k_u z\right)}$$

- Therefore, finally, the generator of the field potential reads,

$$\begin{aligned}\mathscr{G}_V &= -\int_0^{\lambda_u} V^{\text{int}} dz \\ &= \lambda_u \Re\sum_h \frac{e^{ih\theta}}{\gamma}\left[K_{\text{eff}}\int_C^h + p_x\int_1^h + K\int_{zC}^h \partial_z + \frac{K_{\text{eff}}^2}{k_u\gamma}\int_{SC}^h \partial_x\right]\mathbb{K}_h\end{aligned}$$

where $C \equiv \cos\left(k_u z\right)$, $SC \equiv \sin\left(k_u z\right)\cos\left(k_u z\right)$, and the inegration parameter is, for example,

$$\int_C^h \equiv \frac{e^{-ih\theta}}{\lambda_u}\int_0^{\lambda_u} \cos\left(k_u z\right) e^{ih\left(\theta^{\text{int}} - k_u z\right)} dz$$

explicitly....

Field Potential Map

Split Hamiltonians

- We can write the propagated field potential by

$$V^{\text{int}} \equiv \mathscr{S}\mathscr{F}V = -\Re\sum_h \left[\frac{K_{eff}}{\gamma}\cos\left(k_u z\right) + \frac{p_x}{\gamma}\right] K_h^{\text{int}} e^{ih\left(\theta^{\text{int}} - k_u z\right)}$$

- Therefore, finally, the generator of the field potential reads,

$$\begin{aligned}\mathscr{G}_V &= -\int_0^{\lambda_u} V^{\text{int}} dz \\ &= \lambda_u \Re\sum_h \frac{e^{ih\theta}}{\gamma}\left[K_{\text{eff}}\int_C^h + p_x\int_1^h + K\int_{zC}^h \partial_z + \frac{K_{\text{eff}}^2}{k_u\gamma}\int_{SC}^h \partial_x\right]\mathbb{K}_h\end{aligned}$$

where $C \equiv \cos\left(k_u z\right)$, $SC \equiv \sin\left(k_u z\right)\cos\left(k_u z\right)$, and the inegration parameter is, for example,

$$\int_C^h \equiv \frac{e^{-ih\theta}}{\lambda_u}\int_0^{\lambda_u} \cos\left(k_u z\right) e^{ih\left(\theta^{\text{int}} - k_u z\right)} dz$$

explicitly....

Field Potential Map

- We can write the propagated field potential by

$$V^{\text{int}} \equiv \mathscr{S}\mathscr{F}V = -\Re\sum_h \left[\frac{K_{eff}}{\gamma}\cos(k_u z) + \frac{p_x}{\gamma}\right] K_h^{\text{int}} e^{ih\left(\theta^{\text{int}} - k_u z\right)}$$

- Therefore, finally, the generator of the field potential reads,

$$\begin{aligned} \mathscr{G}_V &= -\int_0^{\lambda_u} V^{\text{int}} dz \\ &= \lambda_u \Re\sum_h \frac{e^{ih\theta}}{\gamma}\left[K_{\text{eff}}\int_C^h + p_x\int_1^h + K\int_{zC}^h \partial_z + \frac{K_{\text{eff}}^2}{k_u\gamma}\int_{SC}^h \partial_x\right]\mathbb{K}_h \end{aligned}$$

where $C \equiv \cos(k_u z)$, $SC \equiv \sin(k_u z)\cos(k_u z)$, and the inegration parameter is, for example,

$$\int_C^h \equiv \frac{e^{-ih\theta}}{\lambda_u}\int_0^{\lambda_u} \cos(k_u z)\, e^{ih\left(\theta^{\text{int}} - k_u z\right)}\, dz$$

explicitly....

Integration Parameters

To the 1st order of $\dot{\theta}$, ζ, and $\Delta\xi \equiv \xi - \xi_R$ where $\xi_R \equiv \frac{k_r K^2}{8 k_u \gamma_R^2}$

$$\int_C^h = \frac{1}{2}\left(J^{h\xi_R}_{-\frac{h-1}{2}} + J^{h\xi_R}_{-\frac{h+1}{2}}\right)\left(1 + \frac{ih\dot{\theta}\lambda_u}{2}\right) - \frac{1}{2}\frac{h\dot{\theta}}{k_u}\left(\sum_{l\neq -\frac{h-1}{2}} \frac{J_l^{h\xi_R}}{(2l+h-1)} + \sum_{l\neq -\frac{h+1}{2}} \frac{J_l^{h\xi_R}}{(2l+h+1)}\right)$$
$$-\frac{1}{2}\Delta\xi\left(\frac{h-1}{2\xi_R}J^{h\xi_R}_{-\frac{h-1}{2}} + hJ^{h\xi_R}_{-\frac{h-3}{2}} + \frac{h+1}{2\xi_R}J^{h\xi_R}_{-\frac{h+1}{2}} + hJ^{h\xi_R}_{-\frac{h-1}{2}}\right) + \frac{h\zeta}{2}\frac{1}{2}\left(J^{h\xi_R}_{-\frac{h+2}{2}} - J^{h\xi_R}_{-\frac{h-2}{2}}\right)$$

$$\int_1^h = J^{h\xi_R}_{-\frac{h}{2}}\left(1 + \frac{ih\dot{\theta}\lambda_u}{2}\right) - \frac{h\dot{\theta}}{k_u}\sum_{l\neq -\frac{h}{2}} \frac{J_l^{h\xi_R}}{(2l+h)} - \Delta\xi\left(\frac{h}{2\xi_R}J^{h\xi_R}_{-\frac{h}{2}} + hJ^{h\xi_R}_{-\frac{h}{2}+1}\right) + \frac{h\zeta}{2}\left(J^{h\xi_R}_{-\frac{h+1}{2}} - J^{h\xi_R}_{-\frac{h-1}{2}}\right)$$

$$\int_{zC}^h = \frac{\lambda_u}{4}\left(J^{h\xi_R}_{-\frac{h-1}{2}} + J^{h\xi_R}_{-\frac{h+1}{2}}\right) + \frac{i\lambda_u}{4\pi}\left(\sum_{l\neq -\frac{h-1}{2}} \frac{J_l^{h\xi_R}}{(2l+h-1)} + \sum_{l\neq -\frac{h+1}{2}} \frac{J_l^{h\xi_R}}{(2l+h+1)}\right)$$

$$\int_{SC}^h = \frac{1}{4i}\left(J^{h\xi_R}_{-\frac{h-2}{2}} - J^{h\xi_R}_{-\frac{h+2}{2}}\right)$$

$J_i^{h\xi_R}$: First kind Bessel function of order i and argument $h\xi_R$, only integer i allowed.

$\int_{SC}$ vanishes for *odd* harmonics but can be large for *even* harmonics.

