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MODE-ANALYSIS METHODS FOR THE STUDY OF COLLECTIVE INSTABILITIES IN ELECTRON-STORAGE RINGS*

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Abstract

We review recent progress on the application of mode analysis to the study of collective instabilities in electron storage rings with Higher Harmonic RF Cavities (HHCs). The focus is on transverse instabilities in the presence of a dominant resistive-wall impedance, a problem of particular relevance to the new generation of diffraction-limited light sources. The secular equation is solved after applying a regularizing transformation, a key step to obtain numerically accurate solutions. We provide a demonstration that with vanishing chromaticity and in the absence of radiation damping the beam motion is always unstable. This is in contrast to the classical Transverse-Mode-Coupling Instability (TMCI) without HHCs, which is known to exhibit a well defined instability threshold.

INTRODUCTION

A narrow vacuum chamber to accommodate strong magnets or high-performance Insertion Devices (ID) and use of bunch-lengthening Higher-Harmonic Cavities (HHCs) to reduce intrabeam scattering are two distinctive features of the new generation of storage-ring light sources. This paper concerns itself with the HHC effect on the transverse collective instabilities induced by the Resistive Wall (RW) impedance, which in the new machines is a major, if not the largely dominant, source of transverse impedance due to the small chamber aperture.

HHCs achieve bunch lengthening by introducing an amplitude dependence in the synchrotron oscillation frequency and therefore altering the linear character of the longitudinal motion. The resulting frequency spread is commonly associated with the expectation of a beneficial impact on the beam stability, as alluded by the often-encountered 'Landau cavities' designation. The reality, however, is more nuanced. While HHCs have the potential to reduce or eliminate certain instabilities through the Landau damping mechanism, whether they actually do depends on a number of other factors. In fact, the presence of HHCs can under some circumstances degrade beam stability. This is known although not widely acknowledged for longitudinal multi-bunch instabilities [1–3]. The main point to be made here is that such a degradation can be realized in the transverse plane as well. This paper illustrates the main results reported in [4], to which we refer for the more technical details.

The focus is on developing a mode-analysis theory in the presence of HHCs applicable to single-bunch instabilities at vanishing chromaticities. We base the analysis on the familiar DC-conductivity, RW impedance model for a vacuum chamber with uniform circular cross section of radius b, length L, and conductivity σ_c (cgs units):

$$Z_y(k) = \frac{\mathrm{sign}(k) - i}{\sqrt{|k|}} \frac{L}{b^3} \sqrt{\frac{2}{\pi c \sigma_c}}, \tag{1}$$

with wake-function $W_y(z) = -2L\sqrt{c}/\left(\pi b^3 \sqrt{\sigma_c |z|}\right)$, for $z \leq 0$ (and vanishing otherwise).

THE CLASSICAL TMCI (NO HHCS)

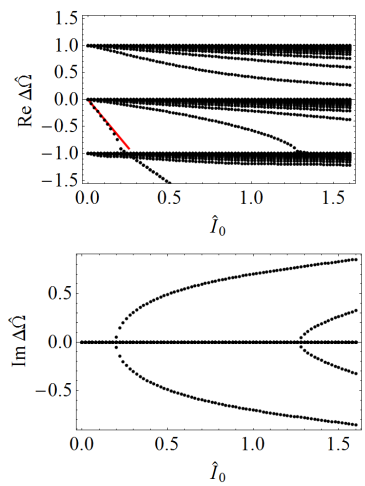

Figure 1: Classical TMCI in the absence of HHCs: real (top) and imaginary (bottom) parts of the mode complex-number frequency shift $\Delta\hat{\Omega} = (\Omega - \omega_y)/\omega_{s0}$ over a bunch-current range. The red line in the top picture is the tuneshift for the rigid dipole mode as given by Eq. (7).

