

A Framework for Maxwell's Equations in Non-Inertial Frames Based on Differential Forms

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Abstract

We set up a consistent framework for the Lagrangian view of (3+1)-dimensional electro-dynamics using the language of differential forms with no need for coordinate systems or reference frames. A natural decomposition mechanism admits the construction of this framework with a minimum of overhead. Employing two observers, one holonomic and the other one locally inertial, opens the possibility to use the simple form of both the Maxwell equations and the constitutive relations simultaneously. Connections to standard results are provided, and the feasibility is further demonstrated by means of a classical application.

INTRODUCTION

In many engineering applications the interaction between the electromagnetic field and moving bodies is of great interest. It is natural to use a Lagrangian description, where the unknowns are defined on a mesh which moves and deforms together with the considered objects. What is the correct form of Maxwell's equations and the constitutive laws under such circumstances? The aim of the present paper is to tackle this question by using the language of differential forms.

We start from Maxwell's equations and constitutive relations in four-dimensional flat Minkowskian space-time. While this essentially constitutes an Eulerian point of view, we proceed to a four-dimensional Lagrangian description by transformation to the canonical reference space. The next step is to introduce a (3+1)-decomposition mechanism which is based exclusively on the pair of a vector field and a 1-form, describing an observer in space-time, [5, 7]. With the help of this mechanism, all fundamental operators like exterior derivative, Hodge, and contraction can be easily decomposed. An application to four-dimensional electro-dynamics yields the common notions of three-velocity and contraction factor, defines the three-dimensional field components, and provides general (3+1)-Maxwell and constitutive equations. It becomes obvious that if the observer responsible for the decomposition is holonomic or locally inertial, then Maxwell's equations or constitutive laws adopt their simple form, respectively. This observation becomes utilized by introducing two observers to the Lagrangian setting, one holonomic and the other locally inertial. In order to profit from their individual advantages, a transformation law connecting the two observers is established, again sim-

ply by plugging in the decomposition mechanism. This concludes a convenient description of (3+1)-dimensional electro-dynamics. As an application of our approach, we consider the classical paradoxon by Schiff, [8].

Due to space limitations, most of our results are stated without proof. Detailed derivations are given in the extended version [4]. There, we also provide connections of our approach to standard results.

SPACE-TIME ELECTRO-DYNAMICS

Our model for physical spacetime is that of a four-dimensional affine space M_4 equipped with a metric g of signature (+ - - -), referred to as Minkowski space. The metric is represented by the mapping $g : \Lambda\mathcal{X}^1(M_4) \rightarrow \Lambda\mathcal{F}^1(M_4)$ from the space of smooth (multi-)vector fields to the space of smooth differential forms, defining the extent of a p -form ω as $\|\omega\| = \sqrt{|(\omega|g^{-1}(\omega))|}$, where $\cdot| \cdot$ is the usual duality product. In M_4 , electro-dynamic phenomena are described by Maxwell's equations, namely,

$$d\underline{F} = 0, \quad \text{and} \quad d\underline{G} = \underline{\mathcal{J}}, \quad (1)$$

where d stands for the exterior differential operator, $\underline{F}, \underline{G} \in \mathcal{F}^2(M_4)$ are the electromagnetic field and excitation, respectively, and $\underline{\mathcal{J}} \in \mathcal{F}^3(M_4)$ the four-current density. The field \underline{F} and the excitation \underline{G} are linked by the constitutive laws, [2],

$$\mathbf{i}_u (* \underline{G} + c_0 \varepsilon \underline{F}) = 0, \quad (2a)$$

$$\mathbf{i}_u (* \underline{F} - c_0 \mu \underline{G}) = 0, \quad (2b)$$

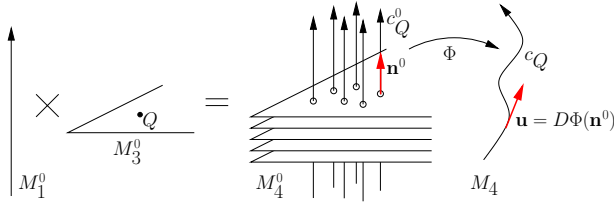
where \mathbf{i}_u denotes the contraction by the four-velocity \mathbf{u} , c_0 the vacuum velocity of light, ε and μ the electric and magnetic permeabilities, and $*$ is the Hodge operator associated with the metric g . The four-velocity vector field \mathbf{u} is time-like and of unit length with respect to g , $\mathbf{i}_u g(\mathbf{u}) = 1$. It is tangent to the world-lines of the volume elements in M_4 . Maxwell's equations (1) and the constitutive laws (2) constitute a complete four-dimensional Eulerian description of electro-dynamics in M_4 .

LAGRANGIAN PERSPECTIVE

The Lagrangian observer describes the events from a reference space M_4^0 , which is the product of a one-dimensional oriented affine space M_1^0 and a three-dimensional oriented affine space M_3^0 , the configuration space, as illustrated in Figure 1. The reference space

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Figure 1: Reference space M_4^0 , placement mapping Φ .

