# Exploring the Validity of the Paraxial Approximation for Coherent Synchrotron Radiation Wake Fields

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# Outline of Talk

- Maxwell's wave equations
  - Effect of slowly-varying envelope approximation on PDEs
- Brief overviews of two methods:
  - CSRDG: Time-domain full Maxwell field solver
  - Paraxial CSR: Frequency-domain impedance code
- Comparisons of CSR wake fields
  - DESY XFEL BC0 fixed-width pipe approximation
  - DESY XFEL BC0 full geometry simulation
- Conclusions and Future Work

# Maxwell's Wave Equations

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- Beam coordinates $(s, x, y)$ and $\tau = ct$
- Assume electron bunch in straight trajectory (for derivation)

$$\frac{\partial^2 \mathbf{E}}{\partial s^2} + \frac{\partial^2 \mathbf{E}}{\partial x^2} + \frac{\partial^2 \mathbf{E}}{\partial y^2} - \frac{\partial^2 \mathbf{E}}{\partial \tau^2} = Z_0 \left( \frac{\partial \mathbf{j}}{\partial \tau} + c \boldsymbol{\nabla} \rho \right),$$

$$\frac{\partial^2 \mathbf{H}}{\partial s^2} + \frac{\partial^2 \mathbf{H}}{\partial x^2} + \frac{\partial^2 \mathbf{H}}{\partial y^2} - \frac{\partial^2 \mathbf{H}}{\partial \tau^2} = -\boldsymbol{\nabla} \times \mathbf{j},$$

- Define Fourier transform with respect to $(s - \tau)$

$$\hat{U}(s, x, y, k) = \int_{-\infty}^{\infty} U(s, x, y, \tau) e^{-ik(s-\tau)} d\tau,$$

$$U(s, x, y, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{U}(s, x, y, k) e^{ik(s-\tau)} dk$$

# Derivation of Paraxial Approximation - 2

- For each field $U(s,x,y,\tau)$ wave equation becomes

$$2ik\frac{\partial \hat{U}}{\partial s}+\frac{\partial^2 \hat{U}}{\partial s^2}+\frac{\partial^2 \hat{U}}{\partial x^2}+\frac{\partial^2 \hat{U}}{\partial y^2}=\hat{S}_U(s,x,y,k)$$

- Slowly-varying envelope or paraxial approximation

  given by $\left|2ik\frac{\partial \hat{U}}{\partial s}\right| \gg \left|\frac{\partial^2 \hat{U}}{\partial s^2}\right|$ to drop $\frac{\partial^2}{\partial s^2}$ term

- New adjusted “paraxial” equation with $\hat{S}_V \equiv \hat{S}_U$

$$2ik\frac{\partial \hat{V}}{\partial s}+\frac{\partial^2 \hat{V}}{\partial x^2}+\frac{\partial^2 \hat{V}}{\partial y^2}=\hat{S}_V(s,x,y,k)$$

- Some freq.-domain methods evolve $\hat{V}(s,x,y,k)$ as IBVP in $s$

# Derivation of Paraxial Approximation - 3

- Inverse FT of paraxial equation for $\hat{V}(s,x,y,k)$ becomes

$$-2\frac{\partial^2 V}{\partial s \partial \tau} + \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} - 2\frac{\partial^2 V}{\partial \tau^2} = S_V(s,x,y,\tau)$$

- Apply Galilean transformation $z = s - \tau$ to $V(s,x,y,\tau)$

$$2\frac{\partial^2 V}{\partial z \partial \tau} + \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} - 2\frac{\partial^2 V}{\partial \tau^2} = S_V(z+\tau,x,y,\tau)$$

- Compare to Galilean transformation for $U(s,x,y,\tau)$

$$2\frac{\partial^2 U}{\partial z \partial \tau} + \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \frac{\partial^2 U}{\partial \tau^2} = S_U(z+\tau,x,y,\tau)$$

- If $2\frac{\partial}{\partial z} - 2\frac{\partial}{\partial \tau} \approx 2\frac{\partial}{\partial z} - \frac{\partial}{\partial \tau}$ or $\left\|\frac{\partial}{\partial z}\right\| \gg \left\|\frac{\partial}{\partial \tau}\right\|$ then expect $U \approx V$

# Brief Overview of CSRDG Code

- CSRDG – MATLAB GPU-enabled Maxwell field solver for modeling CSR with a Discontinuous Galerkin (DG) finite element method [1]
- CSRDG Capabilities and Goals:
  - Compute electromagnetic fields generated by CSR in a given domain such as vacuum chambers
  - Compute wake functions and impedance (by FT of wake)
  - Visualize field and wake evolution throughout a simulation
  - Compare with other CSR methods and establish range of validity for paraxial methods

[1] D. A. Bizzozero, E. Gjonaj, and H. De Gersem "Coherent Synchrotron Radiation and Wake Fields with Discontinuous Galerkin Time Domain Methods", Proceedings of IPAC 17, Copenhagen, Denmark, 2017.

# Brief Overview of Paraxial Method

- Frequency-domain method developed by R. Warnock and D. Bizzozero [2]
- Key idea of method:
  - Starting from paraxial equation for $\hat{V}(s,x,y,k)$
  - Include curvilinear terms dependent on bunch trajectory
  - Add Fourier-series decomposition in $y$-coordinate (mode $p$)
  - Evolve $\hat{E}_{yp}(s,x,k)$ and $\hat{H}_{yp}(s,x,k)$ Schrödinger-type 1D PDEs in $s$ for each wavenumber $k$
  - Longitudinal impedance $\hat{E}_{sp}(s,x,k)$ obtained from $\hat{E}_{yp}, \hat{H}_{yp}$
  - Wake field obtained by inverse Fourier transform in $k$

[2] R. L. Warnock and D. A. Bizzozero , “Efficient computation of coherent synchrotron radiation in a rectangular chamber”, Phys. Rev. Accel. Beams **19**, 090705, September 2016.