In the absence of HHCs the longitudinal motion is linear and at zero chomaticities the beam is susceptible to the Transverse-Mode Coupling Instability (TMCI). The characteristic signature of the instability is the convergence of the dipole ($m = 0$) and head-tail ($m = -1$) azimuthal-mode oscillation frequencies at the critical bunch current [5]. The starting point for the analysis is the linearized Vlasov equation for the perturbation

$$g_1(r, \varphi; t) = e^{-i\Omega t} \sum_{m=-\infty}^{\infty} R_m(r; \Omega) e^{im\varphi}, \tag{2}$$

written as a superposition of azimuthal modes with radial functions R_m and depending on the longitudinal-motion amplitude/angle coordinates (r, φ). The perturbation g_1 has a physical interpretation as the transverse (say vertical) offset of the electrons contained in the infinitesimal

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phase-space area $\Delta r \Delta\varphi$ centered at (r,φ). Mode analysis entails solving an eigenvalue problem in the form of a system of integral equations for the unknown eigenvectors $R_m(\rho) \equiv R_m(\rho\sigma_{z0};\Omega)$ of the form

$$(\Delta\hat{\Omega} - m)R_m(\rho) + i\hat{I}_0 e^{-\rho^2/2} \times \sum_{m'=-\infty}^{\infty} \int_0^{\infty} R_{m'}(\rho')\mathcal{G}_{m,m'}(\rho,\rho')\rho' d\rho' = 0, \quad (3)$$

where we have scaled the radial coordinate by the rms natural bunch length at equilibrium $\rho = r/\sigma_{z0}$, and

$$\hat{I}_0 = \frac{N r_c c}{(2\pi)^{3/2}\gamma\nu_{s0} b^3 \sqrt{c\sigma_c \sigma_{z0}}} \frac{\beta_{y,u} L_u}{2\pi} \quad (4)$$

is a dimensionless current parameter depending on the bunch population N, relativistic factor γ, synchrotron tune $\nu_{s0} = \omega_{s0}/\omega_0$, undulator length L_u, where the relevant source of RW is localized, and betatron function $\beta_{y,u}$ at the undulator. The sought eigenvalue is the complex-number frequency shift $\Delta\hat{\Omega} = (\Omega - \omega_y)/\omega_{s0}$ from the betatron oscillation frequency $\omega_y = \nu_y \omega_0$, scaled by the synchrotron-oscillation frequency ω_{s0}. In Eq. (3) the kernel involves the Bessel functions J_m and has the form

$$\mathcal{G}_{m,m'} = c_{m,m'} \int_0^{\infty} \frac{d\kappa}{\sqrt{\kappa}} J_{|m|}(\kappa\rho) J_{|m'|}(\kappa\rho'), \quad (5)$$

with coefficients $c_{m,m'} = i^{(m-m')}\{[1-(-1)^{m+m'}] - i[1+(-1)^{m+m'}]\} \times [\mathrm{sign}(m)]^m \times [\mathrm{sign}(m')]^{m'}$.

The conventional approach to solving the eigenvalue problem is to discretize Eq. (3) by expanding $R_m(\rho)$ over an orthonormal polynomial basis. Since the bunch equilibrium is gaussian, a natural and efficient choice for this problem is to use Gauss-Legendre polynomials which yield fairly accurate results upon retaining only a few (possibly just one) radial-mode components for the relevant azimuthal modes $|m| \le m_{\max} = 1$.

Alternatively, and for this problem less efficiently, one can introduce a discretization where $R_m(\rho)$ is represented as a step-wise function on a grid with $n_{\max}$ grid points. The problem is reduced to finding the roots of the secular equation

$$\det[\Delta\hat{\Omega} - \boldsymbol{B}] = 0, \quad (6)$$

where $\boldsymbol{B}$ is a $[(2m_{\max}+1)n_{\max}]^2$-dimension square matrix. The eigenvalue-analysis result obtained with a uniform $n_{\max} = 40$ grid and $m_{\max} = 1$ is shown in Fig. 1. A finite bunch current removes the degeneracy of the radial modes and as its value increases the (real) frequency of one radial-mode component after the other (all having azimuthal mode number $m = 0$) is seen to cross with those relative to the head-tail mode $m = -1$, at which point the imaginary part of $\Delta\hat{\Omega}$ becomes positive signaling instability. The lowest-current crossing involves the $m = 0$ mode with $R_0(\rho) \sim e^{-\rho^2/2}$ radial component (rigid dipole) and occurs at $\hat{I}_0 \simeq 0.197$. To good approximation the current dependence of the real-part of the frequency shift is given by (red line in the top picture of Fig. 1)

$$\mathrm{Re}\,\Delta\hat{\Omega} = -\Gamma(1/4)\hat{I}_0, \quad (7)$$

where $\Gamma(1/4) \simeq 3.63$ is Euler's Gamma function.