M_4^0 and the physical space M_4 are linked by a placement mapping in form of a diffeomorphism $\Phi : M_4^0 \rightarrow M_4$. The physical interpretation of Φ is deduced from the observation of a point $Q \in M_3^0$, which defines a curve

$$c_Q^0 = M_1^0 \times Q \quad (3)$$

in M_4^0 . Then the mapped curve $c_Q = \Phi(c_Q^0)$ in M_4 is exactly the world-line of the volume element labelled by Q . The Lagrangian description of electro-dynamics is nothing more than the reformulation of Maxwell's equations (1) and the constitutive laws (2) within the reference space M_4^0 . This is accomplished by pull-back of the involved field quantities and operators via the placement mapping Φ . After setting

$$\underline{F}^0 = \Phi^* \underline{F}, \quad \underline{G}^0 = \Phi^* \underline{G}, \quad \underline{J}^0 = \Phi^* \underline{J},$$

the transformed equations are of the same kind as the original ones, namely,

$$d^0 \underline{F}^0 = 0, \quad \text{and} \quad d^0 \underline{G}^0 = \underline{J}^0, \quad (4a)$$

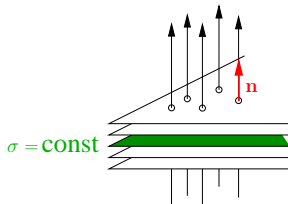
$$\mathbf{i}_{\mathbf{u}^0} (*^0 \underline{G}^0 + c_0 \varepsilon \underline{F}^0) = 0, \quad (4b)$$

$$\mathbf{i}_{\mathbf{u}^0} (*^0 \underline{F}^0 - c_0 \mu \underline{G}^0) = 0, \quad (4c)$$

where $*^0$ indicates the Hodge operator of the pulled-back metric g^0 . With (4), a complete four-dimensional Lagrangian description has been derived, whose properties will be elaborated in the sequel.

(3+1) DECOMPOSITION MECHANISM

Definition In order to keep as much flexibility as possible, we employ the techniques presented in [5, p. 117]. The setting is illustrated in Figure 2. For a general space-

Figure 2: Foliation and fibration of M described by \mathbf{n} and $\sigma = \text{const}$.

time manifold M , let a fibration of M be described by a three-parameter vector field \mathbf{n} , introducing the notion of relative space. Furthermore, let a foliation of M be described by a one-parameter family of hypersurfaces $\sigma =$

const., introducing the notion of relative time. Setting $\sigma = d\sigma$, it is possible to scale \mathbf{n} and σ such that

$$\mathbf{i}_{\mathbf{n}} \sigma = 1. \quad (5)$$

The pair (\mathbf{n}, σ) constructed in this manner constitutes an observer in space-time. In the special case of an inertial observer, the fibration consists of parallel time-like lines with the leafs of the foliation being orthogonal, and we have $\sigma = g(\mathbf{n})$. Without the parallelism of the fibres, a locally inertial observer is obtained provided that $\sigma = g(\mathbf{n})$ in every point. If a frame of reference was attached to an observer (\mathbf{n}, σ) satisfying (5), \mathbf{n} and σ would describe the temporal basis elements of this frame. However, the following (3+1) decomposition mechanism does not rely on the existence of a reference frame. In particular, we construct a projection operator $P = P_{\mathbf{n}, \sigma}$ depending only on the two constitutive parameters \mathbf{n} and σ .

We first consider the space $\mathcal{X}^p = \mathcal{X}^p(M)$ of smooth p -vector fields on M . Setting

$$\mathcal{X}_{\mathbf{w}}^p = \{\mathbf{w} \in \mathcal{X}^p : \mathbf{w} \lrcorner \mathbf{w} = 0\},$$

the projection of a (1-)vector field is defined by

$$P : \mathcal{X}^1 \rightarrow \mathcal{X}_{\sigma}^1 \times \mathcal{X}_{\sigma}^0, \quad \mathbf{w} \mapsto (\mathbf{a}, b),$$

$$b = \sigma \lrcorner \mathbf{w}, \quad \mathbf{a} = \mathbf{w} - b\mathbf{n}.$$

We proceed analogously for decomposing forms. Let $\mathcal{F}^p = \mathcal{F}^p(M)$ denote the space of smooth p -forms on M , and define

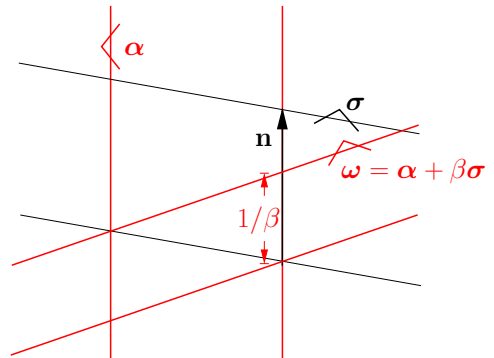
$$\mathcal{F}_{\mathbf{w}}^p = \{\omega \in \mathcal{F}^p : \mathbf{i}_{\mathbf{w}} \omega = 0\}.$$