Integration Parameters

To the 1st order of $\dot{\theta}$, ζ, and $\Delta\xi \equiv \xi - \xi_R$ where $\xi_R \equiv \frac{k_r K^2}{8 k_u \gamma_R^2}$

$$\int_C^h = \frac{1}{2}\left(J^{h\xi_R}_{-\frac{h-1}{2}} + J^{h\xi_R}_{-\frac{h+1}{2}}\right)\left(1 + \frac{ih\dot{\theta}\lambda_u}{2}\right) - \frac{1}{2}\frac{h\dot{\theta}}{k_u}\left(\sum_{l\neq-\frac{h-1}{2}} \frac{J_l^{h\xi_R}}{(2l+h-1)} + \sum_{l\neq-\frac{h+1}{2}} \frac{J_l^{h\xi_R}}{(2l+h+1)}\right)$$
$$-\frac{1}{2}\Delta\xi\left(\frac{h-1}{2\xi_R} J^{h\xi_R}_{-\frac{h-1}{2}} + hJ^{h\xi_R}_{-\frac{h-3}{2}} + \frac{h+1}{2\xi_R} J^{h\xi_R}_{-\frac{h+1}{2}} + hJ^{h\xi_R}_{-\frac{h-1}{2}}\right) + \frac{h\zeta}{2}\frac{1}{2}\left(J^{h\xi_R}_{-\frac{h+2}{2}} - J^{h\xi_R}_{-\frac{h-2}{2}}\right)$$

$$\int_1^h = J^{h\xi_R}_{-\frac{h}{2}}\left(1 + \frac{ih\dot{\theta}\lambda_u}{2}\right) - \frac{h\dot{\theta}}{k_u}\sum_{l\neq-\frac{h}{2}} \frac{J_l^{h\xi_R}}{(2l+h)} - \Delta\xi\left(\frac{h}{2\xi_R} J^{h\xi_R}_{-\frac{h}{2}} + hJ^{h\xi_R}_{-\frac{h}{2}+1}\right) + \frac{h\zeta}{2}\left(J^{h\xi_R}_{-\frac{h+1}{2}} - J^{h\xi_R}_{-\frac{h-1}{2}}\right)$$

$$\int_{zC}^h = \frac{\lambda_u}{4}\left(J^{h\xi_R}_{-\frac{h-1}{2}} + J^{h\xi_R}_{-\frac{h+1}{2}}\right) + \frac{i\lambda_u}{4\pi}\left(\sum_{l\neq-\frac{h-1}{2}} \frac{J_l^{h\xi_R}}{(2l+h-1)} + \sum_{l\neq-\frac{h+1}{2}} \frac{J_l^{h\xi_R}}{(2l+h+1)}\right)$$

$$\int_{SC}^h = \frac{1}{4i}\left(J^{h\xi_R}_{-\frac{h-2}{2}} - J^{h\xi_R}_{-\frac{h+2}{2}}\right)$$

$J_i^{h\xi_R}$: First kind Bessel function of order i and argument $h\xi_R$, only integer i allowed.

$\int_{SC}$ vanishes for *odd* harmonics but can be large for *even* harmonics.

Effective Hamiltonian

- Map not yet practically useful
  - map is not solvable
  - step size is fixed by one undulator period
- Instead, we build effective Hamiltonian using Baker-Campbel-Hausdorff (BCH)

$$\begin{aligned} H_{\text{eff}} &= -\frac{1}{\lambda_u}\left(\mathscr{G}_S + \mathscr{G}_F + \mathscr{G}_V\right) \\ &\quad -\frac{1}{2\lambda_u}\left(:\mathscr{G}_S:\mathscr{G}_F + :\mathscr{G}_S:\mathscr{G}_V + :\mathscr{G}_F:\mathscr{G}_V\right) + \ldots \end{aligned}$$

- Now, it is autonomous → arbitrary step size. Can apply numerical methods like Runge-Kutta (RK)

Effective Hamiltonian

- Map not yet practically useful
  - map is not solvable
  - step size is fixed by one undulator period
- Instead, we build effective Hamiltonian using Baker-Campbel-Hausdorff (BCH)

$$
\begin{aligned}
H_{\text{eff}} &= -\frac{1}{\lambda_u}(\mathscr{G}_S + \mathscr{G}_F + \mathscr{G}_V) \\
&\quad -\frac{1}{2\lambda_u}(:\mathscr{G}_S:\mathscr{G}_F + :\mathscr{G}_S:\mathscr{G}_V + :\mathscr{G}_F:\mathscr{G}_V) + \dots
\end{aligned}
$$

- Now, it is autonomous → arbitrary step size. Can apply numerical methods like Runge-Kutta (RK)

Effective Hamiltonian

- Map not yet practically useful
  - map is not solvable
  - step size is fixed by one undulator period
- Instead, we build effective Hamiltonian using Baker-Campbel-Hausdorff (BCH)

$$H_{\text{eff}} = -\frac{1}{\lambda_u}(\mathscr{G}_S + \mathscr{G}_F + \mathscr{G}_V) - \frac{1}{2\lambda_u}(:\mathscr{G}_S:\mathscr{G}_F + :\mathscr{G}_S:\mathscr{G}_V + :\mathscr{G}_F:\mathscr{G}_V) + \dots$$

- Now, it is autonomous $\rightarrow$ arbitrary step size. Can apply numerical methods like Runge-Kutta (RK)

Effective Hamiltonian

- Map not yet practically useful
  - map is not solvable
  - step size is fixed by one undulator period
- Instead, we build effective Hamiltonian using Baker-Campbel-Hausdorff (BCH)

$$
\begin{aligned}
H_{\text{eff}} &= -\frac{1}{\lambda_u}\left(\mathscr{G}_S + \mathscr{G}_F + \mathscr{G}_V\right) \\
&\quad -\frac{1}{2\lambda_u}\left(:\mathscr{G}_S : \mathscr{G}_F + :\mathscr{G}_S : \mathscr{G}_V + :\mathscr{G}_F : \mathscr{G}_V\right) + \ldots
\end{aligned}
$$

- Now, it is autonomous → arbitrary step size. Can apply numerical methods like Runge-Kutta (RK)

Effective Hamiltonian

- Map not yet practically useful
  - map is not solvable
  - step size is fixed by one undulator period
- Instead, we build effective Hamiltonian using Baker-Campbel-Hausdorff (BCH)

$$
\begin{aligned}
H_{\text{eff}} &= -\frac{1}{\lambda_u}(\mathscr{G}_S + \mathscr{G}_F + \mathscr{G}_V) \\
&\quad -\frac{1}{2\lambda_u}(:\mathscr{G}_S:\mathscr{G}_F + :\mathscr{G}_S:\mathscr{G}_V + :\mathscr{G}_F:\mathscr{G}_V) + \ldots
\end{aligned}
$$

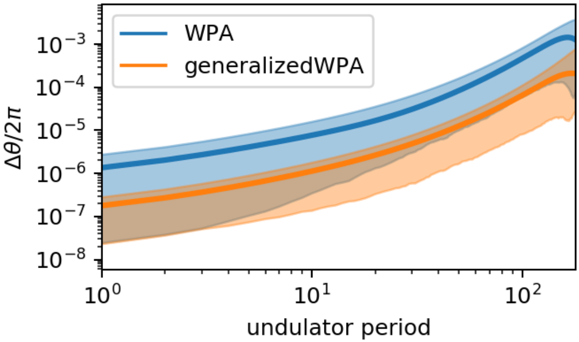
- Now, it is autonomous → arbitrary step size. Can apply numerical methods like Runge-Kutta (RK)

Example

Hamiltonian

Figure
: Pusher Comparison

- $\Delta\theta \equiv |\theta - \theta_{\text{ref}}|$
- $\theta_{\text{ref}} \leftarrow$ using exact H
- Fixed envelope
  $\mathbb{K}_1 = A_0 e^{-x^2/\sigma_x^2} e^{z/L_G}$
  $\sigma_x = 55\,\mu\text{m}$
  $L_G = 50\lambda_u$
- eBeam param
  $\gamma = 1000$
  $\Delta\gamma/\gamma = 2 \times 10^{-4}$



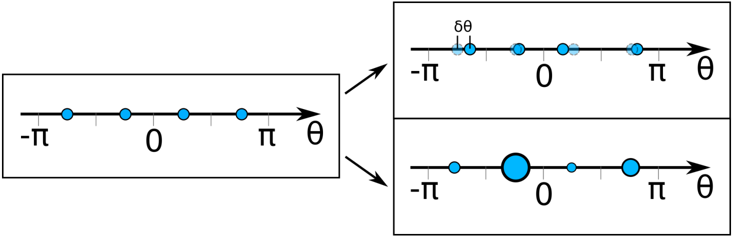
Outline

1. Introduction
2. Generalize WPA
   - How?
   - Review : Perturbative Lie Map
   - Hamiltonian
   - Generator
   - Effective Hamiltonian
3. Improve Shot Noise Modeling
   - Review of shot-noise modeling methods
   - Improved shot-noise modeling methods
   - IMPACT code suit and example
4. Conclusion

1D Model

5D mirroring

- Here, we review two 1D methods by Dr. Fawley and Dr. McNeil et.al.