For a practical illustration loosely based on parameters from the ALS-U design studies [6], assume that RW is the only relevant source of transverse impedance and that it is dominated by aggressively narrow ID vacuum chambers of $b = 3$ mm radius, Table 1. There are 10 straight sections available for IDs and we conservatively assume that the vacuum chamber is identically narrow in all of them. Finally, assuming copper material for the vacuum chamber ($\sigma_c = 5.3 \times 10^{17}$ s^{-1} in cgs units, or 5.9×10^7 Ω^{-1}m^{-1} in MKS units), we find a critical $N_{c0} = 3.3 \times 10^{10}$ bunch population for the instability threshold, equivalent to 8.1 mA single-bunch current, *vs.* a design $I_b = 1.76$ mA.

Table 1: Beam/Machine Parameters Loosely Based on ALS-U

Ring circumference		196.5 m
Beam energy		2 GeV
Design bunch current	I_b	1.76 mA
Vertical tune	ν_y	20.368
Momentum compaction		2.79×10^{-4}
Natural energy spread		0.835×10^{-3}
Energy loss per turn		182 keV
Vertical damping time	τ_y	14.4 ms
Main rf cavity voltage		0.76 MV
Main rf cavity frequency		500 MHz
Harmonic rf cavity frequency		1.5 GHz
Rms bunch length (no HHCs)	σ_{z0}	3.2 mm
Linear synchr. tune (no HHCs)	ν_{s0}	2.3×10^{-3}
Rms bunch length with HHCs	σ_z	13 mm
Avg. synchr. tune with HHCs	$\langle\nu_s\rangle$	0.44×10^{-3}
Total ID length	L_u	40 m
ID vacuum chamber radius	b	3 mm
Avg. beta function along IDs	$\beta_{y,u}$	3 m

STABILITY ANALYSIS WITH HHCS

Some simplifying assumptions are made to represent the single-particle longitudinal motion in the presence of HHCs. The first is to approximate the total RF potential combining main and harmonic cavities as a purely-quartic polynomial function of the particle longitudinal-coordinate z, yielding an exactly linear dependence of the synchrotron-oscillation frequency on the oscillation amplitude r. This is a very good approximation in the regime where the HHCs are tuned for 'optimal' (*i.e.* maximally flat) bunch lengthening. The second approximation is to write $z = r\cos\varphi$, as for an harmonic oscillator. Somewhat surprisingly, for a purely quartic potential this is a fairly good approximation, entailing only a few % error [7]. With these approximations the system of integral equations becomes

$$(\Delta\hat{\Omega} - m\rho)R_m(\rho) + i\hat{I}e^{-h_1\rho^4} \times \sum_{m'=-\infty}^{\infty} \int_0^{\infty} R_{m'}(\rho')\mathcal{G}_{m,m'}(\rho,\rho')\rho'^2 d\rho' = 0, \quad (8)$$

where now the radial coordinate $\rho = r/\sigma_z$ is scaled by the length σ_z of the bunch stretched by the HHCs, the frequency