Then, the projection of a 1-form is given by

$$P : \mathcal{F}^1 \rightarrow \mathcal{F}_{\mathbf{n}}^1 \times \mathcal{F}_{\mathbf{n}}^0, \quad \omega \mapsto (\alpha, \beta),$$

$$\beta = \omega \lrcorner \mathbf{n}, \quad \alpha = \omega - \beta \sigma.$$

In Figure 3, the action of P on a covector ω is sketched in 2D. The domain of P can be extended to \mathcal{F}^p by setting

Figure 3: Projection of a covector ω .

$$P : \mathcal{F}^p \rightarrow \mathcal{F}_{\mathbf{n}}^p \times \mathcal{F}_{\mathbf{n}}^{p-1}, \quad \omega \mapsto (\alpha, \beta), \quad (6)$$

$$\beta = \mathbf{i}_{\mathbf{n}} \omega, \quad \alpha = \omega - \sigma \wedge \beta = \mathbf{i}_{\mathbf{n}} (\sigma \wedge \omega).$$

From the definition above, it is obvious that the projection P has the inverse

$$P^{-1} : \mathcal{F}_{\mathbf{n}}^p \times \mathcal{F}_{\mathbf{n}}^{p-1} \rightarrow \mathcal{F}^p, \quad (\alpha, \beta) \mapsto \omega = \alpha + \sigma \wedge \beta.$$

Decomposition of the exterior derivative We indicate the Lie derivative along the vector field \mathbf{n} as “temporal” derivative

$$\bullet = \mathcal{L}_{\mathbf{n}} = \mathbf{i}_{\mathbf{n}} \circ d + d \circ \mathbf{i}_{\mathbf{n}}.$$

For the exterior differentiation of a projected pair $(\alpha, \beta) \in \mathcal{F}_{\mathbf{n}}^p \times \mathcal{F}_{\mathbf{n}}^{p-1}$, it is natural to use the composition $P \circ d \circ P^{-1}$. We obtain

$$P \circ d \circ P^{-1} = \begin{pmatrix} d_3 & \eta \wedge \\ \bullet & -d_3 + \delta \wedge \end{pmatrix}, \quad (7)$$

where $\delta = \dot{\sigma}$ is the acceleration 1-form and $\eta = d_3 \sigma$ the vorticity 2-form, [3]. In other words, we have that $d\sigma = \eta + \sigma \wedge \delta$ with $(\eta, \delta) = P(d\sigma)$. In the case that (\mathbf{n}, σ) is seen as the temporal basis vector and covector of a reference frame, one obtains a geodesic frame for $\delta = 0$, and an irrotational frame for $\eta = 0$. Whenever $\delta \neq 0$ or $\eta \neq 0$, one speaks of an anholonomic frame. We remark that the composition $d_3 \circ d_3 = -\eta \wedge \bullet$. This does not contradict Stoke’s theorem, since $\eta \neq 0$ yields $\sigma \wedge d\sigma \neq 0$ which violates the Frobenius integrability condition. In this case, a three-dimensional integral submanifold does not exist.

Of special interest will be the case $d\sigma = 0$ yielding $\delta = \eta = 0$. Then, the decomposition of the exterior derivative is given by the simple form

$$P \circ d \circ P^{-1} = \begin{pmatrix} d_3 & 0 \\ \bullet & -d_3 \end{pmatrix}.$$

A direct consequence of these considerations is that the canonical form of the (3+1)D Maxwell equations is only guaranteed for $\delta = \eta = 0$, as will become obvious in Section “Decomposition of Maxwell’s equations”.

Decomposition of the Hodge operator Let

$$s : \Lambda(\mathcal{F}) \rightarrow \Lambda(\mathcal{F}), \quad \omega \mapsto (-1)^{\deg \omega} \omega.$$

The metric isomorphism $g^{-1} : \mathcal{F}^p \rightarrow \mathcal{X}^p$ can be decomposed into

$$Pg^{-1} = (g_3^{-1}s, \cdot), \quad g_3^{-1} : \mathcal{F}_{\mathbf{n}}^p \rightarrow \mathcal{X}_{\sigma}^p, \quad (8)$$

with a positive definite metric g_3 . In particular, we have

$$g^{-1}(\omega) = g_3^{-1}(s\omega) + \mathbf{n} \wedge \chi(s\omega), \quad (9)$$

for a mapping $\chi : \mathcal{F}_{\mathbf{n}}^p \rightarrow \mathcal{X}_{\sigma}^{p-1}$. A justification for (9) is given in [4]. It is possible to express the induced Hodge \ast_3 for the 3-metric g_3 in terms of the 4D Hodge \ast and the vector field \mathbf{n} , namely

$$\ast_3 = \|\mathbf{n}\|^{-1} \mathbf{i}_{\mathbf{n}} \ast.$$

Now, everything is available to decompose the Hodge \ast with respect to the projection P . To this end, define $\lambda \in \mathbb{R}$ and $\mathbf{w} \in \mathcal{X}_{\sigma}^1$ by $(\mathbf{w}, \lambda) = Pg^{-1}(\sigma)$. Considering