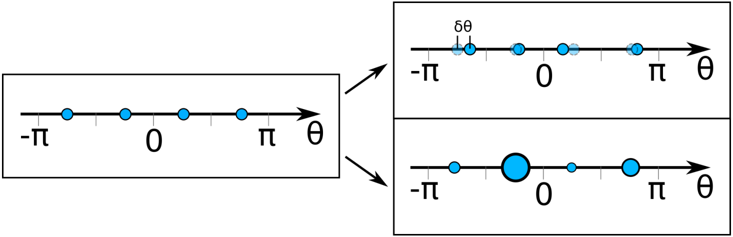


- 1st step : uniform along temporal coordinate → zero bunching factor
- Next step : add (temporal coordinate / charge weight) perturbations
  - to model physical shot noise : correct RMS bunching factor $\langle b_h b_h^* \rangle = 1/N_e$ at least

1D Model

5D mirroring

- Here, we review two 1D methods by Dr. Fawley and Dr. McNeil et.al.

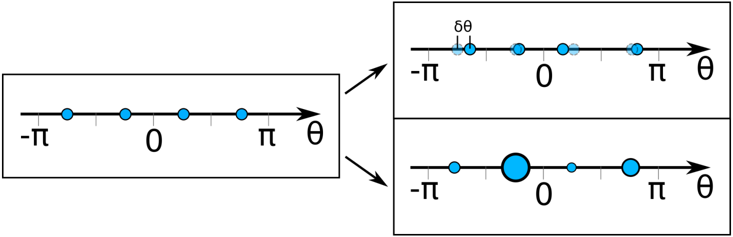


- 1st step : uniform along temporal coordinate → zero bunching factor
- Next step : add (temporal coordinate / charge weight) perturbations
    - to model physical shot noise : correct RMS bunching factor $\langle b_h b_h^* \rangle = 1/N_e$ at least

1D Model

5D mirroring

- Here, we review two 1D methods by Dr. Fawley and Dr. McNeil et.al.

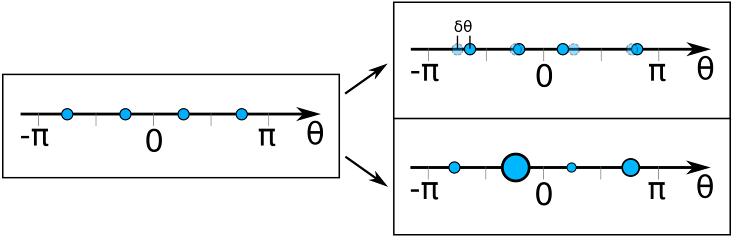


- 1st step : uniform along temporal coordinate → zero bunching factor
- Next step : add (temporal coordinate / charge weight) perturbations
  - to model physical shot noise : correct RMS bunching factor $\langle b_h b_h^* \rangle = 1/N_e$ at least

1D Model

5D mirroring

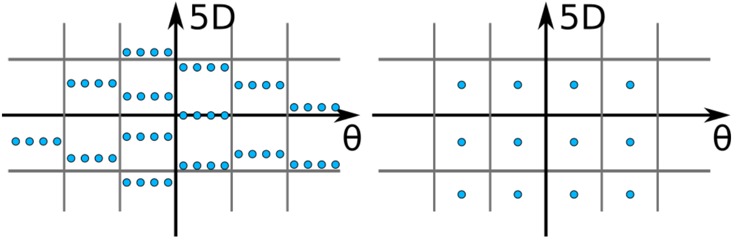
- Here, we review two 1D methods by Dr. Fawley and Dr. McNeil et.al.



- 1st step : uniform along temporal coordinate → zero bunching factor
- Next step : add (temporal coordinate / charge weight) perturbations
  - to model physical shot noise : correct RMS bunching factor $\langle b_h b_h^* \rangle = 1/N_e$ at least

6D extension

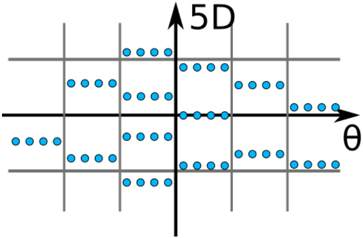
- Here, we review two 6D extension methods of the 1D model.

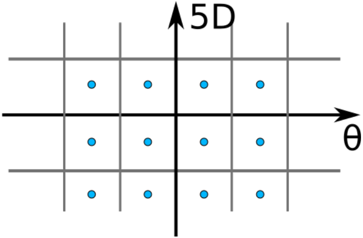


5D mirroring (Fawley) 6D volume division (McNeil et.al.)

6D extension

- Here, we review two 6D extension methods of the 1D model.





5D mirroring (Fawley)

6D volume division (McNeil et.al.)

Remarks

1D Model

- 5D mirroring
  - same 5D coordinates x, y, p_x, p_y,, γ among a set of particles called "beamlet"
    - each beamlet is based on 1D model
  - member particles of a beamlet are **not** statistically independent
    - numerical shot-noise upon particles *migration* across the numerical mesh
- 6D volume division
  - comes with the charge perturbation $\leftarrow$ Poisson principle
  - all particles are statistically indepedent
    - No shot-noise upon particles migration
  - requires a lot of particles as division over 6 dimension can be huge

Remarks

1D Model

- 5D mirroring
  - same 5D coordinates x, y, p_x, p_y, γ among a set of particles called "beamlet"
    - each beamlet is based on 1D model
  - member particles of a beamlet are **not** statistically independent
    - numerical shot-noise upon particles *migration* across the numerical mesh
- 6D volume division
  - comes with the charge perturbation ← Poisson principle
  - all particles are statistically indepedent
    - No numerical shot-noise upon particles migration
  - requires a lot of particles as division over 6 dimension can be huge

Remarks

1D Model

- 5D mirroring
  - same 5D coordinates x, y, p_x, p_y,, γ among a set of particles called "beamlet"
    - each beamlet is based on 1D model
  - member particles of a beamlet are **not** statistically independent
    - numerical shot-noise upon particles *migration* across the numerical mesh
- 6D volume division
  - comes with the charge perturbation ← Poisson principle
  - all particles are statistically indepedent
    - No shot-noise upon particle migration
  - requires a lot of particles as division over 6 dimension can be huge

