Statistical Properties of Schottky Spectra

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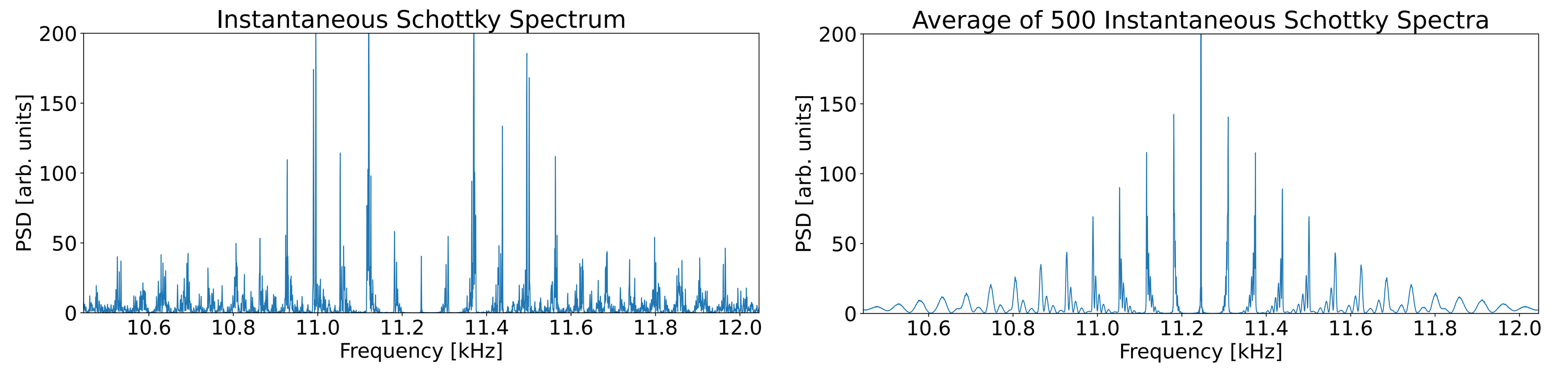
I. Introduction

Schottky signals are used for non-invasive beam diagnostics as they contain information on various beam and machine parameters. The Large Hadron Collider (LHC) Schottky system provides a single spectrum every second, based on the signal acquired over approximately the last 16 000 revolutions. Notably, consecutive spectra exhibit significant dissimilarity, necessitating the aggregation of numerous spectra to attain the mean value. This study explores the variability inherent in consecutive instantaneous Schottky spectra and analyses quantitatively their statistical properties, including the expected value and variance of Schottky power spectra.

Furthermore, we investigate how these quantities evolve with the number of particles in the bunch, the observed harmonic of the revolution frequency, the distribution of synchrotron oscillation amplitudes, and the bunch profile. The theoretical findings are compared against macro-particle simulations as well as Monte Carlo computations.



II. Theoretical description

The longitudinal Schottky spectrum is the power spectral density (PSD) of the beam current.

The intensity signal $i(t)$ of a bunch consisting of N particles is [1] :

$$i(t) = q f_0 \sum_{i=1}^{N} \sum_{n,p=-\infty}^{\infty} J_p(n\omega_0 \hat{\tau}_i)\, e^{j(n\omega_0 t + p\Omega_{s_i} t + p\varphi_{s_i})}$$

Deterministic Signal entirely defined by a set of N amplitudes τ_i and N phases $\varphi_{s,i}$.

With:

- J_p : Bessel function of order p.
- q : charge of the particle.
- $f_0 = \omega_0/2\pi$: revolution frequency.
- $\hat{\tau}_i$: synchrotron time amplitude.
- $\Omega_{s,i}$: synchrotron frequency.
- $\varphi_{s,i}$: synchrotron initial phase.

$$I(t) = q f_0 \sum_{i,n,p} J_p(n\omega_0 \hat{T}_i)\, e^{j(n\omega_0 t + p\Omega_{s_i} t + p\Phi_{s_i})}$$

Random Process defined by random variables:

$\hat{T}_i$: synchrotron time amplitude following a given pdf $g(\hat{\tau})$.

Φ_{si} : synchrotron initial phase following a uniform distribution on $[0,2\pi]$.