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = P \ast P^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = P \ast (\alpha + \sigma \wedge \beta),$$

gives

$$\begin{aligned} \beta' &= \|\mathbf{n}\| \ast_3 \alpha - \|\mathbf{n}\| \mathbf{i}_{\mathbf{w}} \ast_3 s\beta, \\ \alpha' &= -\|\mathbf{n}\| \ast_3 \mathbf{i}_{\mathbf{w}} s\alpha + \|\mathbf{n}\| \lambda \ast_3 s\beta - \|\mathbf{n}\| \mathbf{i}_{\mathbf{w}} \ast_3 \mathbf{i}_{\mathbf{w}} \beta. \end{aligned}$$

Thus, the Hodge \ast decomposes to

$$P \ast P^{-1} = \|\mathbf{n}\| \begin{pmatrix} -\ast_3 \mathbf{i}_{\mathbf{w}} s & \lambda \ast_3 s - \mathbf{i}_{\mathbf{w}} \ast_3 \mathbf{i}_{\mathbf{w}} \\ \ast_3 & -\mathbf{i}_{\mathbf{w}} \ast_3 s \end{pmatrix}. \quad (10)$$

Of special interest is the case $\mathbf{w} = 0$. If (\mathbf{n}, σ) is extended to a reference frame, this frame is said to be time-orthogonal. Then, the decomposition simplifies to

$$P \ast P^{-1} = \|\mathbf{n}\| \begin{pmatrix} 0 & \lambda \ast_3 s \\ \ast_3 & 0 \end{pmatrix}.$$

Moreover, it follows for $\mathbf{w} = 0$ that $g^{-1}(\sigma) = \lambda \mathbf{n}$, thus,

$$\lambda = \sigma |g^{-1}(\sigma) = \lambda g(\mathbf{n}) | \lambda \mathbf{n} = \lambda^2 \|\mathbf{n}\|^2,$$

yielding $\lambda = \|\mathbf{n}\|^{-2}$. Therefore, the decomposition of the Hodge operator in time-orthogonal frames is given by

$$P \ast P^{-1} = \begin{pmatrix} 0 & \|\mathbf{n}\|^{-1} \ast_3 s \\ \|\mathbf{n}\| \ast_3 & 0 \end{pmatrix}. \quad (11)$$

The consequence of these observations is that simple (3+1)D constitutive laws only occur for $\mathbf{w} = 0$, as will be shown in the next section.

Decomposition of the contraction The projection P also enables us to decompose the contraction of a p -form $\omega = \alpha + \sigma \wedge \beta$ by a vector field $\mathbf{u} \in \mathcal{X}$. We can write \mathbf{u} as

$$\mathbf{u} = \mathbf{a} + b\mathbf{n}, \quad \mathbf{a} \in \mathcal{X}_{\sigma}^1, \quad b \in \mathcal{X}^0.$$

Then, the contraction $\mathbf{i}_{\mathbf{u}}$ decomposes to

$$P \mathbf{i}_{\mathbf{u}} P^{-1} = \begin{pmatrix} \mathbf{i}_{\mathbf{a}} & b \\ 0 & -\mathbf{i}_{\mathbf{a}} \end{pmatrix}. \quad (12)$$

(3+1) DECOMPOSITION OF ELECTRO-DYNAMICS

Decomposition of the four-velocity Projecting the four-velocity \mathbf{u} defines the three-velocity \mathbf{v} measuring the velocity in three-space \mathcal{X}_{σ}^1 of the observer given by (\mathbf{u}, μ) relative to (\mathbf{n}, σ) . Additionally, one obtains the contraction factors γ_1, γ_2 , and the 1-form ν by setting

$$\gamma_1 \begin{pmatrix} \mathbf{v}/c_0 \\ 1 \end{pmatrix} = P\mathbf{u}, \quad \gamma_2 \begin{pmatrix} \nu/c_0 \\ 1 \end{pmatrix} = P\mu. \quad (13)$$

Setting $-\beta^2 = c_0^{-2} \nu | \mathbf{v}$, we obtain

$$-\gamma_1 \gamma_2 \beta^2 = 1 - \gamma_1 \gamma_2.$$

For the contraction factor γ defined by $\gamma^2 = \gamma_1 \gamma_2$, we observe that

$$\gamma = (1 - \beta^2)^{-1/2}. \quad (14)$$

In our setting, we will consider the special case of locally inertial observers,

$$\mathbf{n} = g^{-1}(\boldsymbol{\sigma}), \quad \mathbf{u} = g^{-1}(\boldsymbol{\mu}). \quad (15)$$