Remarks

1D Model

- 5D mirroring
  - same 5D coordinates $x, y, p_x, p_y,, \gamma$ among a set of particles called "beamlet"
    - each beamlet is based on 1D model
  - member particles of a beamlet are **not** statistically independent
    - numerical shot-noise upon particles *migration* across the numerical mesh
- 6D volume division
  - comes with the charge perturnbation $\leftarrow$ Poisson principle
  - all particles are statistically indepedent
    - No shot-noise upon particles *migration*
  - requires a lot of particles as division over 6 dimension can be huge

Remarks

1D Model

- 5D mirroring
  - same 5D coordinates x, y, p_x, p_y, γ among a set of particles called "beamlet"
    - each beamlet is based on 1D model
  - member particles of a beamlet are **not** statistically independent
    - numerical shot-noise upon particles *migration* across the numerical mesh
- 6D volume division
  - comes with the charge perturbation ← Poisson principle
  - all particles are statistically indepedent
    - No shot-noise upon particles migration
  - requires a lot of particles as division over 6 dimension can be huge

Remarks

1D Model

- 5D mirroring
  - same 5D coordinates $x, y, p_x, p_y,, \gamma$ among a set of particles called "beamlet"
    - each beamlet is based on 1D model
  - member particles of a beamlet are **not** statistically independent
    - numerical shot-noise upon particles *migration* across the numerical mesh
- 6D volume division
  - comes with the charge perturnbation $\leftarrow$ Poisson principle
  - all particles are statistically indepedent
    - No shot-noise upon particles *migration*
  - requires a lot of particles as division over 6 dimension can be huge

Remarks

1D Model

- 5D mirroring
  - same 5D coordinates $x, y, p_x, p_y,, \gamma$ among a set of particles called "beamlet"
    - each beamlet is based on 1D model
  - member particles of a beamlet are **not** statistically independent
    - numerical shot-noise upon particles *migration* across the numerical mesh
- 6D volume division
  - comes with the charge perturnbation ← Poisson principle
  - all particles are statistically indepedent
    - No shot-noise upon particles *migration*
  - requires a lot of particles as division over 6 dimension can be huge

Remarks

1D Model

- 5D mirroring
  - same 5D coordinates $x, y, p_x, p_y, , \gamma$ among a set of particles called "beamlet"
    - each beamlet is based on 1D model
  - member particles of a beamlet are **not** statistically independent
    - numerical shot-noise upon particles *migration* across the numerical mesh
- 6D volume division
  - comes with the charge perturnbation $\leftarrow$ Poisson principle
  - all particles are statistically indepedent
    - No shot-noise upon particles *migration*
  - requires a lot of particles as division over 6 dimension can be huge

Remarks

1D Model

- 5D mirroring
  - same 5D coordinates $x, y, p_x, p_{y,}, \gamma$ among a set of particles called "beamlet"
    - each beamlet is based on 1D model
  - member particles of a beamlet are **not** statistically independent
    - numerical shot-noise upon particles *migration* across the numerical mesh
- 6D volume division
  - comes with the charge perturnbation $\leftarrow$ Poisson principle
  - all particles are statistically indepedent
    - No shot-noise upon particles *migration*
  - requires a lot of particles as division over 6 dimension can be huge

Remarks

1D Model

- 5D mirroring
  - same 5D coordinates $x, y, p_x, p_y,, \gamma$ among a set of particles called "beamlet"
    - each beamlet is based on 1D model
  - member particles of a beamlet are **not** statistically independent
    - numerical shot-noise upon particles *migration* across the numerical mesh
- 6D volume division
  - comes with the charge perturnbation $\leftarrow$ Poisson principle
  - all particles are statistically indepedent
    - No shot-noise upon particles *migration*
  - requires a lot of particles as division over 6 dimension can be huge

Beamlet : a statistically independent entity

- We adopt the 5D mirroring strategy
- Our idea is to interpret one beamlet as one statistically independent entity
  - based on the fact that member particles are not statistically indepedent and
    - motion of beamlets are macroscopic ($\gtrsim \lambda_r$)
    - motion of member particles are microscopic ($\lesssim \lambda_r$)
  - phase-space coordinate of a beamlet is given by the average over the member particles in it
  - This allows natural loading method :
    - random number of particle density functions or
    - external uptream tracking code

Beamlet : a statistically independent entity

- We adopt the 5D mirroring strategy
- Our idea is to interpret one beamlet as one statistically independent entity
  - based on the fact that member particles are not statistically indepedent and
    - motion of beamlets are macroscopic ($\gtrsim \lambda_r$)
    - motion of member particles are microscopic ($\lesssim \lambda_r$)
  - phase-space coordinate of a beamlet is given by the average over the member particles in it
  - This allows natural loading method :
    - random number of particle density functions or
    - external uptream tracking code

Beamlet : a statistically independent entity

- We adopt the 5D mirroring strategy
- Our idea is to interpret one beamlet as one statistically independent entity
  - based on the fact that member particles are not statistically indepedent and
    - motion of beamlets are macroscopic ($\gtrsim \lambda_r$)
    - motion of member particles are microscopic ($\lesssim \lambda_r$)
  - phase-space coordinate of a beamlet is given by the average over the member particles in it
  - This allows natural loading method :
    - random number of particle density functions or
    - external uptream tracking code

Beamlet : a statistically independent entity

- We adopt the 5D mirroring strategy
- Our idea is to interpret one beamlet as one statistically independent entity
  - based on the fact that member particles are not statistically indepedent and
    - motion of beamlets are macroscopic ($\gtrsim \lambda_r$)
    - motion of member particles are microscopic ($\lesssim \lambda_r$)
  - phase-space coordinate of a beamlet is given by the average over the member particles in it
  - This allows natural loading method :
    - random number of particle density functions or
    - external uptream tracking code

Beamlet : a statistically independent entity

- We adopt the 5D mirroring strategy
- Our idea is to interpret one beamlet as one statistically independent entity
    - based on the fact that member particles are not statistically indepedent and
        - motion of beamlets are macroscopic ($\gtrsim \lambda_r$)
        - motion of member particles are microscopic ($\lesssim \lambda_r$)
    - phase-space coordinate of a beamlet is given by the average over the member particles in it
    - This allows natural loading method :
        - random number of particle density functions or
        - external uptream tracking code

Beamlet : a statistically independent entity

- We adopt the 5D mirroring strategy
- Our idea is to interpret one beamlet as one statistically independent entity
  - based on the fact that member particles are not statistically indepedent and
    - motion of beamlets are macroscopic ($\gtrsim \lambda_r$)
    - motion of member particles are microscopic ($\lesssim \lambda_r$)
  - phase-space coordinate of a beamlet is given by the average over the member particles in it
  - This allows natural loading method :
    - random number of particle density functions or
    - external uptream tracking code

Beamlet : a statistically independent entity

- We adopt the 5D mirroring strategy
- Our idea is to interpret one beamlet as one statistically independent entity
  - based on the fact that member particles are not statistically indepedent and
    - motion of beamlets are macroscopic ($\gtrsim \lambda_r$)
    - motion of member particles are microscopic ($\lesssim \lambda_r$)
  - phase-space coordinate of a beamlet is given by the average over the member particles in it
  - This allows natural loading method :
    - random number of particle density functions or
    - external uptream tracking code