The **PSD** of a random process $X(t)$ is defined by:

$$P(\omega) = \lim_{T\to\infty} \frac{1}{2T} |X_T(\omega)|^2$$

where $X_T(\omega)$ is the Fourier transform of the truncated process: $X_T(\omega) = \int_{-T}^{T} X(t) e^{-j\omega t} dt$

Longitudinal Schottky spectrum (i.e. the PSD of $I(t)$)

$$P(\omega) = q^2 f_0^2 \sum_{\substack{i,n,p\\ i',p'}} J_p(n\omega_0 \hat{T}_i) J_{p'}(n\omega_0 \hat{T}_{i'}) e^{j(p\Phi_{s_i} - p'\Phi_{s_{i'}})} 2\pi\delta(n\omega_0 + p\Omega_{s_i} - \omega)\, \delta_K(p\Omega_{s_i} - p'\Omega_{s_{i'}})$$

With δ the Dirac delta and $\delta_K(x) = \lim_{T\to\infty} \frac{\sin(xT)}{xT} = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise.} \end{cases}$

$p = 0$ — Central Satellite

$$P_{(0)} = q^2 f_0^2 \sum_{i=1}^{N} \sum_{i'=1}^{N} J_0(n_h\omega_0 \hat{T}_i) J_0(n_h\omega_0 \hat{T}_{i'}) = q^2 f_0^2 \left(\sum_{i=1}^{N} Z_i\right)^2$$

with $Z_i = J_0(n_h\omega_0 \hat{T}_i)$

This equation is formally identical to the squared distance of a **1D random walk** of N steps of variable length Z_i .

Expected power and **variance** are given by:

$$\frac{E(P_{(0)})}{q^2 f_0^2} = N\beta + N(N-1)\alpha^2$$

$$\frac{Var(P_{(0)})}{q^4 f_0^4} = N\delta + (4N^2 - 4N)\gamma\alpha + (2N^2 - 3N)\beta^2 + (4N^3 - 16N^2 + 12N)\beta\alpha^2 + (-4N^3 + 10N^2 - 6N)\alpha^4$$

With α, β, γ and δ, the first four moments of Z_i.

$p \neq 0$ — Non-Central Satellites

$$P_{(p)} = q^2 f_0^2 \sum_{i,i'} J_p(n_h\omega_0 \hat{T}_i) J_p(n_h\omega_0 \hat{T}_{i'}) e^{jp(\Phi_{s_i} - \Phi_{s_{i'}})} = q^2 f_0^2 \left|\sum_i Z_{i,(p)} e^{jp\Phi_{s_i}}\right|^2$$

with $Z_{i,(p)} = J_p(n_h\omega_0 \hat{T}_i)$

This equation is formally identical to the squared distance of a **2D random walk** of N steps of variable length $Z_{i,(p)}$ and direction $p\Phi_{s,i}$.

Expected power and variance are given by: $E(P_{(p)}) = q^2 f_0^2\, N\beta_{(p)}$

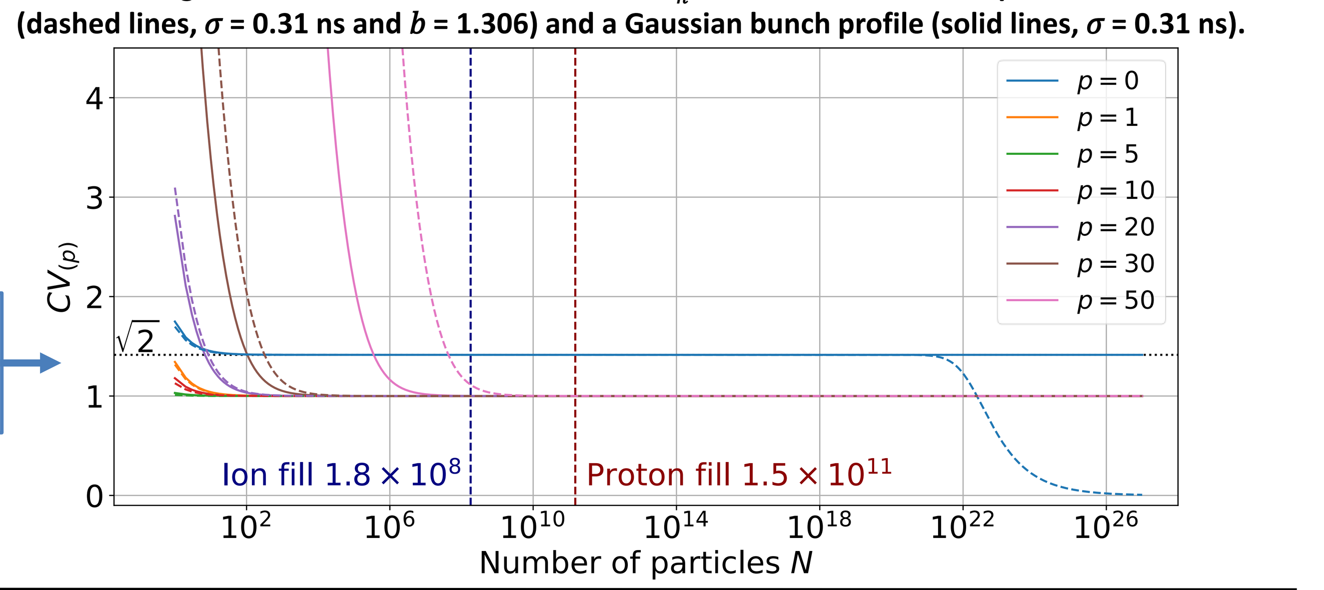
$$Var(P_{(p)}) = q^4 f_0^4 \left(N\delta_{(p)} + (N^2 - 2N)\beta_{(p)}^2\right)$$

