Geometry of Electromagnetism and its Implications in Field and Wave Analysis*

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Abstract

During the recent years so called geometric techniques have become popular in computational electromagnetism. In this paper, exploiting differential geometry and manifolds, we first give a meaning to what is meant by geometric approaches. Therafter we examine some implications of such geometry in numerical analysis of electromagnetic field and wave problems.

INTRODUCTION

Recently, so called geometric approaches have become rather popular in generating numerical solutions for electromagnetic boundary value problems. In this context the word *geometric* is typically chosen to emphasize the idea that the degrees of freedom are not related only to points, but instead to "edges", "faces", and "volumes" as well. Then, thanks to the *generalized Stokes theorem*, differentiation can be understood as evaluation of functions over the boundaries of such cells. Eventually, the approach results in an elegant geometric reinterpretation of finite elements and finite difference techniques. Moreover, as the basics are rather easy to capture, the approach possess some pedagogical advantages. [1], [2], [3], [4], [5], [6]

Such a computational view, however, does not yet explain what the geometry of electromagnetism is all about. For this reason, in this paper we will first focus in more details to geometry and only thereafter examine what is implied to field and wave analysis.

BACKGROUND

The notions of electric and magnetic fields are not meaningful without taking sides to the underlying spatial (and possibly temporal) domain. For, in the formal sense the field notitons are mappings between a spatial domain and some vector space. Geometry is informally about a study of spatial relationships, and consequently, it is a tool to characterize what kind of properties the domain of electromagnetic fields has. To further develop this idea, we need first to introduce some tools.

Manifold

Let us start from *manifolds*. In short, a manifold is a *set* and a non-empty collection of maps from $\mathbb{R}^n$ to the underlying set. These maps are known by name *charts*.

To build a prelimanary view, let's say the laboratory, i.e., the "real spatial space" in which we want to know the electromagnetic fields is modelled as a *point set* Ω. The set of our manifold M is some subset $X \subset \Omega$.[1]

By definition, a chart of our manifold should then be a map from $\mathbb{R}^3$ to set X. For example, the model of some object one builds in the preprocessor of a finite element code is meant to point from $\mathbb{R}^3$ to X (i.e., to "reality"). Consequently, we may consider the model in the preprocessor as the domain of a chart from $\mathbb{R}^3$ to set X, see Fig. 1 and 2.

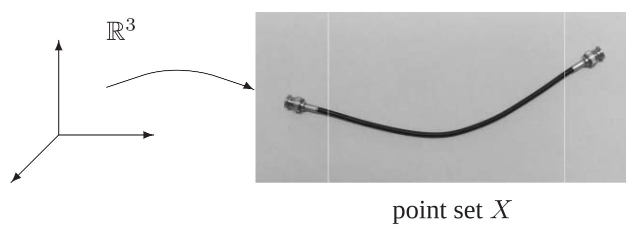

Figure 1: An elementary example of a primitive manifold. Point set X is here a coaxial cable and the chart maps from $\mathbb{R}^3$ to X.

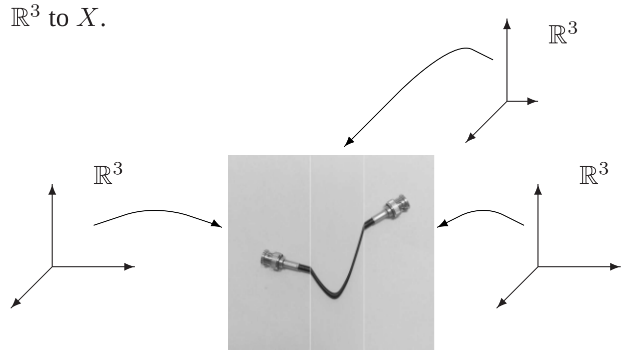

Figure 2: Another collection of charts yielding coordinates for the same coaxial cable as in Fig. 1. But now, due a different choice of charts the model appears visually bit different.

The very idea of such a chart is to express set X by coordinates, or more precisely, by n-*tuples* of real numbers. (In our example, one has $n = 3$, and the 3-tuples are triplets of real numbers.) Thereafter, thanks to the real numbers, *arithmetics* in set X becomes possible.