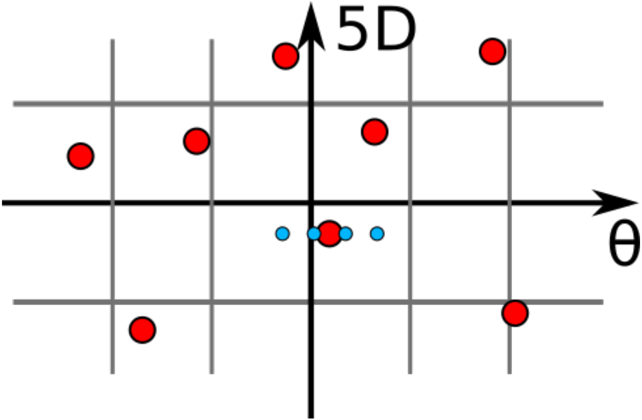
Beamlet : a statistically independent entity

- We adopt the 5D mirroring strategy
- Our idea is to interpret one beamlet as one statistically independent entity
  - based on the fact that member particles are not statistically indepedent and
    - motion of beamlets are macroscopic ($\gtrsim \lambda_r$)
    - motion of member particles are microscopic ($\lesssim \lambda_r$)
  - phase-space coordinate of a beamlet is given by the average over the member particles in it
  - This allows natural loading method :
    - random number of particle density functions or
    - external uptream tracking code

Beamlet : a statistically independent entity

- We adopt the 5D mirroring strategy
- Our idea is to interpret one beamlet as one statistically independent entity
  - based on the fact that member particles are not statistically indepedent and
    - motion of beamlets are macroscopic ($\gtrsim \lambda_r$)
    - motion of member particles are microscopic ($\lesssim \lambda_r$)
  - phase-space coordinate of a beamlet is given by the average over the member particles in it
  - This allows natural loading method :
    - random number of particle density functions or
    - external uptream tracking code

Particle Loading



- ● : beamlet
- ● : member particle

Benchmark / shot-noise from migration

- Migrate all member particles when a beamlet migrate
- Smoother numerical discretization
  - weight and shape functions are evaluated **at the beamlet position**

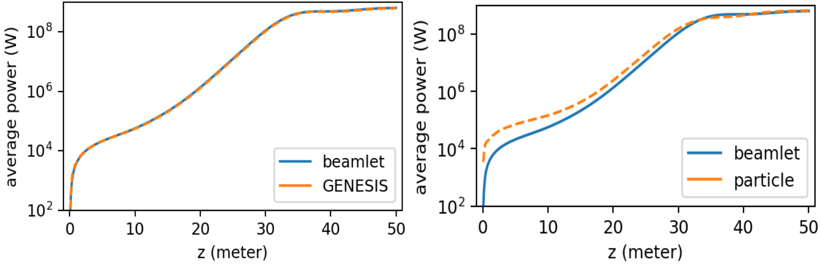


Figure. Benchmark : Beamlet vs individual particle migration. NGLS-like parameters are used.

Benchmark / shot-noise from migration

- Migrate all member particles when a beamlet migrate
- Smoother numerical discretization
  - weight and shape functions are evaluated **at the beamlet position**

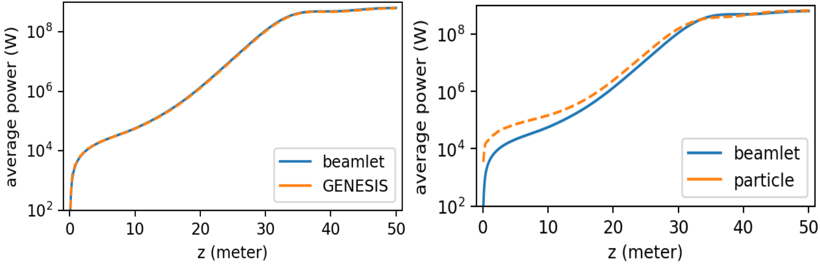


Figure. Benchmark : Beamlet vs individual particle migration. NGLS-like parameters are used.

Benchmark / shot-noise from migration

- Migrate all member particles when a beamlet migrate
- Smoother numerical discretization
  - weight and shape functions are evaluated **at the beamlet position**

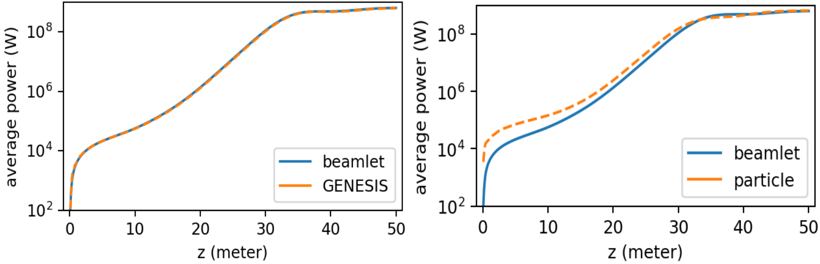


Figure. Benchmark : Beamlet vs individual particle migration. NGLS-like parameters are used.

Benchmark / shot-noise from migration

- Migrate all member particles when a beamlet migrate
- Smoother numerical discretization
  - weight and shape functions are evaluated **at the beamlet position**

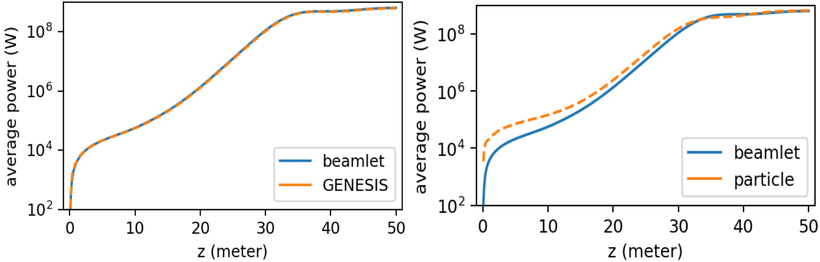


Figure. Benchmark : Beamlet vs individual particle migration. NGLS-like parameters are used.

Slippage Resolution

- Typical implementation of slippage is to copy the field data from the previous temporal mesh point to the next temporal mesh point
- Beamlet migration enables arbitrary slippage resolution through moving window
  - It also allows natural slippage modeling through arbitrary length of non-resonant tranport like drift

Data Copy from previous slice

```
for i in [0,1,2,...,nt]:
    Fld.data[:,:,nt-i] = Fld.data[:,:,nt-i-1]
```

Moving wondow : change of domain

```
Fld.domain.theta[:]
  = Fld.domain.theta[:]+dtheta
```

Slippage Resolution

- Typical implementation of slippage is to copy the field data from the previous temporal mesh point to the next temporal mesh point
- Beamlet migration enables arbitrary slippage resolution through moving window
  - It also allows natural slippage modeling through arbitrary length of non-resonant tranport like drift

Data Copy from previous slice

```
for i in [0,1,2,...,nt]:
    Fld.data[:,:,nt-i] = Fld.data[:,:,nt-i-1]
```

Moving wondow : change of domain

```
Fld.domain.theta[:]
  = Fld.domain.theta[:]+dtheta
```

Slippage Resolution

- Typical implementation of slippage is to copy the field data from the previous temporal mesh point to the next temporal mesh point
- Beamlet migration enables arbitrary slippage resolution through moving window
  - It also allows natural slippage modeling through arbitrary length of non-resonant tranport like drift