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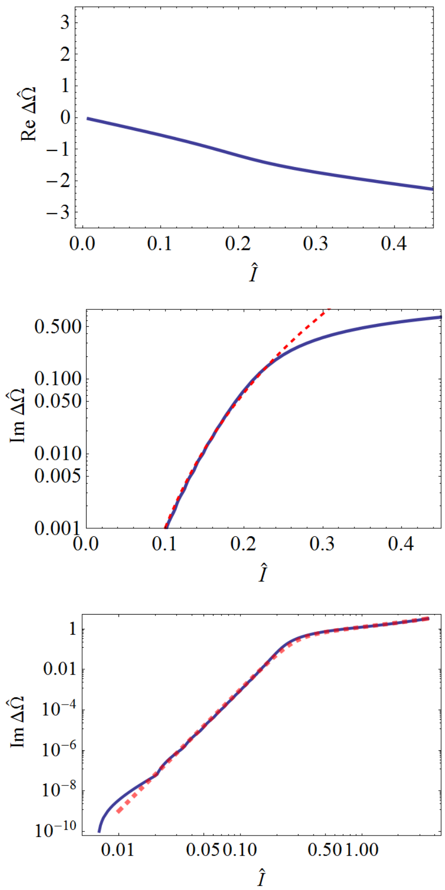

Figure 2: Stability analysis in the presence of HHCs after applying the regularizing transformation to the eigenvalue problem. Real (top) and imaginary (middle and bottom) parts of the root with largest imaginary part of the secular equation (10), as functions of the current parameter $\hat{I}$. The bottom picture contains the same data as the middle picture but on a double-log scale. In the limit of small $\hat{I}$ the numerical solution is consistent with the power law Im $\Delta\hat{\Omega} = (2^{5/3}\hat{I})^6$, red dashed curve in the middle picture. Overall, the numerical solution is reasonably well fitted by Eq. (11), red dashed curve in the bottom picture. Calculation done with $n_{\max} = 40$, $m_{\max} = 1$, and $\rho_{\max} = 3$.

shift $\Delta\hat{\Omega} = (\Omega - \omega_y)/(h_2\langle\omega_s\rangle)$ is scaled by the synchrotron-oscillation frequency averaged over the bunch $\langle\omega_s\rangle$, and the dimensionless current parameter reads

$$\hat{I} \;=\; \frac{N r_c c}{\pi^{5/2}\gamma\langle\nu_s\rangle b^3\sqrt{c\sigma_c\sigma_z}}\frac{\beta_{y,u}L_u}{2\pi}. \qquad (9)$$

exhibiting $\langle\nu_s\rangle$, the average of the synchrotron-oscillation tune over the bunch, in place of ν_{s0} appearing in Eq. (4). The quantities h_1 and h_2 are numerical coefficients, $h_1 = 2\pi^2/\Gamma(1/4)^4 \simeq 0.114$ and $h_2 = 2^{3/4}\pi^{3/2}/\Gamma(1/4)^2 \simeq 0.712$.

As for the integral equation, the main difference from Eq. (3) is the appearance of the ρ dependence in the factor multiplying $R_m(\rho)$ in the first term. This term is familiar from the analysis of Landau damping in plasma waves or longitudinal instabilities, raising a flag that care should be taken to handle the singularity occurring when the above factor vanishes for certain values of ρ. Because of the singularity, the eigenfunctions of Eq. (8) are in general not ordinary functions but distributions in the sense of Dirac [8, 9] and finite-dimension approximations of the problem are not guaranteed to converge [10].

The proper way to proceed is to 'sweep the singularity under the (integral) rug' by introducing a simple change of variable of the unknown function [11], $R_m(\rho) \longrightarrow S_m(\rho) = (\Delta\hat{\Omega} - m\rho)R_m(\rho)e^{h_1\rho^4}$, yielding the transformed integral equations

$$S_m(\rho)+i\hat{I}\sum_{m'=-\infty}^{\infty}\int_0^{\infty}\frac{S_{m'}(\rho')e^{-h_1\rho'^4}}{\Delta\hat{\Omega}-m'\rho'}\mathcal{G}_{m,m'}(\rho,\rho')\rho'^2 d\rho' = 0.$$

These equations can now be safely discretized with the prescription that the integration path in ρ should be deformed to go under the the pole if $\Delta\hat{\Omega}$ is real or has negative imaginary part. Since we are primarily interested in establishing the condition for instability (Im $\Delta\hat{\Omega} > 0$) we can do without the path deformation, provided that we take numerical care to insure the necessary accuracy when the imaginary part of $\Delta\hat{\Omega}$ is positive but small. An effective integration strategy is to approximate the numerator in the integral by a piece-wise linear or quadratic polynomial, in which case the integral can be carried out analytically.