It is plain that the charts cannot be arbitrary mappings, but instead they must have certain properties [7]: A *differentiable manifold* of dimension n is a connected topological space X and a collection of charts (also called a

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[1] The universal point set Ω is introduced only for the reason to enable us to take sides on shapes of bodies.

collection of coordinate systems) such that

1. each coordinate system is a homeomorphism of an open set of $\mathbb{R}^n$ into a subset of X,
2. set X can be covered by a finite or denumerable number of the chart codomains,
3. if the intersection of two chart codomains $\mathrm{cod}(c)$ and $\mathrm{cod}(c')$ is not empty, then the so called *transition map* (or transformation of coordinates) $c^{-1} \circ c'$ is smooth and regular. In this case one says charts c and c' are mutually compatible.

Atlas

An *atlas* is the collection of all coordinate systems of X. Thus, an atlas is an *equivalence class*; If all charts $c \in \mathcal{C}$ and $c' \in \mathcal{C}'$ of manifolds $\{X, \mathcal{C}\}$ and $\{X, \mathcal{C}'\}$ are mutually compatible, then $\mathcal{C}$ and $\mathcal{C}'$ are equivalent [8].

The very idea is to put emphasis on the property that coordinate systems for X exists instead of focusing on some specific charts. This is a tiny, but here, simultaneously a significant move. For, the idea is that the electromagnetic theory should not depend on the coordinate systems chosen by the modeller.

In computational electromagnetism one typically assumes almost without any further consideration that the domain of electromagnetic fields can be expressed with coordinates at will. However, be aware, formally such a move should first be justified, and for this, we may use Euclidean geometry.

GEOMETRY

The classical *Euclidean geometry* [9] can be postulated in five basic axioms which have to do with straight lines and circles. Here, instead of taking sides to these axioms, we'll rather focus on these straight lines and circles. For, at first, the very idea is that a *ruler* (i.e. straight edge without any grading) and a *compass* are enough to create coordinate systems for our laboratory point set X. (We assume dimension n is at most three.)

In other words, no other tools than a ruler and a compass are needed to create locally the *structure* of *linear space* to set X. More precisely, introducing such a structure locally is about constructing the so called *tangent spaces*.

The compass enables us also to recognize the most symmetrical *barrels* –or, in more technical words, barrels with *maximal isotropy groups* [10]. Such barrels generate *norms* which have the special property that they do not privilidge any direction and they stem for a *scalar product*, Fig. 3. Consequently, they deserve a dedicated name: an *Euclidean norm*.

There is no absolute choice of the most symmetric barrel. For, the the same ellipsoid generated with a compass –that is, with the object we have chosen to call a *rigid body*– appears different depending on our choice of coordinate system. And now, as there is no choice of a canonical coordinate system, it is not possible to say which ellipsoid is a sphere in the absolute sense, Fig. 3. Still, we may talk of the Euclidean geometry, for all Euclidean structures (the affine space equipped with an Euclidean norm) are equivalent up to a linear transformation. [10]

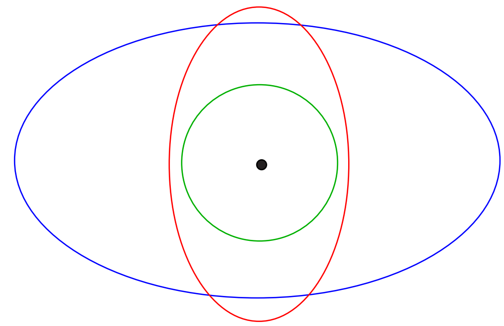

Figure 3: Examples of one and the same ellipsoid around a point represented in different coordinate systems. The one which we recognize as a sphere depends on the choice of chart.

Summing up, Euclidean geometry provides us with a collection $\mathcal{G}$ of charts, which we'll call by name "Euclidean chart collection".

GEOMETRY OF ELECTROMAGNETISM

We have now enough background to specify a proper meaning for term "geometry of electromagnetism". The key idea is: electromagnetic fields can be interpreted as another collection of charts mapping to set X. To explain the idea, notice first that we know already that $\{X, \mathcal{G}\}$ is a *Riemannian manifold*; Locally every tangent space of $x \in X$ has the Euclidean structure.