# CSRDG Comparison for DESY BC0 - 1

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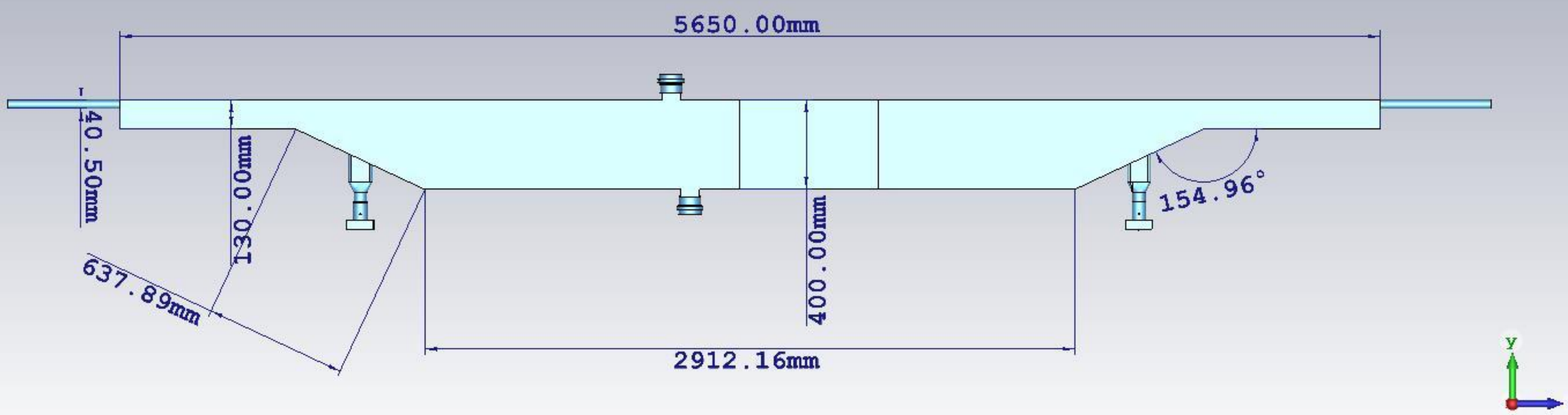
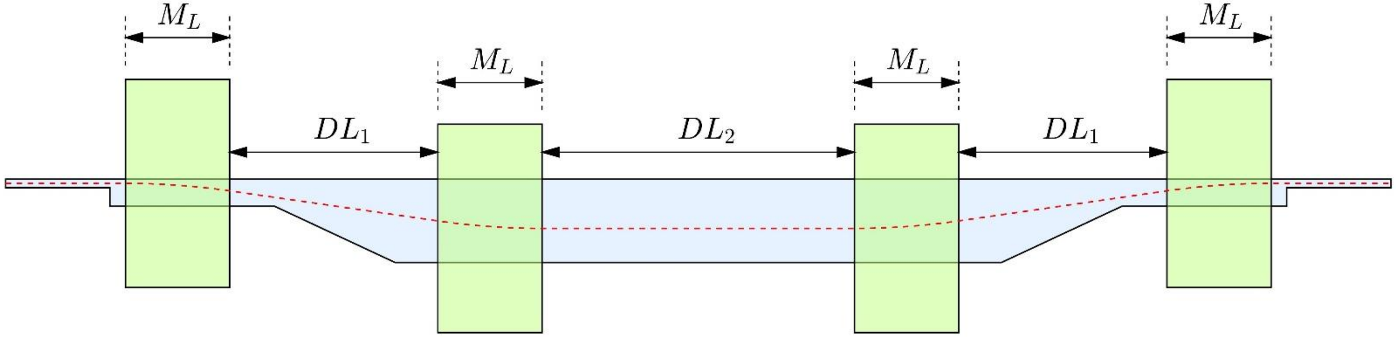
## DESY XFEL Bunch Compressor 0 Layout (CST)



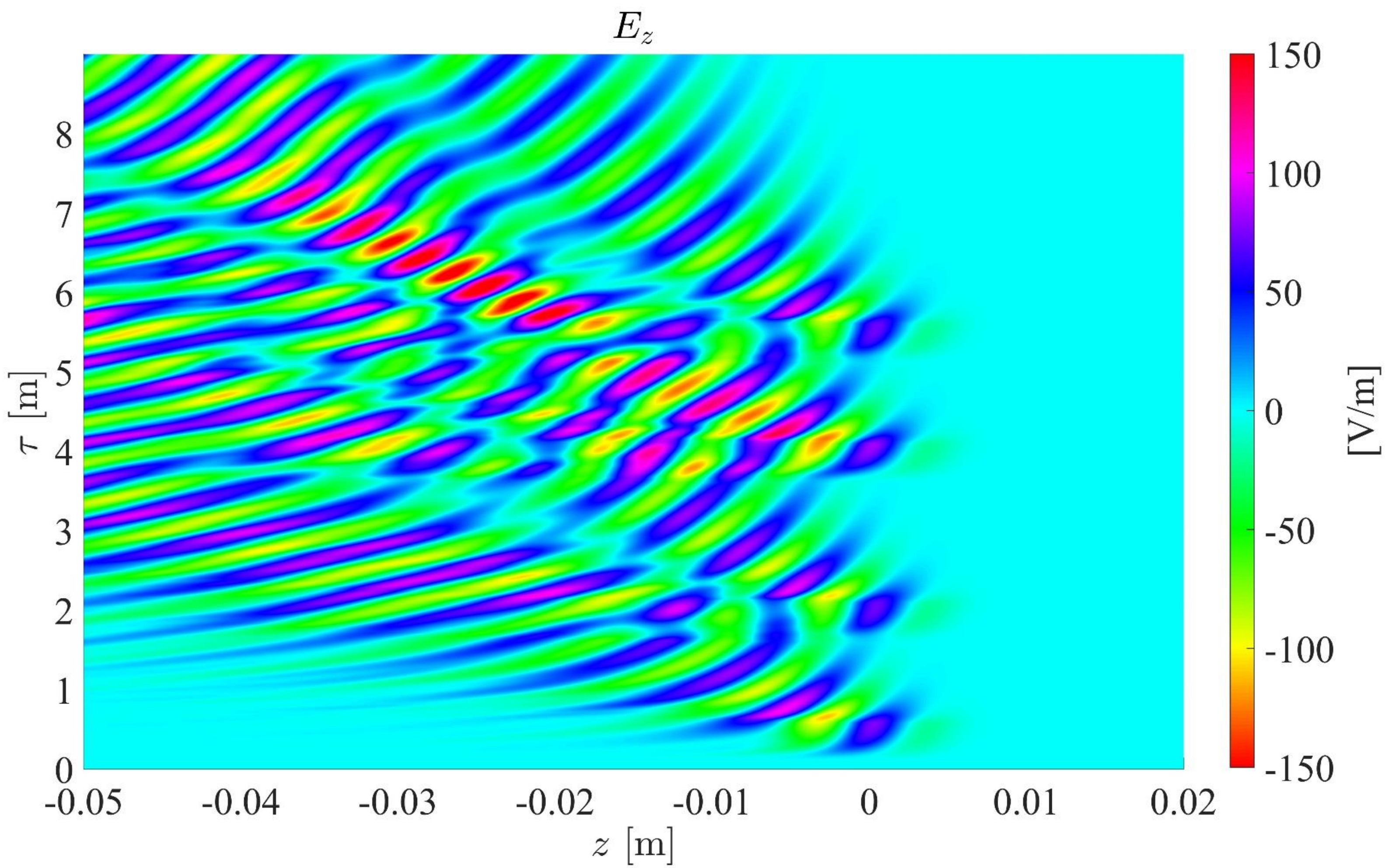
# CSRDG Comparison for DESY BC0 - 2

- DESY XFEL BC0 full geometry model
  - CSR and geometry generates wake
  - Source size: $\sigma_s = 2\,\mathrm{mm},\ \sigma_y = 0.1\,\mathrm{mm}$