Then we have

$$\gamma_1 = \boldsymbol{\sigma} | \mathbf{u} = \boldsymbol{\sigma} | g^{-1}(\boldsymbol{\mu}) = \boldsymbol{\sigma} \cdot \boldsymbol{\mu} = \boldsymbol{\mu} | g^{-1}(\boldsymbol{\sigma}) = \boldsymbol{\mu} | \mathbf{n} = \gamma_2,$$

thus, $\gamma = \gamma_1 = \gamma_2$. From (13), we conclude

$$\mathbf{u} = \gamma(\mathbf{n} + \mathbf{v}/c_0). \quad (16)$$

Decomposition of Maxwell's equations With the projection operator P , it is possible to decompose the four-dimensional electro-dynamic quantities into their three-dimensional components. The four-potential $\underline{\Phi}$, the excitation \underline{G} , the electromagnetic field \underline{E} , and the four-current density \underline{J} decompose to

$$\begin{pmatrix} \underline{A} \\ -\underline{\varphi}/c_0 \end{pmatrix} = P\underline{\Phi}, \quad \begin{pmatrix} \underline{D} \\ \underline{H}/c_0 \end{pmatrix} = P\underline{G}, \quad (17a)$$

$$\begin{pmatrix} \underline{B} \\ -\underline{E}/c_0 \end{pmatrix} = P\underline{E}, \quad \begin{pmatrix} \underline{\rho} \\ -\underline{J}/c_0 \end{pmatrix} = P\underline{J}. \quad (17b)$$

These projections define the conventional three-dimensional electromagnetic field quantities, [5, p. 118ff.]. Projecting the four-dimensional Maxwell equations (1) and substituting $P^{-1}P$ for the identity yields

$$P \, dP^{-1}P\underline{E} = 0, \quad \text{and} \quad P \, dP^{-1}P\underline{G} = P\underline{J}.$$

Using the decompositions (7) and (17) of the four-differential and of the field quantities, respectively, we obtain the (3+1)-Maxwell equations

$$\begin{aligned} d_3 \underline{H} &= \underline{J} + \dot{\underline{D}} + \boldsymbol{\delta} \wedge \underline{H}, & d_3 \underline{B} &= 0 + \boldsymbol{\eta} \wedge \underline{E}, \\ d_3 \underline{E} &= -\dot{\underline{B}} + \boldsymbol{\delta} \wedge \underline{E}, & d_3 \underline{D} &= \underline{\rho} - \boldsymbol{\eta} \wedge \underline{H}, \end{aligned}$$

the continuity equation $d_3 \underline{J} = 0$ yields

$$d_3 \underline{J} + \dot{\underline{\rho}} = \boldsymbol{\delta} \wedge \underline{J},$$

and the definition of the four-potential $d\underline{\Phi} = \underline{E}$ gives the potential equations

$$\underline{E} = -d_3 \underline{\varphi} - \dot{\underline{A}} + \boldsymbol{\varphi} \wedge \boldsymbol{\delta}, \quad \underline{B} = d_3 \underline{A} - \boldsymbol{\varphi} \wedge \boldsymbol{\eta}.$$

It becomes obvious that it will not be convenient to operate in anholonomic settings, i.e. whenever $\boldsymbol{\delta}$ or $\boldsymbol{\eta}$ are different from zero.

Decomposition of the constitutive laws We proceed in the same manner to derive (3+1)-dimensional constitutive laws. The projection of (2) gives

$$P \mathbf{i}_{\mathbf{u}} P^{-1} (P * P^{-1} P\underline{G} + c_0 \varepsilon P\underline{E}) = 0, \quad (18a)$$

$$P \mathbf{i}_{\mathbf{u}} P^{-1} (P * P^{-1} P\underline{E} - c_0 \mu P\underline{G}) = 0. \quad (18b)$$

Using the decompositions (12), (10), and (17) of the contraction, the Hodge star, and the field quantities, respectively, yields (3+1)-dimensional constitutive laws. A particularly easy situation is obtained by introducing a locally inertial observer, in which the considered material element is instantaneously at rest. Then, the three-velocity \mathbf{v} is zero, hence, $\mathbf{u} = \mathbf{n}$ by (16) and $\mathbf{n} = g^{-1}(\boldsymbol{\sigma})$. As a consequence, we have $\mathbf{a} = 0$, $b = 1$ in (12) and, due to the time-orthogonality, the Hodge star decomposes to the simple form (11), with $\|\mathbf{n}\| = 1$. Therefore, indicating the such projected field components with a prime, relations (18) imply that

$$*_3 \underline{D}' = \varepsilon \underline{E}', \quad *_3 \underline{B}' = \mu \underline{H}'. \quad (19)$$

In the more general setting of (15) and (16), we obtain

$$\begin{aligned} \underline{D} &= *_3 \varepsilon (\underline{E} - (1 - c^2/c_0^2) \mathbf{i}_{\mathbf{v}} \underline{B}) + O(\beta^2), \\ \underline{B} &= *_3 \mu (\underline{H} + (1 - c^2/c_0^2) \mathbf{i}_{\mathbf{v}} \underline{D}) + O(\beta^2), \end{aligned}$$

which are the well-known relations confirmed by Wilson and Röntgen/Eichwald, respectively. If the decomposition was not time-orthogonal, we would have to deal with the more involved expression (10) for the decomposed Hodge operator, which would lead to a quite complicated form of the constitutive laws. The advantage of our approach is that we will not have to consider this situation.