Now, let us employ electrostatics as a simplifying example and say c is a topologically trivial charged conductor. The electrostatic field is the pair $\{e, d\}$, where e is the 1-form called electric field and d the 2-form called electric flux. Thanks to the properties of manifold $\{X, \mathcal{G}\}$, in the complement of c in X we may say what pair $\{e, d\}$ is precisely: Assuming appropriate boundary conditions pair $\{e, d\}$ is the solution of equations

$$\mathrm{d}e = 0, \tag{1}$$
$$\mathrm{d}d = 0, \tag{2}$$
$$d = \star_\epsilon e \tag{3}$$

where d is the *exterior derivative* and $\star_\epsilon$ is the *Hodge operator* including permittivity. Notice, although we do not select explicitly some coordinate system, the pair $\{e, d\}$ fulfilling (1), (2), and (3) is already fully meaningfull.

As soon as we may talk of the electrostatic field $\{e, d\}$, we may also define equipotential layers and field lines:

Definition 1: A 2-dimensional connected submanifold S of $M = \{X, \mathcal{G}\}$ is an *equipotential layer*, if for all $x \in S$ and $v \in T_x S$ (i.e, vector v is in the tangent space of S at point x) implies $e_x(v) = 0$.

Definition 2: A 1-dimensional connected submanifold F of M is a *field line*, if for all $x \in F$ properties $u \in T_x F$ and $v \in T_x X$ imply $d_x(u \wedge v) = 0$.

And now, in the same manner as a ruler and a compass was exploited on the geometric side, also the field lines and equipotential layers can be employed to introduce a new chart collection of $X \backslash c$. We call this by name "electric chart collection" $\mathcal{E}$.

In this spirit the pair $\{X \backslash c, \mathcal{E}\}$ is called by name *electric geometry*. The electric geometry has its own metric structure, which we'll name ϵ-metric. Distances between pairs of points in $X \backslash c$ get a proper meaning as electric *geodesics*.

In a more general setting name *electromagnetic geometry* will consequently refer to pair $\{X, \mathcal{E}_M\}$, where the collection of charts $\mathcal{E}_M$ is obtained from the electromagnetic phenomenon. Notice that chart collections $\mathcal{E}_M$ and $\mathcal{G}$ are two representatives of one and the same atlas.

SOLUTIONS OF BOUNDARY VALUE PROBLEMS

Practical electromagnetic design needs solutions of electromagnetic boundary value problems. The solutions of such problems –be they analytic or numerical– are mappings from some coordinate system to vector spaces modelling the electromagnetic notions. Consequently, in terms of manifolds, the solution of an electromagnetic boundary value problem can be understood as a transition map

$$c_{em}^{-1} \circ c_g : \mathbb{R}^n \to X \to \mathbb{R}^n$$

from the domain of some geometric chart c_g to some domain of an electromagnetic chart c_{em}, Fig. 4.

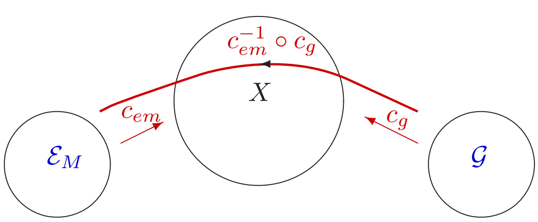

Figure 4: Solutions of electromagnetic boundary value problems can be understood as transition maps from the domain of a geometric chart to the domain of an electromagnetic one.

Example 1: Let us say we solve in a 2d domain X eqs. (1), (2), and (3) in terms of potentials using finite elements. As a solution we get potentials $\{\varphi, \psi\}$, and formally, the question is of generating a map $(x, y) \to (\varphi, \psi)$. Now, (x, y) are Euclidean coordinates of point $\omega \in \Omega$, and (φ, ψ) are electric coordinates of this very same point. Thus, map $(x, y) \to (\varphi, \psi)$ is a transition map from one chart to another.

SOFTWARE SYSTEMS

Once the solutions of boundary value problems are interpreted as transition maps, then, consequently, a finite element, finite difference, or finite integration software system becomes a machinery to construct transition maps between the geometric and electromagnetic charts chosen by the user. In developing finite element kind of software systems one can exploit this kind of geometric approach in several ways. Let us next consider a few examples.

Twisted objects

Let us first consider "twisted objects" such as superconducting wires made of twisted superconducting filaments imbedded in some conductor. The magnetic field generated by such a wire is a two dimensional problem, which can be solved with standard 2d finite element software provided the system allowed the user to introduce the transition map from the twisted wire to a straight one. To see the point, interpret the two images of Fig. 5 to represent one and the same object in domains of two different charts. (The one on the left is in the domain of an Euclidean chart.)