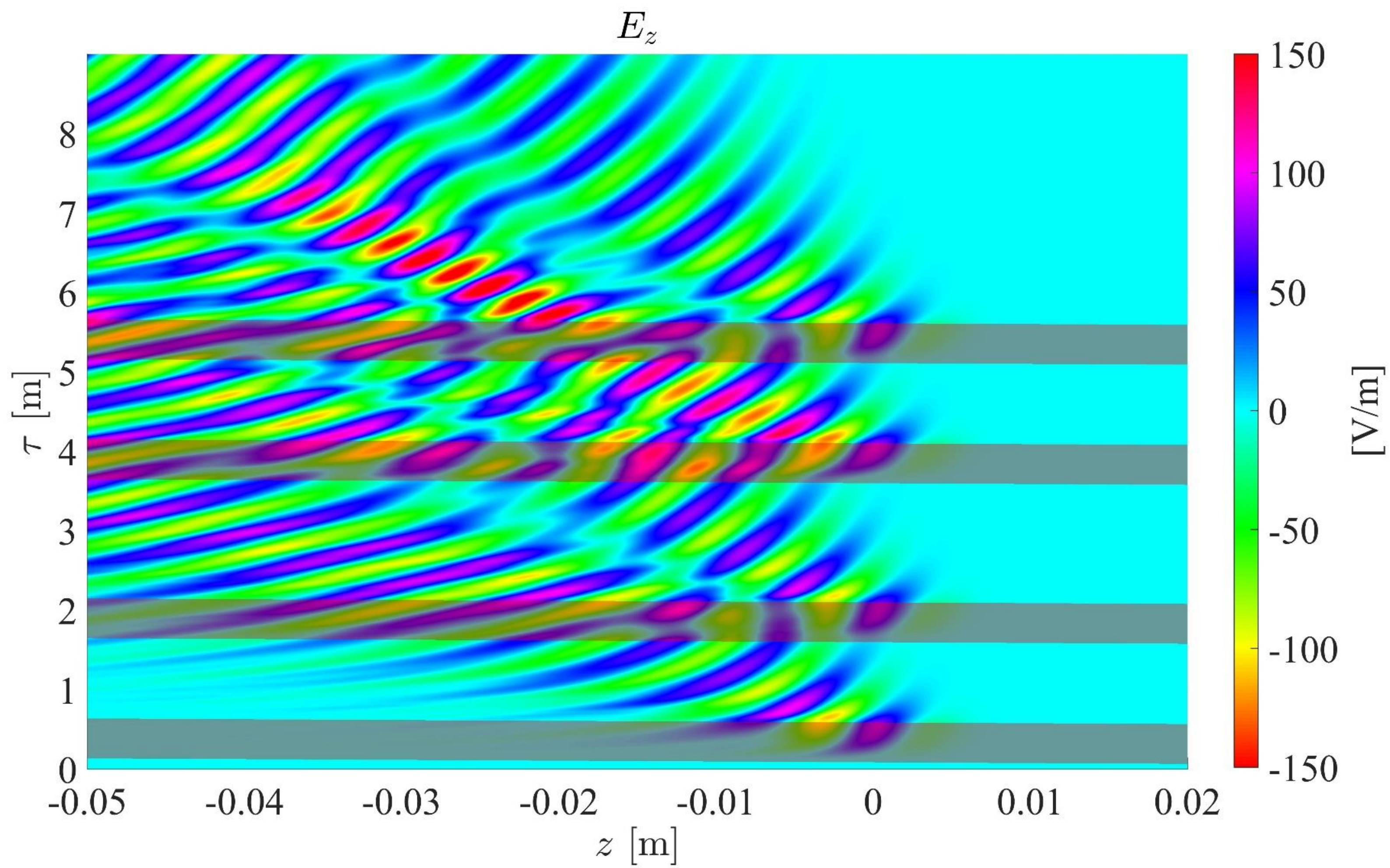



- DESY XFEL BC0 fixed width model

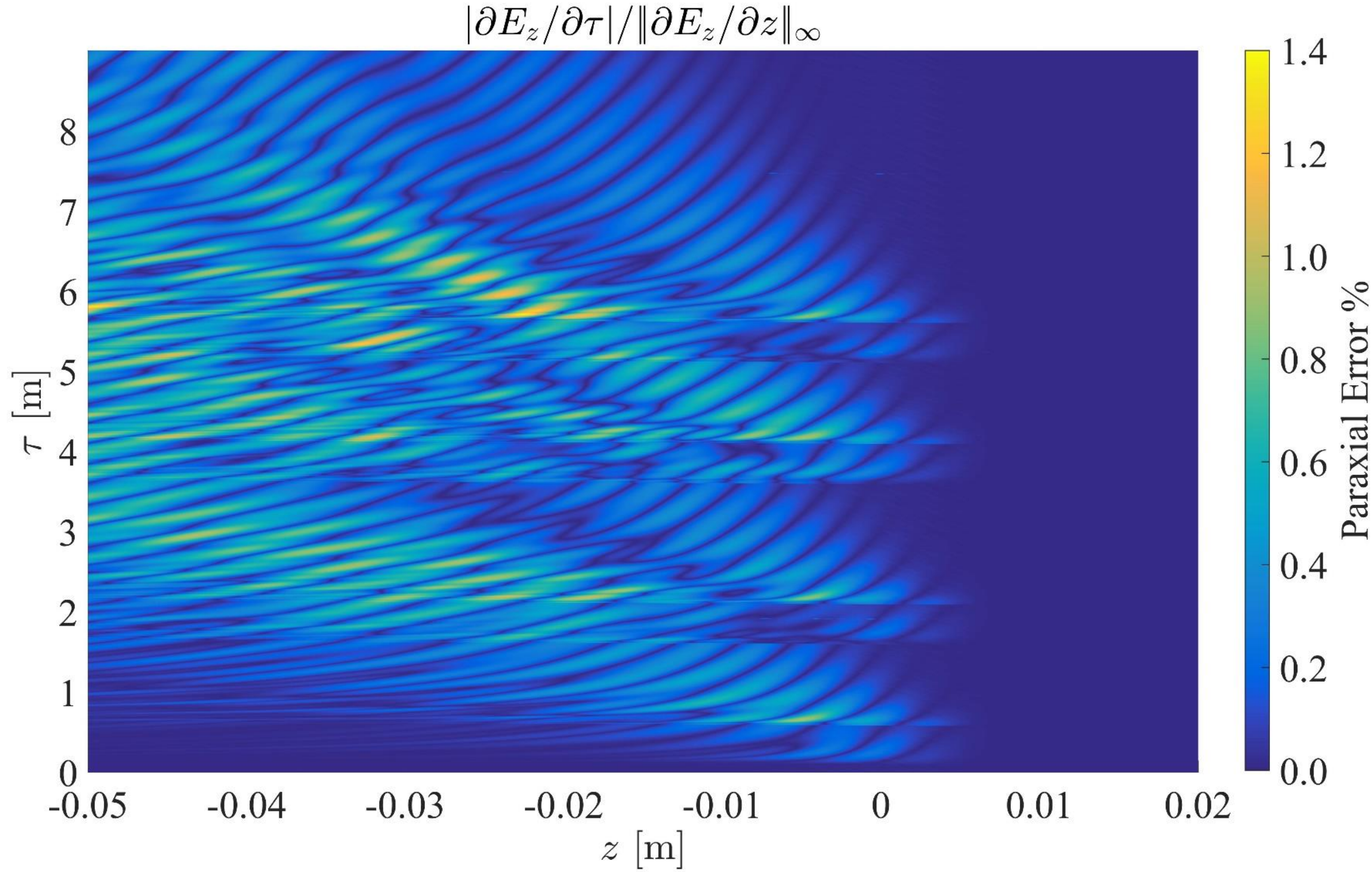
# DESY XFEL BC0 Fixed Width Model



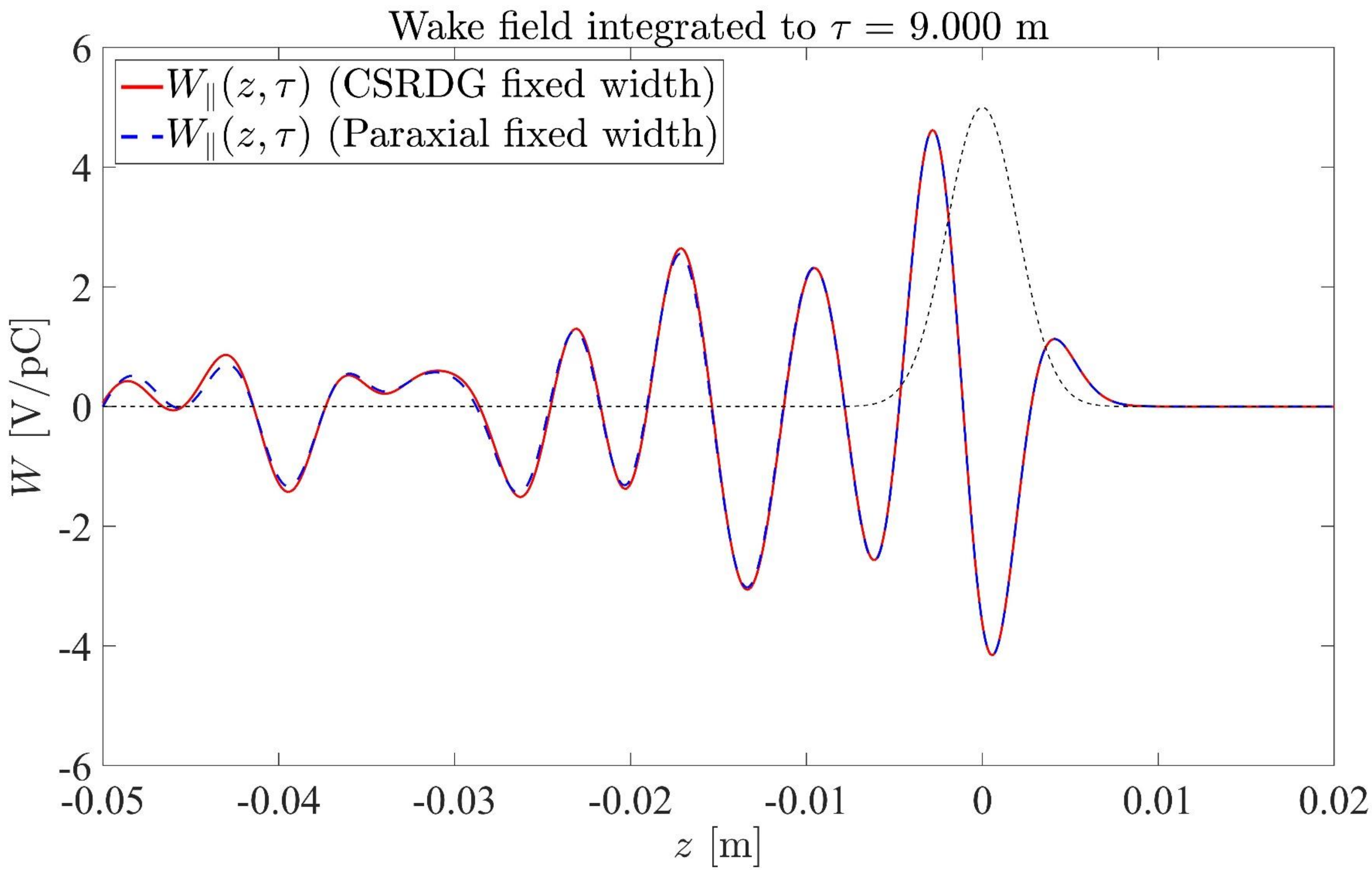