APPLICATION TO THE LAGRANGIAN PERSPECTIVE

In the following, we establish two (3+1)-decompositions of the reference space M_4^0 . To this end, we parametrize the curves c_Q^0 defined in (3) by arc-length with respect to the pulled-back metric g^0 . This admits the introduction of a coordinate t_Q^0 on c_Q^0 and, proceeding like this for all Q in M_3^0 , of a coordinate t^0 in M_4^0 . The set described by $t_Q^0 = 0$, $Q \in M_3^0$, constitutes a leaf of the foliation of M_4^0 . The pair $(\mathbf{n}^0, \boldsymbol{\sigma}^0)$ given by

$$\mathbf{n}^0 = \partial_{c_0 t^0}, \quad \boldsymbol{\sigma}^0 = d(c_0 t^0),$$

represents the natural foliation of M_4^0 . By construction, we have $\boldsymbol{\sigma}^0 | \mathbf{n}^0 = 1$, thus, it is possible to apply the projection formalism. We remark that the choice of $(\mathbf{n}^0, \boldsymbol{\sigma}^0)$ is unique. Although for any non-zero $\alpha \in \mathcal{F}^0(M_4^0)$, the pair $(\alpha \mathbf{n}^0, \alpha^{-1} \boldsymbol{\sigma}^0)$ describes the same foliation, the choice $\alpha = 1$ is fixed by the fact that the arc-length parametrization requires $|\alpha| = 1$, while the orientation of M_1^0 sets the sign.

By construction, we have that $D\Phi(\mathbf{n}^0) = \mathbf{u}$, i.e., $\mathbf{n}^0 = \mathbf{u}^0$ resulting in $\mathbf{v}^0 = 0$, as it is suggested by a Lagrangian description. Moreover, we have $d\boldsymbol{\sigma}^0 = 0$ by construction, thus, the projected Maxwell equations are of their simple form. We remark that the condition $g(\mathbf{n}^0) | \mathbf{n}^0 > 0$ reduces the possibilities for admissible placement mappings Φ , requiring that the curves c_Q^0 and c_Q have to be inside the light cone defined by the metric g^0 and g , respectively.

The metric admits the definition of a second, metric-compatible observer, by setting

$$\mathbf{n}' = \mathbf{n}^0, \quad \boldsymbol{\sigma}' = g^0(\mathbf{n}').$$

We immediately observe

$$\boldsymbol{\sigma}'|\mathbf{n}' = g(\mathbf{n}')|\mathbf{n}' = g(\mathbf{n}^0)|\mathbf{n}^0 = \|\mathbf{n}^0\|^2 = 1,$$

thus, the projection mechanism may be applied again. Moreover, since $\mathbf{v}^0 = 0$, $\|\mathbf{n}'\| = 1$, and due to the time-orthogonality, the projected constitutive laws are of their simple form (19). The push-forward $D\Phi(\mathbf{n}', \boldsymbol{\sigma}')$ defines a locally inertial observer in M_4 . As a consequence, if $\boldsymbol{\omega}' \in \mathcal{F}_{\mathbf{n}'}^p(M_4^0)$ is a field quantity with respect to the projection defined by $(\mathbf{n}', \boldsymbol{\sigma}')$, then $\Lambda D\Phi(\boldsymbol{\omega}') \in \mathcal{F}_{\mathbf{u}}^p(M_4)$ is the corresponding measurable physical quantity, according to the hypothesis of locality, [6].

In order to suitably relate the two foliations, one has to formulate a transformation law which establishes a one-to-one correspondence between the spaces $\mathcal{F}_{\mathbf{n}^0}^p \times \mathcal{F}_{\mathbf{n}^0}^{p-1}$ and $\mathcal{F}_{\mathbf{n}'}^p \times \mathcal{F}_{\mathbf{n}'}^{p-1}$. We remark that both spaces are identical, since $\mathbf{n}^0 = \mathbf{n}'$. The correspondence is simply given by the mapping $P^0 \circ (P')^{-1}$. One obtains

$$\begin{pmatrix} \boldsymbol{\alpha}^0 \\ \boldsymbol{\beta}^0 \end{pmatrix} = \begin{pmatrix} 1 & (\boldsymbol{\sigma}' - \boldsymbol{\sigma}^0)^\wedge \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}' \\ \boldsymbol{\beta}' \end{pmatrix}. \quad (20)$$

As pointed out before, the advantage of employing both decompositions is that we can employ simple (3 + 1)-Maxwell equations with respect to $(\mathbf{n}^0, \boldsymbol{\sigma}^0)$ and simple constitutive laws with respect to $(\mathbf{n}', \boldsymbol{\sigma}')$. The transformation law (20) completes a convenient description of (3+1)-dimensional electro-dynamics.