Next, recall that the exterior derivative is meaningful as soon as there is topology on set X. By definition, the underlying set of a manifold has topology. Thus, Maxwell's equations do not depend on the choice of chart. That is, in case of Fig. 5, the Maxwell equations can be established before the choice of the coordinate system. So, what remains is to introduce a transition map from the Euclidean chart(on the left in Fig. 5) to the coordinate system on the right of Fig. 5 to give status to the constitutive law and metric. (The numeric values of permeability and other material parameters are specified on Euclidean charts.) As a result one gets a 2d boundary value problem which is fully meaningful in the domain of the coordinate system on the right hand side of Fig. 5.

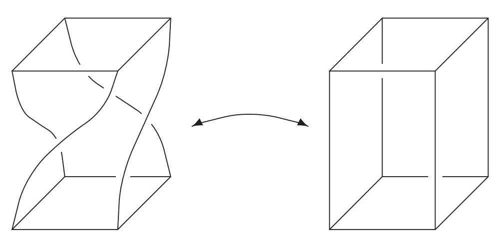

Figure 5: To compute fields caused by twisted objects, all one needs is to give meaning to the constitutive law by introducing a transition map between a straight bar and the twisted object.

Mesh generation

Typically, the existing software systems include only one single chart up to scaling. Furthermore, the user has no choice of the barrel generating Euclidean metric.

To illustrate the idea, how the choice of charts and barrels could be exploited, let us consider a simple electric power line as shown in Fig. 6. It is well known that mesh generation algorithms tend to get in difficulties if the domain has tiny details. The electric power line is an example of such a case. The radius of the wires and of the thickness of the poles are rather small compared to the distance between the poles. The very difficulty lies in creating the first finite element mesh within the system. However, if the end user is able to select the charts and barrels employed, then he may also insert the geometric data in coordinate system(s) which helped the mesh generator to bypass the known pitfalls, see Fig. 6 and 7. [11]

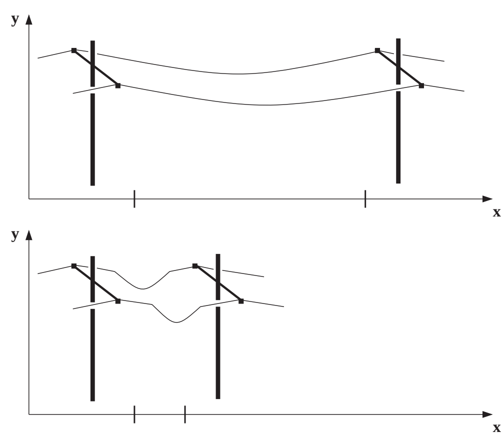

Figure 6: One and the same electric power line (point set) represented in different coordinate systems. The difference lies in the metric of $\mathbb{R}^n$ which is not chosen the same way in the region between the poles –i.e., in the area between the ticks on the x-axis. Still, both charts map to one and the same point set. Consequently, both charts can be employed to solve the electromagnetic fields generated by the power line. However, generating the mesh is easier in the lower domain.

Force computation

The *Maxwell stress tensor* can be expressed without metric structures, see [12]. One may, for instance, exploit this in optimization problems to insert the data of the integration domain directly in terms of the indexes of the mesh entities. Furthermore, the metric independent form of the stress tensor enables one to pre-compute the cost function in order to quickly evaluate its value each time the solution of the field problem is available.

REVERTED WORKING DIRECTION

By definition the transition maps are *diffeomorphic*. This implies, one does not have to restrict oneself to work from the geometric chart collection towards the electromagnetic one. Instead, one may revert the reasoning and work from the electromagnetic charts towards the geometric ones.