# DESY XFEL BC0 Fixed Width Model



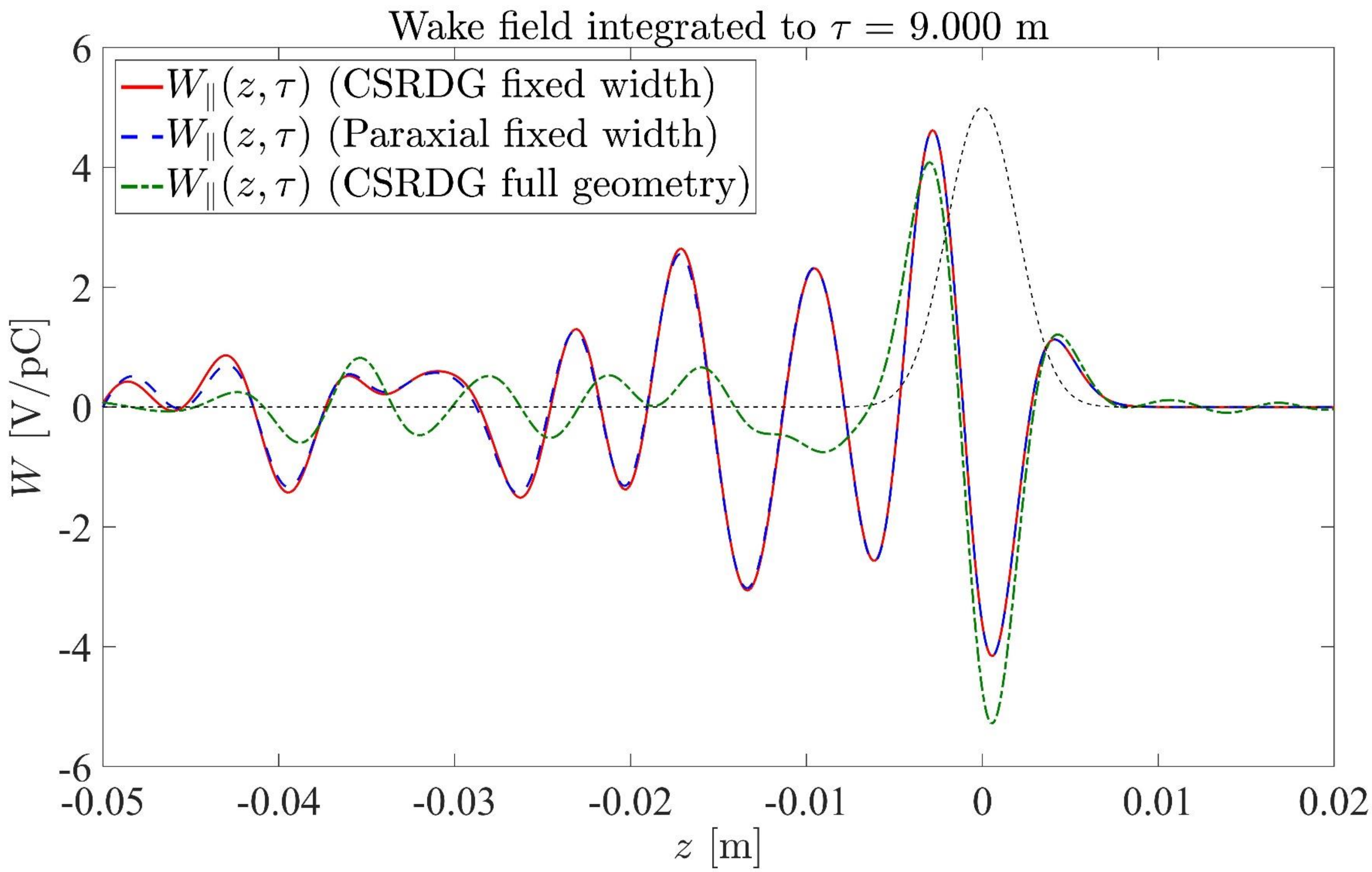
# DESY XFEL BC0 Fixed Width Model



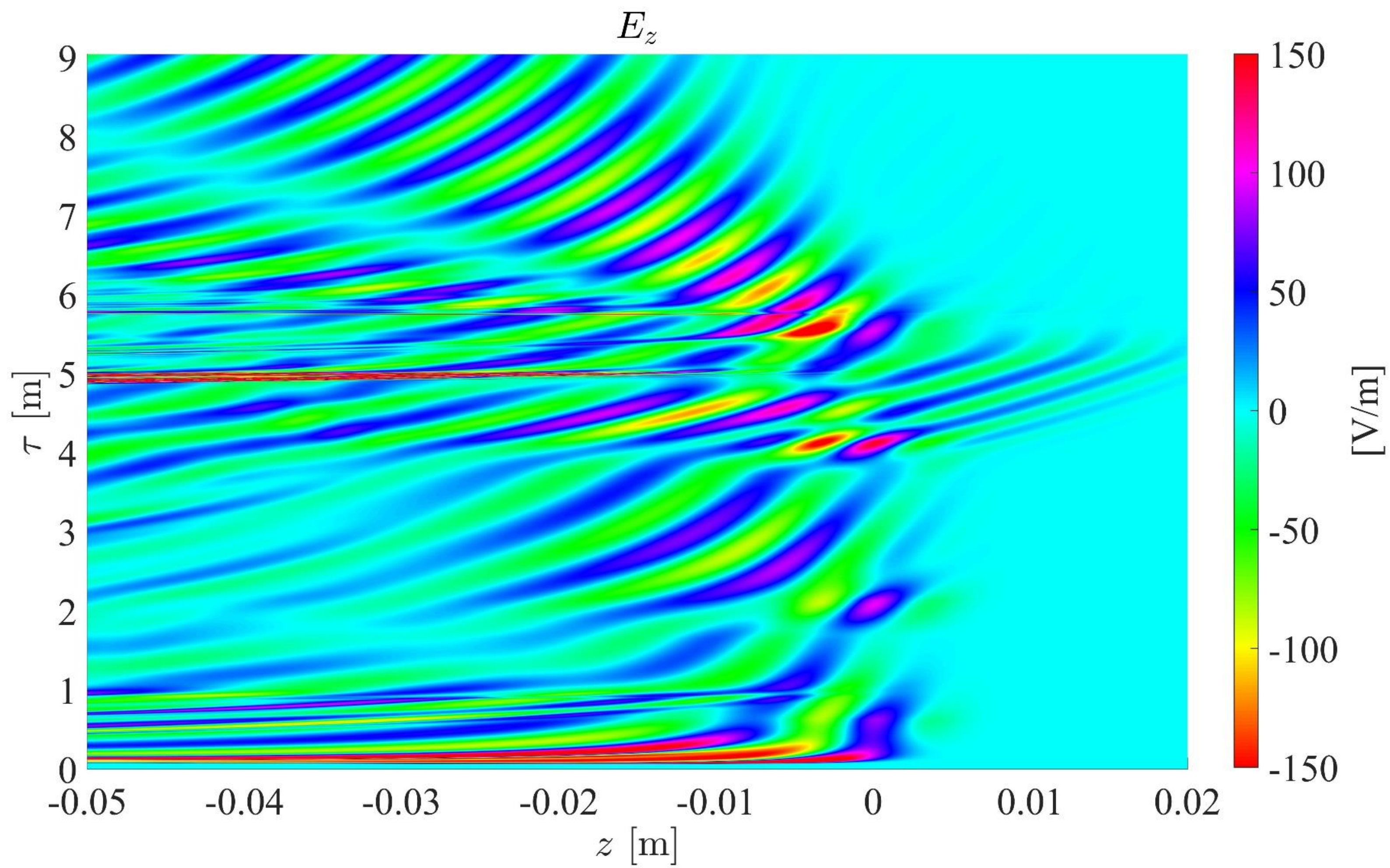