Upon discretization, the above equation is reduced to the form $[\mathbf{1} + \boldsymbol{B}(\Delta\hat{\Omega})]\vec{S} = 0$, where, $\boldsymbol{B}(\Delta\hat{\Omega})$ is now a $\Delta\hat{\Omega}$-dependent, $[(2m_{\max}+1)n_{\max}]^2$-dim matrix. Unlike Eq. (6), the resulting secular equation

$$\det[\mathbf{1} + \boldsymbol{B}(\Delta\hat{\Omega})] = 0 \qquad (10)$$

is a transcendental (*vs.* polynomial) equation in the frequency shift $\Delta\hat{\Omega}$ and in principle more difficult to solve. In practice, we found that a Newton-method search appropriately initiated never failed to converge. The result of our numerical analysis is shown Fig. 2, reporting real and imaginary parts of the frequency shift of the most unstable mode in a calculation using $n_{\max} = 40$ radial grid points and $m_{\max} = 1$. The main result of this analysis is that transverse single-bunch motion in the presence of the RW impedance is unstable at any current.

Over a large current range the imaginary part of the frequency of the most unstable mode is well fitted by the function (red dashed line in the bottom picture of Fig. 2)

$$\text{Im}\,\Delta\hat{\Omega} = \frac{(2^{5/3}\hat{I})^6}{1 + 0.55\times(4\hat{I})^5[1+\tanh(\hat{I}/2)]}. \qquad (11)$$

It is tempting to make the conjecture that Im $\Delta\hat{\Omega} = (2^{5/3}\hat{I})^6$ may be the exact asymptotic limit for $\hat{I} \longrightarrow 0$. It is seen to track the numerical data quite accurately for $\hat{I} \lessapprox 0.2$.

Having argued that for proper numerical treatment of the problem it is important to introduce a regularizing transformation, it is nonetheless instructive to naively apply the discretization method employed when HHCs are absent. Effectively, this is equivalent to studying a modified physics model where the unperturbed beam distribution in phase space consists of a set of $n_{\max}$ equally spaced, concentric,