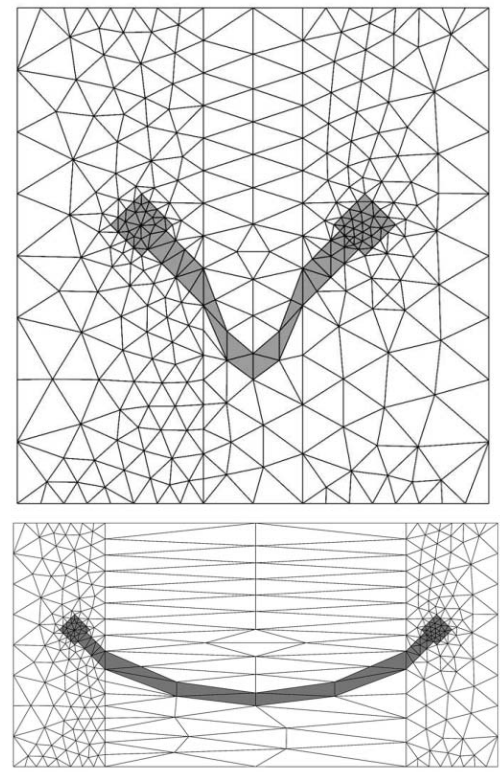

Figure 7: Example of one and the same finite element mesh generated for in the domain of two different charts. The upper domain appears shorter than the lower one implying mesh generation is easier in the upper one.

Engineering design is, in fact, often about working in this direction. For example, in optimization some specification for the electromagnetic field is given a priori, and the task is to find out which geometry produces this field. The common work flow is to make a first attempt, and then the engineer or the system iterates the design until a solution meeting the specification is found. Now, the geometric idea is, once the first solution is found –that is, the first electromagnetic coordinates of set X are given– then one may solve for a desired design without changing this field solution.

For an example see Fig. 8, which can be interpreted to represent the electric field –the field lines and the equipotentials– of some parallel plate capacitor. Say, the horizontal lines are the equipotentials. Next, let's fix the electric coordinates of the points in the (sub)domain between the plates of the capacitor.

Now, we may solve for new Euclidean coordinates in this domain without changing the electric chart. This corresponds to changing the shapes of the capacitor plates and to solving for another capacitors which have the same electric field, see Fig. 9. (The same electric field means the charge and the voltage of the capacitor are fixed.) The practical advantage is that all the new solutions of the Euclidean coordinates are obtained with the same system matrix. Varying the shape corresponds to changing the right hand side of the system of equations, as if the Dirichlet boundary condition is changed in ordinary finite element solution process of the Laplace equation. This implies, if the system matrix is inversed, then all the new Euclidean coordinates are found by simple matrix-vector products.

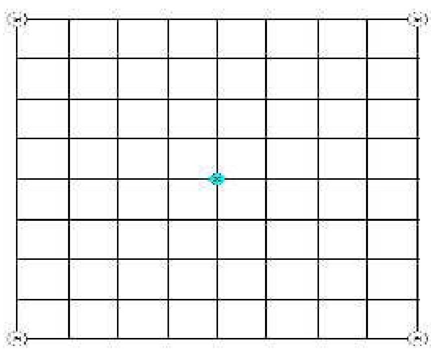

Figure 8: Field lines and equipotentials between a parallel plate capacitor.

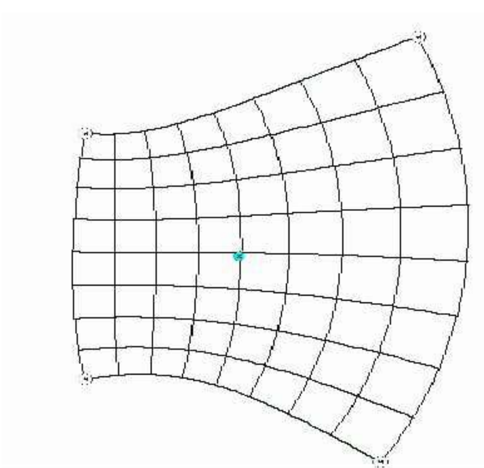

Figure 9: The same field but the shape of the plates has changed. This new shape is found by solving for Euclidean coordinates without changing the electric chart.

CONCLUSION

The basic reasoning behind commonly employed computational techniques and software systems still rely on ideas which were developed a long time ago before the birth of first computers. Computers, however, function in a very different way than human beings. Consequently, the techniques which fitted well the needs of analytic design do not necessarily lend themselves to modern computing. Thus, there is a good reason to re-examine how the old ideas can be expressed in a more general setting. Especially, differential geometry and the theory of manifolds are found to be useful in developing a deeper understanding of electromagnetic field and wave analysis.

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