# DESY XFEL BC0 Wake Comparison



# DESY XFEL BC0 Wake Comparison



# DESY XFEL BC0 Full Geometry Model



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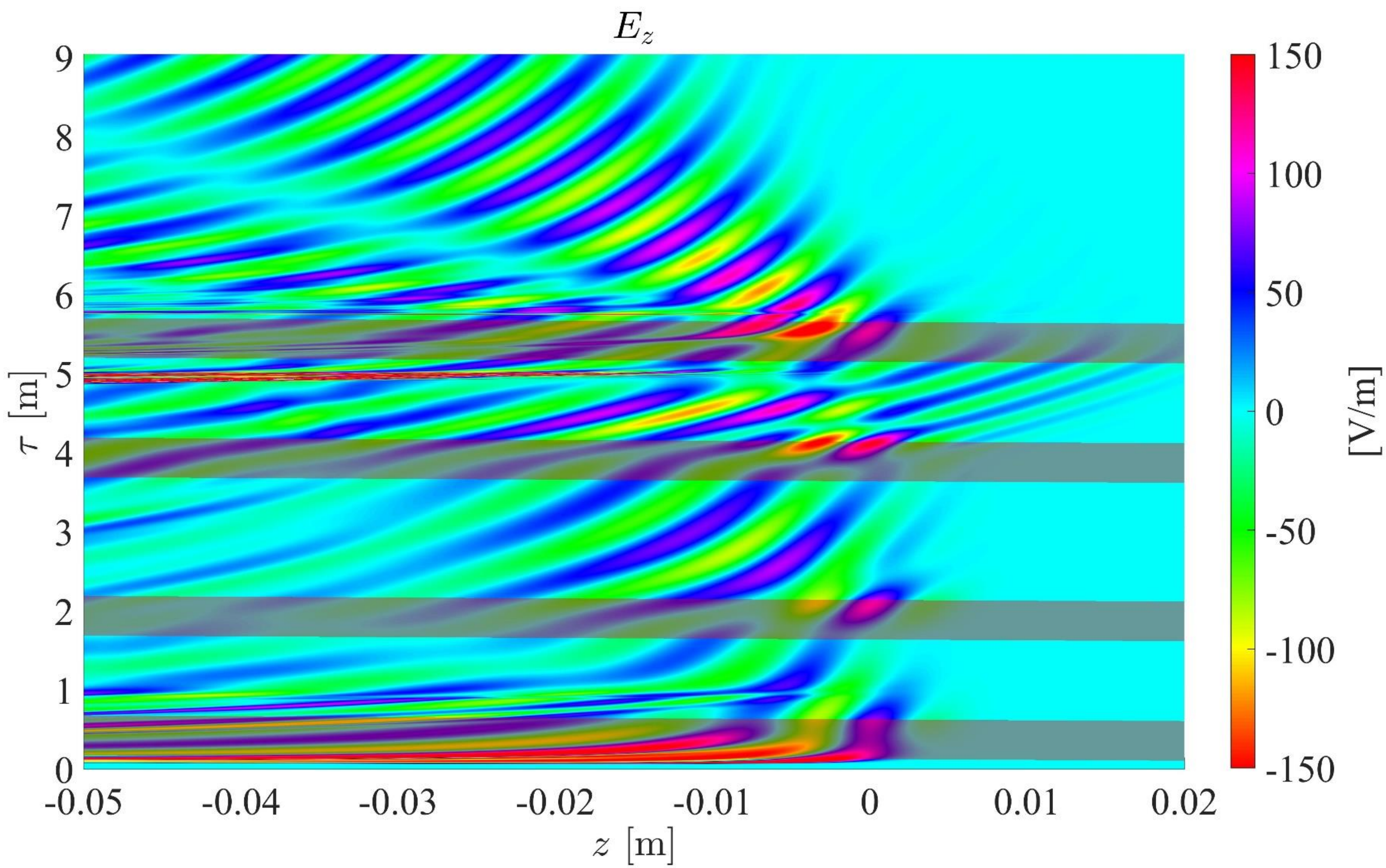