EXAMPLE

We present an application of the above formalism to the classical paradoxon by Schiff [8]. Originally, the setting is given in terms of two conducting spheres. For the sake of simplicity and in order to illustrate the idea, we investigate an analogue in terms of an infinitely long cylinder, see [1, p. 320]. The following field quantities are chosen with respect

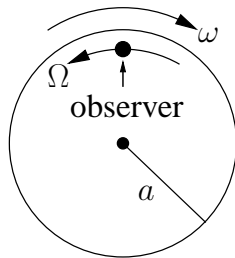


Figure 4: Paradoxon by Schiff (simplified): homogeneously charged cylinder, two different situations.

to an inertial observer attached to the axis of the cylinder. The cylinder of radius a is homogeneously charged with a

surface charge distribution $\sigma = q/(2\pi a)$. It may rotate at an angular speed ω , causing an azimuthal convection current $j_\varphi = -\omega q \delta(r - a)/(2\pi)$, where δ denotes the delta distribution. We install an observer which may rotate at an angular speed Ω . We consider two different situations in which the observer measures the field inside the cylinder. In the first one, only the cylinder moves, whereas in the second one, only the observer moves:

- $\Omega = 0$, $\omega = \omega_0$: As a result of the convection current, the interior of the cylinder is filled by an electromagnetic field with radial displacement $D_r = 0$ and axial magnetic field $H_z = -\omega q/(2\pi)$.
- $\Omega = \omega_0$, $\omega = 0$: The interior is free of an electromagnetic field, which has to hold for arbitrary observers. Therefore, the observer should find $D_r' = H_z' = 0$. However, should not this case be identical to the first one from the observer's point of view? He always experiences the same kinematics: a cylinder moving at the same angular speed.

The resolution of this apparent paradox results from considering the correct measurable fields for the second case. One of the benefits of our approach is now that we consider Maxwell's equations with respect to the observer $(\mathbf{n}^0, \boldsymbol{\sigma}^0)$ where they basically keep their usual form. Nevertheless, they have to be transformed into rotating cylinder coordinates, as found for example in [1, p. 268]. In M_4 , we choose coordinates $(c_0 t, r, \varphi, z)$ yielding the natural bases $(c_0^{-1} \partial_t, \partial_r, \partial_\varphi, \partial_z)$ and $(c_0 dt, dr, d\varphi, dz)$ of the tangent and cotangent spaces, respectively. The metric g in matrix notation is given by $\text{diag}(1, -1, -r^2, -1)$. For the reference space M_4^0 , the coordinates and bases are completely analogous, namely $(c_0 t^0, r^0, \varphi^0, z^0)$, $(c_0^{-1} \partial_{t^0}, \partial_{r^0}, \partial_{\varphi^0}, \partial_{z^0})$, and $(c_0 dt^0, dr^0, d\varphi^0, dz^0)$.

The placement mapping $\Phi : M_4^0 \rightarrow M_4$ is given by

$$\begin{aligned} c_0 t &= \gamma c_0 t^0, & r &= r^0, & z &= z^0 \\ \varphi &= \varphi^0 + \frac{\Omega}{c_0} \gamma c_0 t^0 = \varphi^0 + \Omega t, \end{aligned}$$

with $\gamma = (1 - (\Omega r^0/c_0)^2)^{-1/2}$. The foliation-induced observer is simply given by $(\mathbf{n}^0, \boldsymbol{\sigma}^0) = (c_0^{-1} \partial_{t^0}, c_0 dt^0)$. For the local inertial observer, we have $\mathbf{n}' = \mathbf{n}^0 = c_0^{-1} \partial_{t^0}$ and

$$\boldsymbol{\sigma}' - \boldsymbol{\sigma}^0 = g^0(\mathbf{n}') - \boldsymbol{\sigma}^0 = -\gamma r^2 \Omega / c_0 d\varphi^0.$$

We now turn to the source data of the given problem. In M_4 , we have

$$\underline{\rho} = \frac{q}{2\pi} \delta(r-a) dr \wedge d\varphi \wedge dz, \quad \underline{\mathcal{J}} = 0,$$

with respect to the projection $P = P_{c_0^{-1} \partial_{t^0}, c_0 dt^0}$. By pull-back to M_4^0 and decomposition with respect to P^0 , we obtain

$$\begin{pmatrix} \underline{\rho}^0 \\ -\underline{\mathcal{J}}^0/c_0 \end{pmatrix} = P^0 \underline{\mathcal{J}}^0 = \frac{q}{2\pi} \delta(r^0-a) \begin{pmatrix} dr^0 \wedge d\varphi^0 \wedge dz^0 \\ \gamma \Omega c_0^{-1} dz^0 \wedge dr^0 \end{pmatrix}. \quad (21)$$