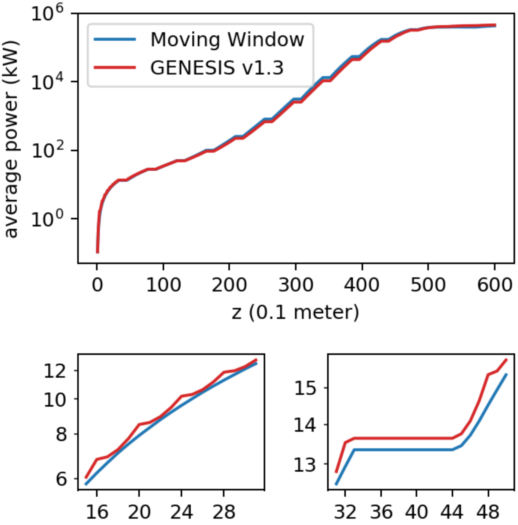
Data Copy from previous slice

```
for i in [0,1,2,...,nt]:
    Fld.data[:,:,nt-i] = Fld.data[:,:,nt-i-1]
```

Moving wondow : change of domain

```
Fld.domain.theta[:]
  = Fld.domain.theta[:]+dtheta
```

Copying Data vs Moving Window



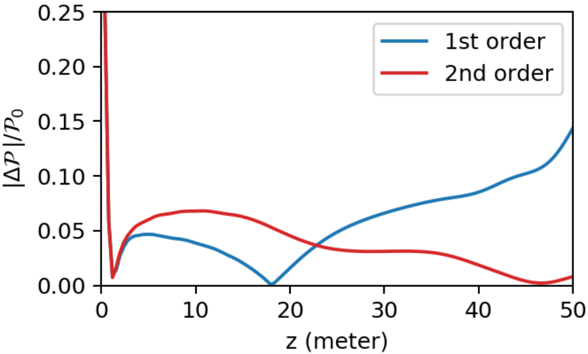
- integration step size $= 5\lambda_u$
- temporal mesh size $= 20\lambda_r$
- One slippage operation every 4 step when copying data is used.

Split and Composition

- Field solver can be split into two operation - diffusion $\mathscr{F}_\perp$ and slippage $\mathscr{F}_\parallel$
- 2nd order composition method

$$\mathscr{F}_\parallel\left(\frac{\Delta z}{2}\right)\mathscr{F}_\perp(\Delta z)\mathscr{F}_\parallel\left(\frac{\Delta z}{2}\right)$$

is possbile due to arbitrary slippage resolution.

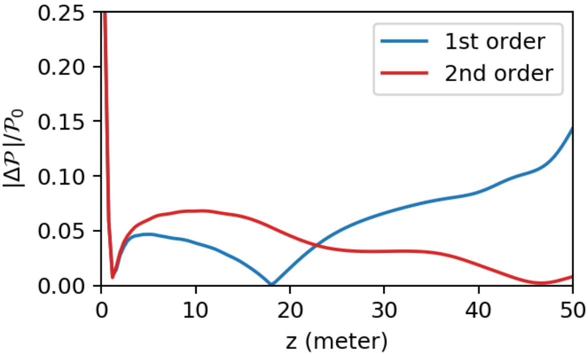


Split and Composition

- Field solver can be split into two operation - diffusion $\mathscr{F}_\perp$ and slippage $\mathscr{F}_\parallel$
- 2nd order composition method

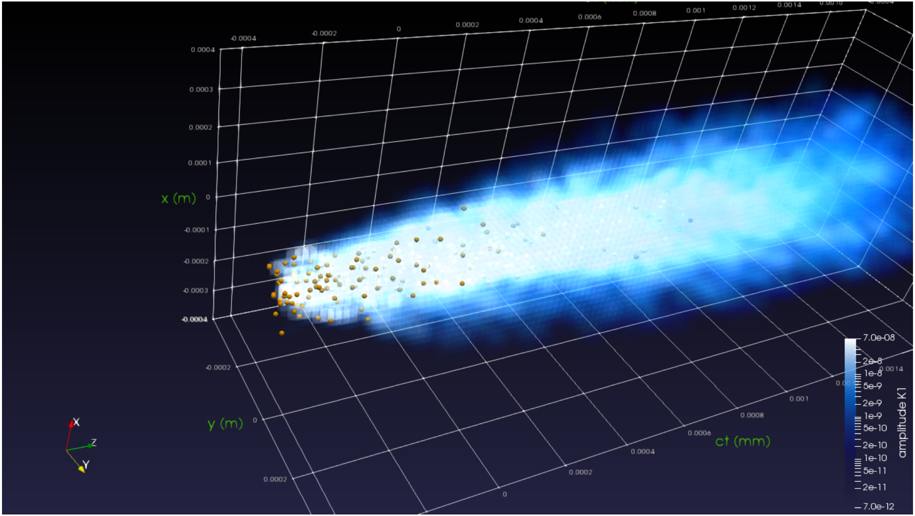
$$\mathscr{F}_\parallel\left(\frac{\Delta z}{2}\right)\mathscr{F}_\perp(\Delta z)\,\mathscr{F}_\parallel\left(\frac{\Delta z}{2}\right)$$

is possbile due to arbitrary slippage resolution.



IMPACT code suit and example

All the methods presented are implemented in IMPACT code suite. Example:



Outline

Conlucsion

- Advances in numerical methods for FEL simulation under the WPA are presented
- We generalized WPA perturbatively using Lie map method
- We improved numerical shot-noise modeling method
  - supressed artificial shot-noise upon migration
  - enabled smoother numerical descretization
    - arbitrary mesh size, weight/shape functions, slippage resolution
  - increased field solver accuracy
- All these methods are implemented in beam dynamics simulation framework IMPACT code suite

Thank you for your attention!

Conlucsion

- Advances in numerical methods for FEL simulation under the WPA are presented
- We generalized WPA perturbatively using Lie map method
- We improved numerical shot-noise modeling method
  - supressed artificial shot-noise upon migration
  - enabled smoother numerical descretization
    - arbitrary mesh size, weight/shape function, slippage resolution
  - increased field solver accuracy
- All these methods are implemented in beam dynamics simulation framework IMPACT code suite

Thank you for your attention!

Conlucsion

- Advances in numerical methods for FEL simulation under the WPA are presented
- We generalized WPA perturbatively using Lie map method
- We improved numerical shot-noise modeling method
    - supressed artificial shot-noise upon migration
    - enabled smoother numerical descretization
        - arbitrary mesh size, weight/shape function, slippage resolution
    - increased field solver accuracy
- All these methods are implemented in beam dynamics simulation framework IMPACT code suite

Thank you for your attention!

Conlucsion

- Advances in numerical methods for FEL simulation under the WPA are presented
- We generalized WPA perturbatively using Lie map method
- We improved numerical shot-noise modeling method
  - supressed artificial shot-noise upon migration
  - enabled smoother numerical descretization
    - arbitrary mesh size, weight/shape function, slippage resolution
  - increased field solver accuracy
- All these methods are implemented in beam dynamics simulation framework IMPACT code suite

Thank you for your attention!

Conlucsion

- Advances in numerical methods for FEL simulation under the WPA are presented
- We generalized WPA perturbatively using Lie map method
- We improved numerical shot-noise modeling method
  - supressed artificial shot-noise upon migration
  - enabled smoother numerical descretization
    - arbitrary mesh size, weight/shape function, slippage resolution
  - increased field solver accuracy
- All these methods are implemented in beam dynamics simulation framework IMPACT code suite

Thank you for your attention!

Conlucsion

- Advances in numerical methods for FEL simulation under the WPA are presented
- We generalized WPA perturbatively using Lie map method
- We improved numerical shot-noise modeling method
  - supressed artificial shot-noise upon migration
  - enabled smoother numerical descretization
    - arbitrary mesh size, weight/shape function, slippage resolution
  - increased field solver accuracy
- All these methods are implemented in beam dynamics simulation framework IMPACT code suite

Thank you for your attention!