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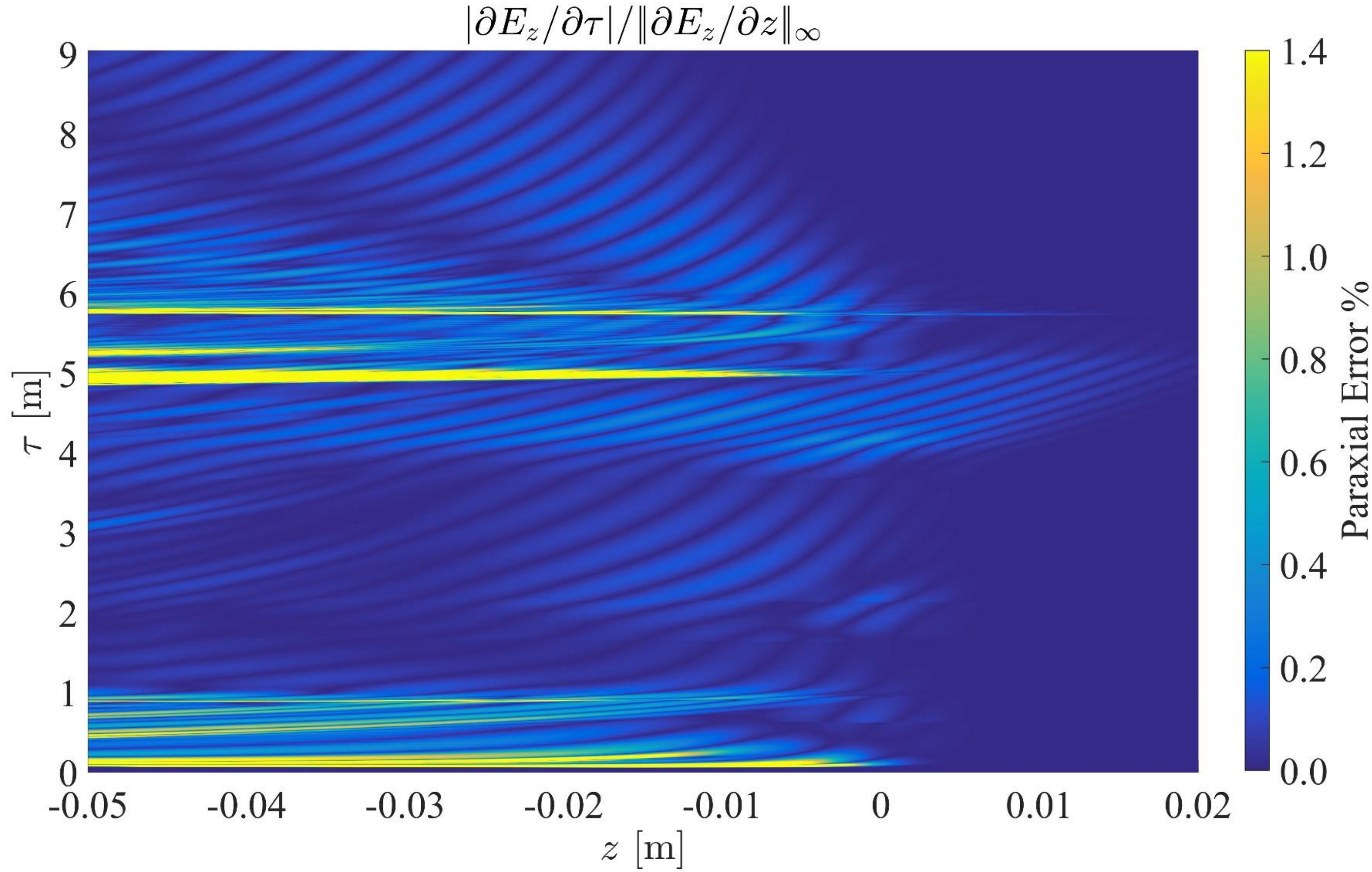