We remark that due to the axial symmetry of the domain and the data, all involved quantities depend only on r^0 , not on φ^0 or z^0 . Therefore, the three-derivative is given by $d_3 = dr^0 \wedge \partial_{r^0}$. Maxwell's equations reduce to

$$-\partial_{r^0} H_z^0 = j_\varphi^0, \quad (22a)$$

$$\partial_{r^0} D_r^0 = \rho^0. \quad (22b)$$

From (21), we deduce for the solutions of (22)

$$H_z^0 = \frac{q}{2\pi} \begin{cases} c_2 & r^0 < a, \\ c_2 + \gamma\Omega & r^0 > a, \end{cases} \quad (23)$$

$$D_r^0 = \frac{q}{2\pi} \begin{cases} c_1 & r^0 < a, \\ c_1 + 1 & r^0 > a. \end{cases} \quad (24)$$

In order to derive physically observable pulled-back quantities, we employ the transformation (20). This results in

$$\underline{D}' = (D_r^0 + \frac{\gamma r^2 \Omega}{c_0^2} H_z^0) d\varphi^0 \wedge dz^0,$$

$$\underline{H}' = \underline{H}^0 = H_z^0 dz^0.$$

If \underline{D}' and \underline{H}' are written in coordinates with respect to the coordinate basis $(c_0 dt^0, dr^0, d\varphi^0, dz^0)$, we have

$$D_r' = D_r^0 + \frac{\gamma r^2 \Omega}{c_0^2} H_z^0, \quad H_z' = H_z^0.$$

From (23) and (24), we obtain

$$D_r' = \frac{q}{2\pi} \begin{cases} c_1 + c_2 \frac{\gamma (r^0)^2 \Omega}{c_0^2} & r^0 < a, \\ c_1 + c_2 \frac{\gamma (r^0)^2 \Omega}{c_0^2} + \gamma^2 & r^0 > a. \end{cases} \quad (25)$$

Using the constitutive laws (19) requires the calculation of the three-dimensional Hodge $*_3^0$. It turns out that the coefficients of the induced metric (g_3^0) are given by $\text{diag}(1, (\gamma r)^2, 1)$. This motivates the explicit construction of the Hodge $*_3^0$ by replacing r by γr^0 in the expression for the three-Hodge in cylindrical coordinates. From (19), we observe

$$D_r' = \varepsilon \gamma r^0 E_r'.$$

Now, we can deduce the integration constants c_1 and c_2 in (23) and (25), respectively. Since D_r' has to remain bounded for $r^0 \rightarrow \infty$, we have $c_2 = 0$. Since $E_r' = D_r' / (\varepsilon \gamma r^0)$ also has to remain bounded for $r^0 \rightarrow 0$, we have $c_1 = 0$. This finally gives the solution

$$H_z' = \frac{\gamma q}{2\pi} \begin{cases} 0 & r^0 < a, \\ \Omega & r^0 > a, \end{cases} \quad (26)$$

and for the solution of (22b)

$$D_r' = \frac{\gamma^2 q}{2\pi} \begin{cases} 0 & r^0 < a, \\ 1 & r^0 > a. \end{cases} \quad (27)$$

Thus, the interior is free of an electromagnetic field.

In order to validate our result, we push-forward our solution from M_4^0 to M_4 , and decompose the result with respect to the inertial observer $(c_0^{-1} \partial dt, c_0 dt)$. As result, we expect the electro-static field of a line charge on the cylinder axis for $r > a$. In M_4^0 , the excitation \underline{G}^0 is given by

$$\underline{G}^0 = \frac{\gamma q}{2\pi} (\gamma^{-1} d\varphi^0 \wedge dz^0 + \Omega dt^0 \wedge dz^0),$$

for $r^0 > a$. The pushed-forward excitation \underline{G} in M_4 amounts to

$$\underline{G} = D\Phi \underline{G}^0 = \frac{q}{2\pi} d\varphi \wedge dz.$$

Applying the projection P with respect to $(c_0^{-1} \partial dt, c_0 dt)$ yields

$$\left(\frac{\underline{D}}{\underline{H}/c_0} \right) = P \underline{G} \Rightarrow \underline{D} = \frac{q}{2\pi} d\varphi \wedge dz, \quad \underline{H} = 0,$$

as has been expected.

CONCLUSION

Summing up, our approach sets up a consistent framework for the Lagrangian view of (3+1)-dimensional electro-dynamics using the language of differential forms with no need for coordinate systems or reference frames. The decomposition mechanism, [5], admits the construction of this framework with a minimum of overhead, only relying on the notion of an observer. Employing two observers, one holonomic and the other locally inertial, opens the possibility to use the simple form of both the Maxwell equations and the constitutive relations simultaneously. The feasibility and usefulness of the approach is demonstrated by means of a classical example, [8].

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