Conlucsion

- Advances in numerical methods for FEL simulation under the WPA are presented
- We generalized WPA perturbatively using Lie map method
- We improved numerical shot-noise modeling method
    - supressed artificial shot-noise upon migration
    - enabled smoother numerical descretization
        - arbitrary mesh size, weight/shape function, slippage resolution
    - increased field solver accuracy
- All these methods are implemented in beam dynamics simulation framework IMPACT code suite

Thank you for your attention!

Conlucsion

- Advances in numerical methods for FEL simulation under the WPA are presented
- We generalized WPA perturbatively using Lie map method
- We improved numerical shot-noise modeling method
  - supressed artificial shot-noise upon migration
  - enabled smoother numerical descretization
    - arbitrary mesh size, weight/shape function, slippage resolution
  - increased field solver accuracy
- All these methods are implemented in beam dynamics simulation framework IMPACT code suite

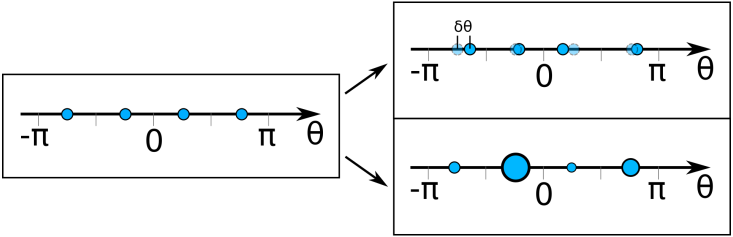
Thank you for your attention!

Conlucsion

- Advances in numerical methods for FEL simulation under the WPA are presented
- We generalized WPA perturbatively using Lie map method
- We improved numerical shot-noise modeling method
  - supressed artificial shot-noise upon migration
  - enabled smoother numerical descretization
    - arbitrary mesh size, weight/shape function, slippage resolution
  - increased field solver accuracy
- All these methods are implemented in beam dynamics simulation framework IMPACT code suite

Thank you for your attention!

1D Model

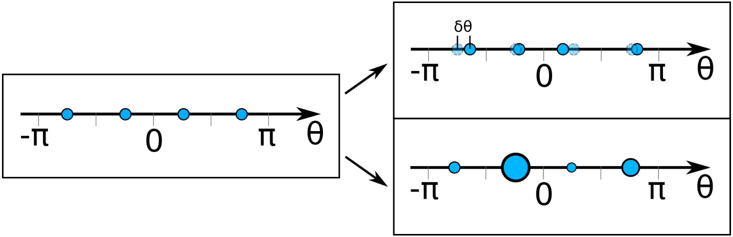


- 1st step : uniform along temporal coordinate → zero bunching factor

$$b_h^0 = \frac{1}{N_e} \sum_{j=1}^{M} m_j e^{ih\theta_j} = 0$$

- Next step is to add perturbations to model physical shot noise $:\langle b_h b_h^* \rangle = 1/N_e$

1D Model



- 1st step : uniform along temporal coordinate → zero bunching factor

$$b_h^0 = \frac{1}{N_e} \sum_{j=1}^{M} m_j e^{ih\theta_j} = 0$$

- Next step is to add perturbations to model physical shot noise :$\langle b_h b_h^* \rangle = 1/N_e$

Temporal coordinate perturbation

- Temporal coordinate perturbation (Fawley)

$$\delta\theta_j = \sum_{h'=1}^{M/2} \xi_{h'} e^{-ih'\theta_j}$$

- Bunching Factor becomes

$$b_h = \frac{1}{N_e} \sum_{j=1}^{M} m_j e^{ih\left(\theta_j + \delta\theta_j\right)} \simeq ih\xi_h$$

- Therefore, RMS becomes

$$\langle b_h b_h^* \rangle \simeq h^2 \langle \xi_h \xi_h^* \rangle = \frac{1}{N_e}$$

  - provided that

$$\langle \xi_h \xi_h^* \rangle = 1/\left(h^2 N_e\right)$$

Temporal coordinate perturbation

- Temporal coordinate perturbation (Fawley)

$$\delta\theta_j = \sum_{h'=1}^{M/2} \xi_{h'} e^{-ih'\theta_j}$$

- Bunching Factor becomes

$$b_h = \frac{1}{N_e} \sum_{j=1}^{M} m_j e^{ih\left(\theta_j + \delta\theta_j\right)} \simeq ih\xi_h$$

- Therefore, RMS becomes

$$\langle b_h b_h^* \rangle \simeq h^2 \langle \xi_h \xi_h^* \rangle = \frac{1}{N_e}$$

  - provided that

$$\langle \xi_h \xi_h^* \rangle = 1/\left(h^2 N_e\right)$$

Temporal coordinate perturbation

- Temporal coordinate perturbation (Fawley)

$$\delta\theta_j = \sum_{h'=1}^{M/2} \xi_{h'} e^{-ih'\theta_j}$$

- Bunching Factor becomes

$$b_h = \frac{1}{N_e} \sum_{j=1}^{M} m_j e^{ih\left(\theta_j + \delta\theta_j\right)} \simeq ih\xi_h$$

- Therefore, RMS becomes

$$\langle b_h b_h^* \rangle \simeq h^2 \langle \xi_h \xi_h^* \rangle \equiv \frac{1}{N_e}$$

  - provided that

$$\langle \xi_h \xi_h^* \rangle = 1/\left(h^2 N_e\right)$$

Temporal coordinate perturbation

- Temporal coordinate perturbation (Fawley)

$$\delta\theta_j = \sum_{h'=1}^{M/2} \xi_{h'} e^{-ih'\theta_j}$$

- Bunching Factor becomes

$$b_h = \frac{1}{N_e} \sum_{j=1}^{M} m_j e^{ih\left(\theta_j + \delta\theta_j\right)} \simeq ih\xi_h$$

- Therefore, RMS becomes

$$\langle b_h b_h^* \rangle \simeq h^2 \langle \xi_h \xi_h^* \rangle \equiv \frac{1}{N_e}$$

  - provided that

$$\langle \xi_h \xi_h^* \rangle = 1/\left(h^2 N_e\right)$$

Charge weight perturbation

- Charge weight perturbation (McNeil et.al.)
  - Let $\tilde{m}_j \equiv m_j + \delta m_j$ such that

$$\left\langle \tilde{m}_j \right\rangle = \left\langle \tilde{m}_j^2 \right\rangle - \left\langle \tilde{m}_j \right\rangle^2 = \frac{N_e}{M}$$

  - Then, from $b_h = \frac{1}{N_e} \sum_{j=1}^{M} \tilde{m}_j e^{ih\theta_j}$

$$\begin{aligned} \langle b_h b_h^* \rangle &= \frac{1}{N_e^2} \sum_j^M \sum_k^M \left\langle \tilde{m}_j \tilde{m}_k \right\rangle e^{ih\left(\theta_j - \theta_k\right)} \\ &= \frac{1}{N_e^2} \sum_j^M \left(\left\langle \tilde{m}_j^2 \right\rangle - \left\langle \tilde{m}_j \right\rangle^2 \right) = \frac{1}{N_e} \end{aligned}$$

Charge weight perturbation

- Charge weight perturbation (McNeil et.al.)
  - Let $\tilde{m}_j \equiv m_j + \delta m_j$ such that

$$\left\langle \tilde{m}_j \right\rangle = \left\langle \tilde{m}_j^2 \right\rangle - \left\langle \tilde{m}_j \right\rangle^2 = \frac{N_e}{M}$$