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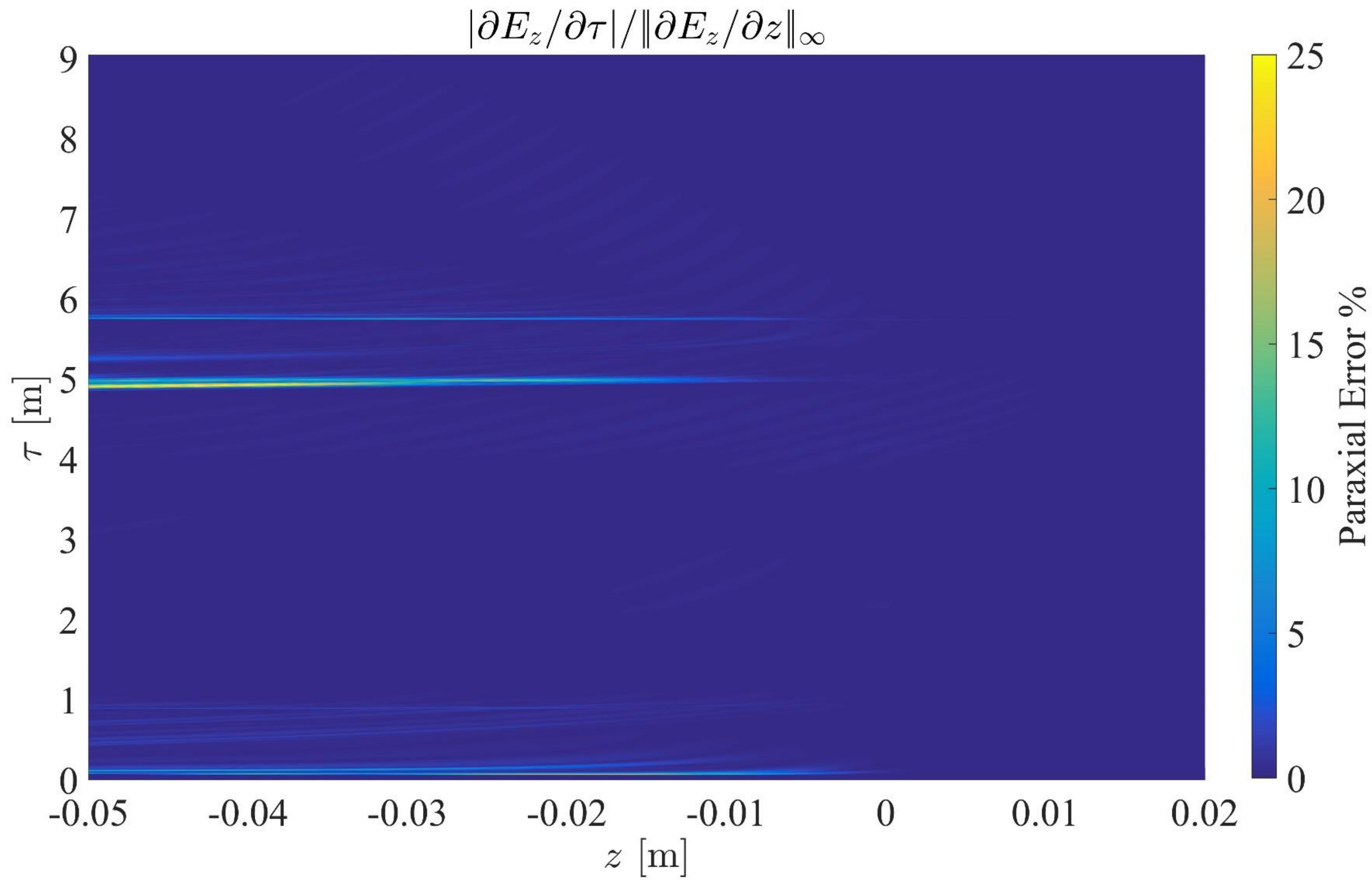
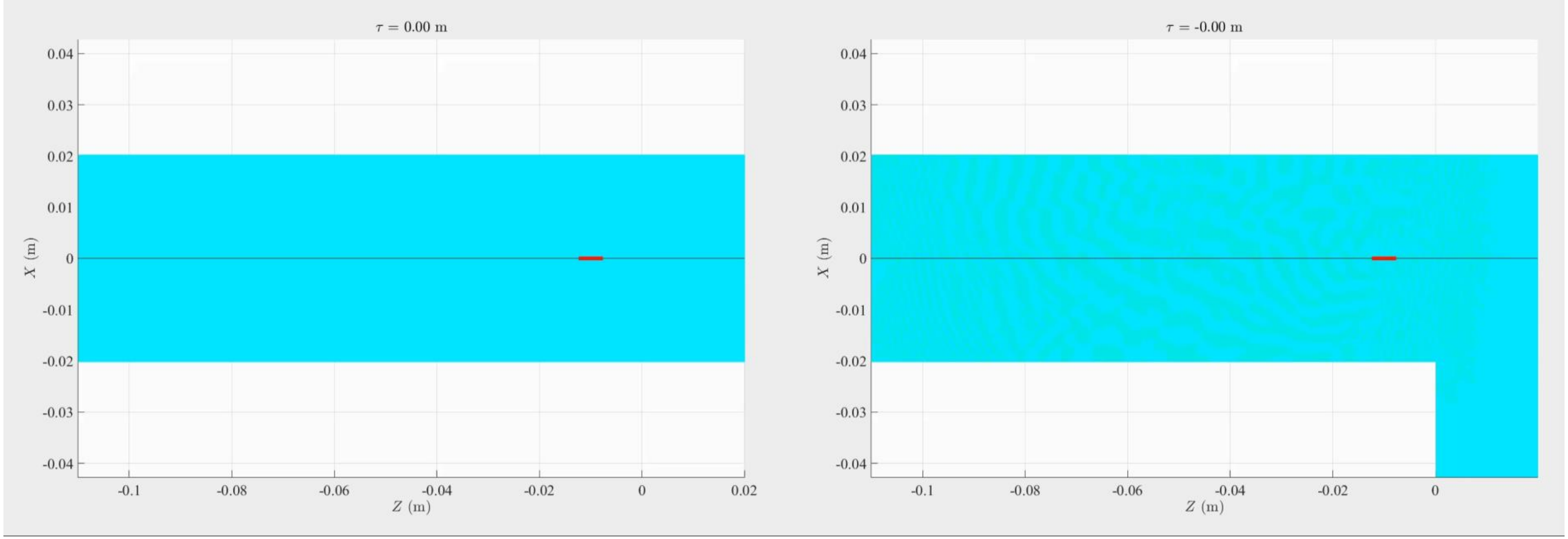