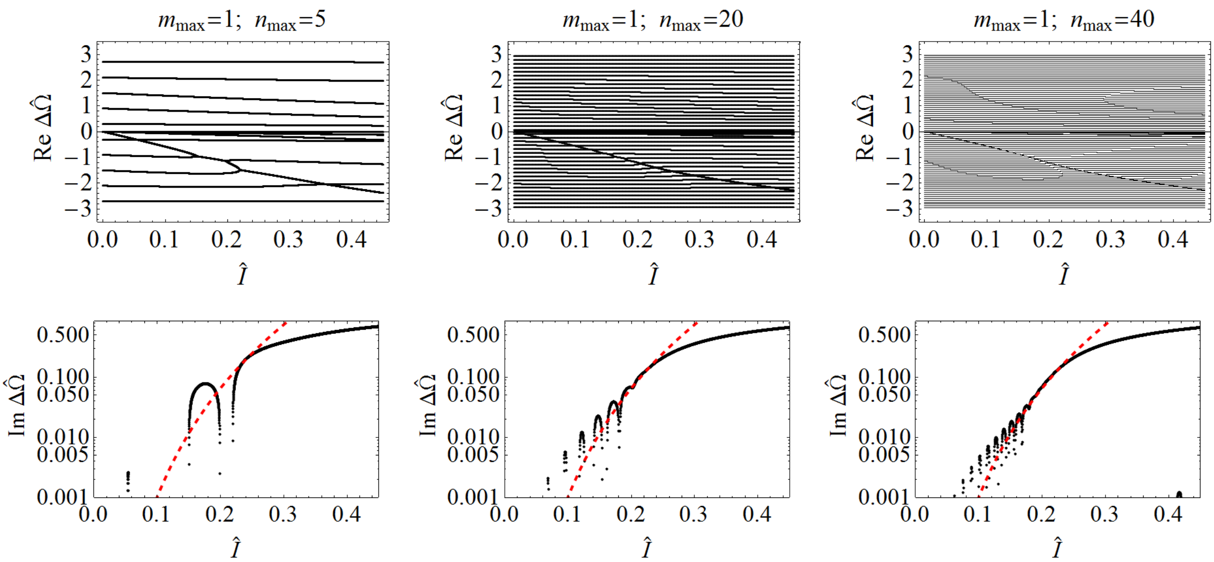

Figure 3: Stability analysis in the presence of HHCs using the conventional eigenvalue-method without the regularizing transformation. The top (bottom) pictures show the real (imaginary) parts of the modes complex-number frequency shifts $\Delta\hat{\Omega} = (\Omega - \omega_y)/(h_2\langle\omega_s\rangle)$ as functions of the current parameter $\hat{I}$, for increasingly finer (left to right) grids in the radial variable ρ, as indicated. The bottom pictures are in log scale and report only the frequencies with positive imaginary part (unstable modes). Particularly at small $\hat{I}$, convergence to what we believe is the exact asymptotic solution of the infinite-dimension problem Im $\Delta\hat{\Omega} = (2^{5/3}\hat{I})^6$, valid for $\hat{I} \lessapprox 0.2$, red dashed curve, appears slow if not outright questionable.

invariant shells. Results are shown in Fig. 3. While it is apparent that these pictures do not extrapolate well into the continuum limit, they provide valuable insight and suggest that the basic mechanism of mode coupling is still at play. First, notice that all the radial modes relative to azimuthal mode $m = 0$ are degenerate but not those relative to $m \neq 0$, even at vanishing current. This is in contrast to the longitudinal linear-motion case (no HHCs), where at zero current all radial modes for any m are degenerate. The reason, of course, is related to the fact that particles on different radial shells have different winding (synchrotron oscillation) frequencies. The emergence of instability is triggered by the convergence of one of the $m = 0$ radial modes frequency with that of one of the $m = -1$ radial modes. In analogy to the linear case, the offensive $m = 0$ radial mode has the form of the bunch equilibrium $R_0(\rho) \sim e^{-h_1\rho^4}$ (rigid dipole). The difference with the linear case is that coupling can now occur at arbitrarily low current as we allow for a finer and finer resolution of the radial beam distribution. For currents less than $\hat{I} \sim 0.25$, regions of instability appear interleaved with regions of stability, with the extent of the latter progressively reduced when we increase the number of grid points n_{max}. The ~ 0.25 edge corresponds to the radial extension ($\rho \sim 1$) of the beam distribution (outer shells become quickly underpopulated for $\rho > 1$ and do not contribute to the coupling).

THE TAKE-HOME RESULT

In electron storage rings radiation damping will eventually prevail if the bunch current is not too high. The condition Im $\Omega = \tau_y^{-1}$, where τ_y is the vertical radiation damping time, defines the critical current parameter $\hat{I} = \hat{I}_c$ as follows: Im $\Omega = h_2\langle\omega_s\rangle$Im $\Delta\hat{\Omega} = h_2\langle\omega_s\rangle(2^{5/3}\hat{I}_c)^6 = \tau_y^{-1}$, having restricted our analysis to the regime where the Im $\Delta\hat{\Omega} \propto \hat{I}^6$ power law applies. We have

$$\hat{I}_c = \frac{2^{-5/3}}{(h_2\tau_y\langle\omega_s\rangle)^{1/6}} \simeq 0.245 \times \left(\frac{T_0}{\tau_y\langle\nu_s\rangle}\right)^{1/6}. \quad (12)$$

More expressively, we can relate N_c, the critical bunch population in the presence of HHCs, and N_{c0}, the critical bunch population in the absence of HHCs, when all the relevant machine parameters are kept unchanged while the HHCs are turned on and off. Combining Eqs. (4), (9) and (12) gives

$$N_c = N_{c0} \times \frac{\pi}{8 \times 2^{1/6}\hat{I}_{c0}} \left(\frac{1}{\tau_y h_2\langle\omega_s\rangle}\right)^{1/6} \frac{\langle\nu_s\rangle}{\nu_{s0}} \left(\frac{\sigma_z}{\sigma_{z0}}\right)^{1/2}, \quad (13)$$

where $\hat{I}_{c0} \simeq 0.197$ is the critical current parameter for the onset of the TMC-Instability in the linear case.