  - Then, from $b_h = \frac{1}{N_e} \sum_{j=1}^{M} \tilde{m}_j e^{ih\theta_j}$

$$\begin{aligned} \langle b_h b_h^* \rangle &= \frac{1}{N_e^2} \sum_j^M \sum_k^M \langle \tilde{m}_j \tilde{m}_k \rangle e^{ih(\theta_j - \theta_k)} \\ &= \frac{1}{N_e^2} \sum_j^M \left(\left\langle \tilde{m}_j^2 \right\rangle - \left\langle \tilde{m}_j \right\rangle^2 \right) = \frac{1}{N_e} \end{aligned}$$

Charge weight perturbation

- Charge weight perturbation (McNeil et.al.)
  - Let $\tilde{m}_j \equiv m_j + \delta m_j$ such that

$$\langle \tilde{m}_j \rangle = \langle \tilde{m}_j^2 \rangle - \langle \tilde{m}_j \rangle^2 = \frac{N_e}{M}$$

  - Then, from $b_h = \frac{1}{N_e} \sum_{j=1}^{M} \tilde{m}_j e^{ih\theta_j}$

$$\begin{aligned} \langle b_h b_h^* \rangle &= \frac{1}{N_e^2} \sum_j^M \sum_k^M \langle \tilde{m}_j \tilde{m}_k \rangle e^{ih\left(\theta_j - \theta_k\right)} \\ &= \frac{1}{N_e^2} \sum_j^M \left(\langle \tilde{m}_j^2 \rangle - \langle \tilde{m}_j \rangle^2 \right) = \frac{1}{N_e} \end{aligned}$$

References

- Qiang, J. and Ding, Y. and Emma, P. and Huang, Z. and Ratner, D. and Raubenheimer, T. O. and Venturini, M. and Zhou, F.
  Phys. Rev. Accel. Beams., **20**, 054402 (2017)
- Hwang, Kilean and Qiang, Ji
  "Advances in numerical methods under wiggler period averaging for free electron laser simulation", *Phys. Rev. Accel. Beams.*, (submitted)
- William M. Fawley
  Phys. Rev. ST Accel. Beams., **5**, 070701 (2002)
- B.W. J. McNeil, M.W. Poole, and G. R.M. Robb
  Phys. Rev. ST Accel. Beams., **6**, 070701 (2003)
- S. Reiche
  Nucl. Instrum. Methods Phys. Res., A **429**, p.243 (1999)
- J. Qiang, R. Ryne, S. Habib, V. Decyk
  J. Comp. Phys. **163**, 434 (2000)

References

Qiang, J. and Ding, Y. and Emma, P. and Huang, Z. and Ratner, D. and Raubenheimer, T. O. and Venturini, M. and Zhou, F.
Phys. Rev. Accel. Beams., **20**, 054402 (2017)

Hwang, Kilean and Qiang, Ji
"Advances in numerical methods under wiggler period averaging for free electron laser simulation", *Phys. Rev. Accel. Beams.*, (submitted)

William M. Fawley
Phys. Rev. ST Accel. Beams., **5**, 070701 (2002)

B.W. J. McNeil, M.W. Poole, and G. R.M. Robb
Phys. Rev. ST Accel. Beams., **6**, 070701 (2003)

S. Reiche
Nucl. Instrum. Methods Phys. Res., A **429**, p.243 (1999)

J. Qiang, R. Ryne, S. Habib, V. Decyk
J. Comp. Phys. **163**, 434 (2000)

References

- Qiang, J. and Ding, Y. and Emma, P. and Huang, Z. and Ratner, D. and Raubenheimer, T. O. and Venturini, M. and Zhou, F.
  Phys. Rev. Accel. Beams., **20**, 054402 (2017)
- Hwang, Kilean and Qiang, Ji
  "Advances in numerical methods under wiggler period averaging for free electron laser simulation", *Phys. Rev. Accel. Beams.*, (submitted)
- William M. Fawley
  Phys. Rev. ST Accel. Beams., **5**, 070701 (2002)
- B.W. J. McNeil, M.W. Poole, and G. R.M. Robb
  Phys. Rev. ST Accel. Beams., **6**, 070701 (2003)
- S. Reiche
  Nucl. Instrum. Methods Phys. Res., A **429**, p.243 (1999)
- J. Qiang, R. Ryne, S. Habib, V. Decyk
  J. Comp. Phys. **163**, 434 (2000)

References

Qiang, J. and Ding, Y. and Emma, P. and Huang, Z. and Ratner, D. and Raubenheimer, T. O. and Venturini, M. and Zhou, F.
Phys. Rev. Accel. Beams., **20**, 054402 (2017)

Hwang, Kilean and Qiang, Ji
"Advances in numerical methods under wiggler period averaging for free electron laser simulation", *Phys. Rev. Accel. Beams.*, (submitted)

William M. Fawley
Phys. Rev. ST Accel. Beams., **5**, 070701 (2002)

B.W. J. McNeil, M.W. Poole, and G. R.M. Robb
Phys. Rev. ST Accel. Beams., **6**, 070701 (2003)

S. Reiche
Nucl. Instrum. Methods Phys. Res., A **429**, p.243 (1999)

J. Qiang, R. Ryne, S. Habib, V. Decyk
J. Comp. Phys. **163**, 434 (2000)

References

- Qiang, J. and Ding, Y. and Emma, P. and Huang, Z. and Ratner, D. and Raubenheimer, T. O. and Venturini, M. and Zhou, F.
  Phys. Rev. Accel. Beams., **20**, 054402 (2017)

- Hwang, Kilean and Qiang, Ji
  "Advances in numerical methods under wiggler period averaging for free electron laser simulation", *Phys. Rev. Accel. Beams.*, (submitted)

- William M. Fawley
  Phys. Rev. ST Accel. Beams., **5**, 070701 (2002)

- B.W. J. McNeil, M.W. Poole, and G. R.M. Robb
  Phys. Rev. ST Accel. Beams., **6**, 070701 (2003)

- S. Reiche
  Nucl. Instrum. Methods Phys. Res., A **429**, p.243 (1999)

- J. Qiang, R. Ryne, S. Habib, V. Decyk
  J. Comp. Phys. **163**, 434 (2000)

References

Qiang, J. and Ding, Y. and Emma, P. and Huang, Z. and Ratner, D. and Raubenheimer, T. O. and Venturini, M. and Zhou, F.
Phys. Rev. Accel. Beams., **20**, 054402 (2017)

Hwang, Kilean and Qiang, Ji
"Advances in numerical methods under wiggler period averaging for free electron laser simulation", *Phys. Rev. Accel. Beams.*, (submitted)

William M. Fawley
Phys. Rev. ST Accel. Beams., **5**, 070701 (2002)

B.W. J. McNeil, M.W. Poole, and G. R.M. Robb
Phys. Rev. ST Accel. Beams., **6**, 070701 (2003)

S. Reiche
Nucl. Instrum. Methods Phys. Res., A **429**, p.243 (1999)

J. Qiang, R. Ryne, S. Habib, V. Decyk
J. Comp. Phys. **163**, 434 (2000)

References

- E. Hairer and C. Lubich and G. Wanner
  Geometric Numerical Integration", *Springer*, (2006)
- Alex J. Dragt
  https://www.physics.umd.edu/dsat/

References

E. Hairer and C. Lubich and G. Wanner
Geometric Numerical Integration", *Springer*, (2006)

Alex J. Dragt
https://www.physics.umd.edu/dsat/