# CSRDG Field Comparisons of Models



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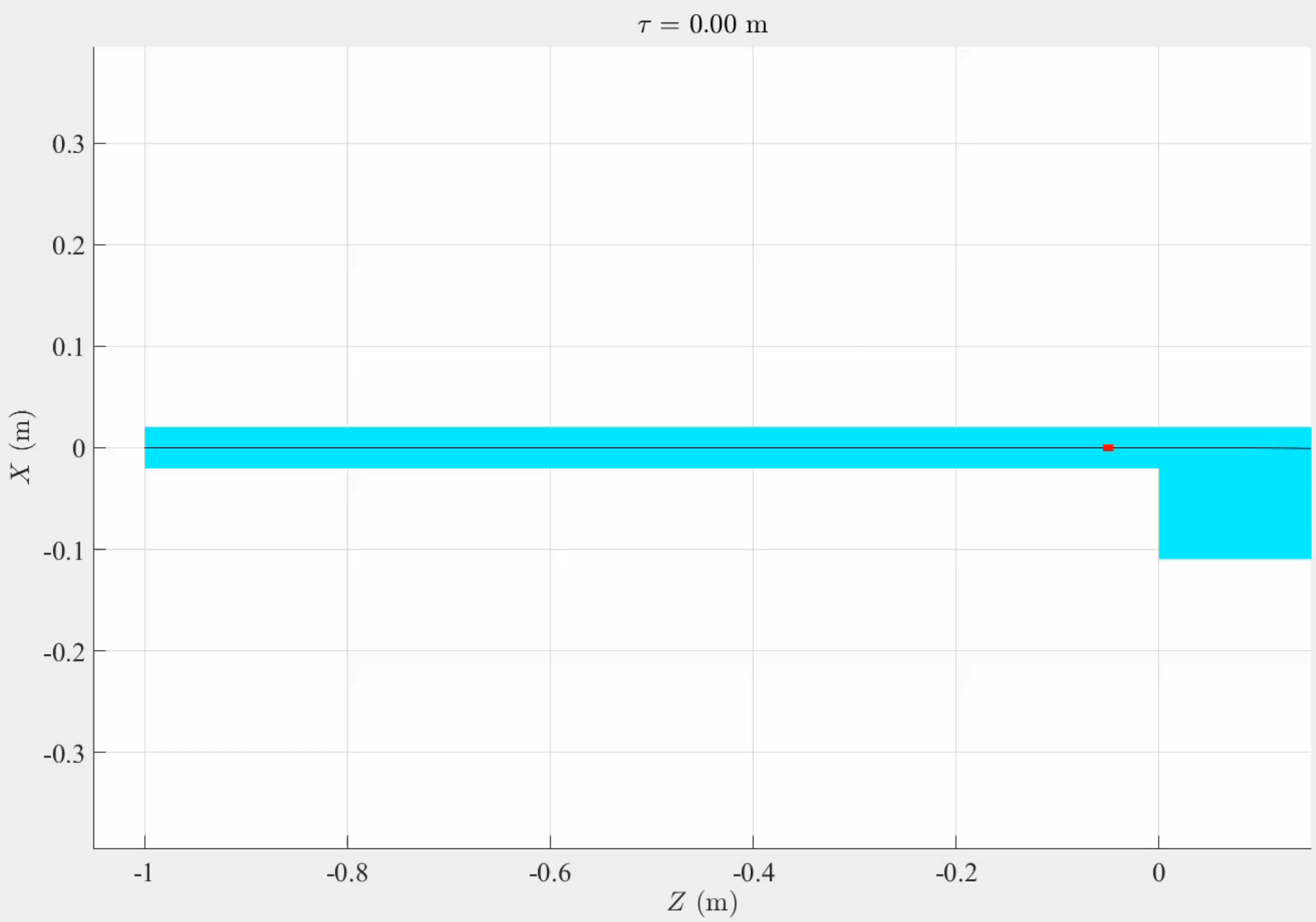

# Conclusions and Future Work

- Performed wake field comparisons between CSRDG and a paraxial method
- Discrepancy in wake fields due to geometry effects
  - Paraxial methods good for CSR component of wake
  - Paraxial methods cannot handle complex geometry
- CSRDG can localize regions of paraxial validity
- Next steps:
  - Consider paraxial simulation with slowly-varying geometry
  - Further examine time-domain paraxial equation behavior
  - Analyze paraxial solution “paraxial error” as self-check

# Thank you for your attention!

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