Making use of the relationship between synchrotron tunes and bunch lengths with and without HHC for the specific case of third-harmonic cavities, see [4], we obtain the final result

$$N_c \simeq 1.15 \times N_{c0} \left(\frac{T_0}{\tau_y\nu_{s0}}\right)^{1/6} \left(\frac{\sigma_{z0}}{\sigma_z}\right)^{1/3}. \quad (14)$$

Note that the quantity elevated to the 1/6 power now depends on ν_{s0} not $\langle\nu_s\rangle$. Using the machine parameters from the ALS-U example (Table 1), we find a critical current $\hat{I}_c \simeq 0.168 < 0.2$ placing the system in the regime of the validity of the Im $\hat{\Omega} \propto \hat{I}^6$ scaling, see Fig 2. Finally, from Eq. (14), we conclude $N_c/N_{c0} \simeq 0.37$, corresponding to $I_b = 3$ mA, *i.e.* the instability threshold with HHCs is less than 40% of that without. More in detail, $\left[T_0/(\tau_y\nu_{s0})\right]^{1/6} \simeq 0.52$ and

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$(\sigma_{z0}/\sigma_z)^{1/3} \simeq 4^{-1/3} \simeq 0.62$. A macroparticle simulation with *elegant* [12] confirms the $\sim I_b^6$ scaling, Fig. 4, and overall is reasonably close to the theory.

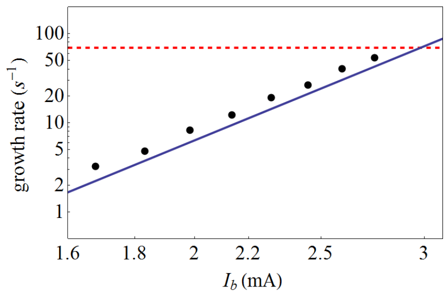

Figure 4: The TMC-Instability growth rate in the presence of HHCs *vs.* bunch current from macroparticle simulations (dots) tracks reasonably well the theory (solid line). The simulation does not include radiation damping but for reference the expected radiation damping rate (red dashed line) is also reported. ALS-U parameters as in Table 1.

CONCLUSION

In summary, we have provided a demonstration that, in the absence of radiation damping the transverse motion at vanishing chromaticities is always unstable, regardless of bunch current, with growth rate varying from a Im $\Omega \sim I_b^6$ dependence at small bunch current I_b to Im $\Omega \sim I_b$ for larger I_b, the former being more likely to be encountered in the physical systems of interest. Because of the strong 6th-power dependence, macroparticle-simulations results could be easily misinterpreted as indicating the existence of a current threshold if the simulation time is not sufficiently long [13]. Finally, we caution that the formulas in the last section are strictly dependent on the RW nature of the assumed impedance model. Work to analyze impedances of different form is left to future studies. The study presented here is for vanishing-chromaticities. Finite chromaticities have a known stabilizing effects. Interestingly, macroparticle simulation work indicates that their stabilizing effect is magnified not reduced by the presence of HHC, see [14] for multi-bunch and Fig. 5 for a single-bunch study. Extension of the theory to multi-bunch instabilities and finite chromaticities will be addressed elsewhere.

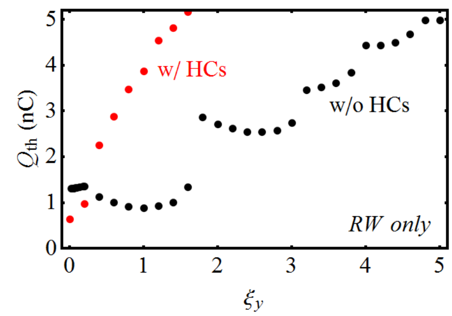

Figure 5: Single-bunch instability threshold *vs.* chromaticity with and without HHCs (ALS-U studies). *Elegant* [12] simulation. The presence of HHCs considerably enhances the stabilizing effect of a finite positive chromaticity. Radiation damping included.

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