$$CV_{(p)} = \frac{\sqrt{Var(P_{(p)})}}{E(P_{(p)})}$$

For typical LHC conditions, the coefficients of variation (CV) converge to:

- $\sqrt{2}$ for the central satellite.
- **1** for non-central satellites.

CV of the longitudinal satellites around harmonic n_h = 427 725 for a Rice amplitude distribution (dashed lines, σ = 0.31 ns and b = 1.306) and a Gaussian bunch profile (solid lines, σ = 0.31 ns).



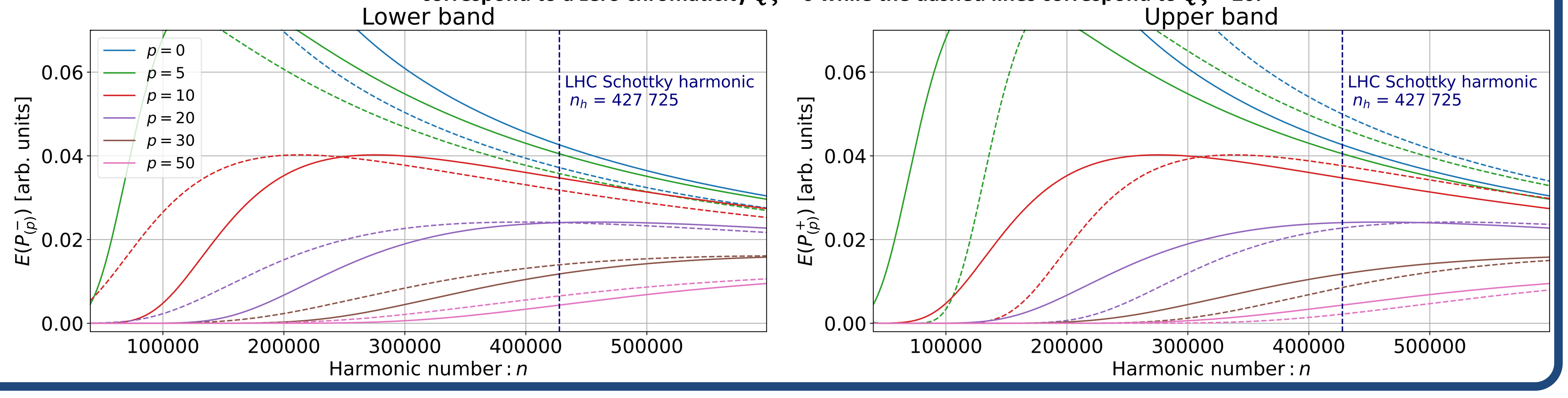
Transverse Schottky spectrum

The transverse Schottky spectrum is the power spectral density (PSD) of the dipole moment of the beam.

A similar development can be conducted for the transverse spectra. The figures on the right present how the expected power of transverse satellites vary with the harmonic number and the chromaticity.

- As the harmonic number increases, the power of lower order satellites decreases, while the opposite occurs for higher order satellites.
- The effect of chromaticity can be observed, resulting in a higher and narrower upper band compared to the lower band.

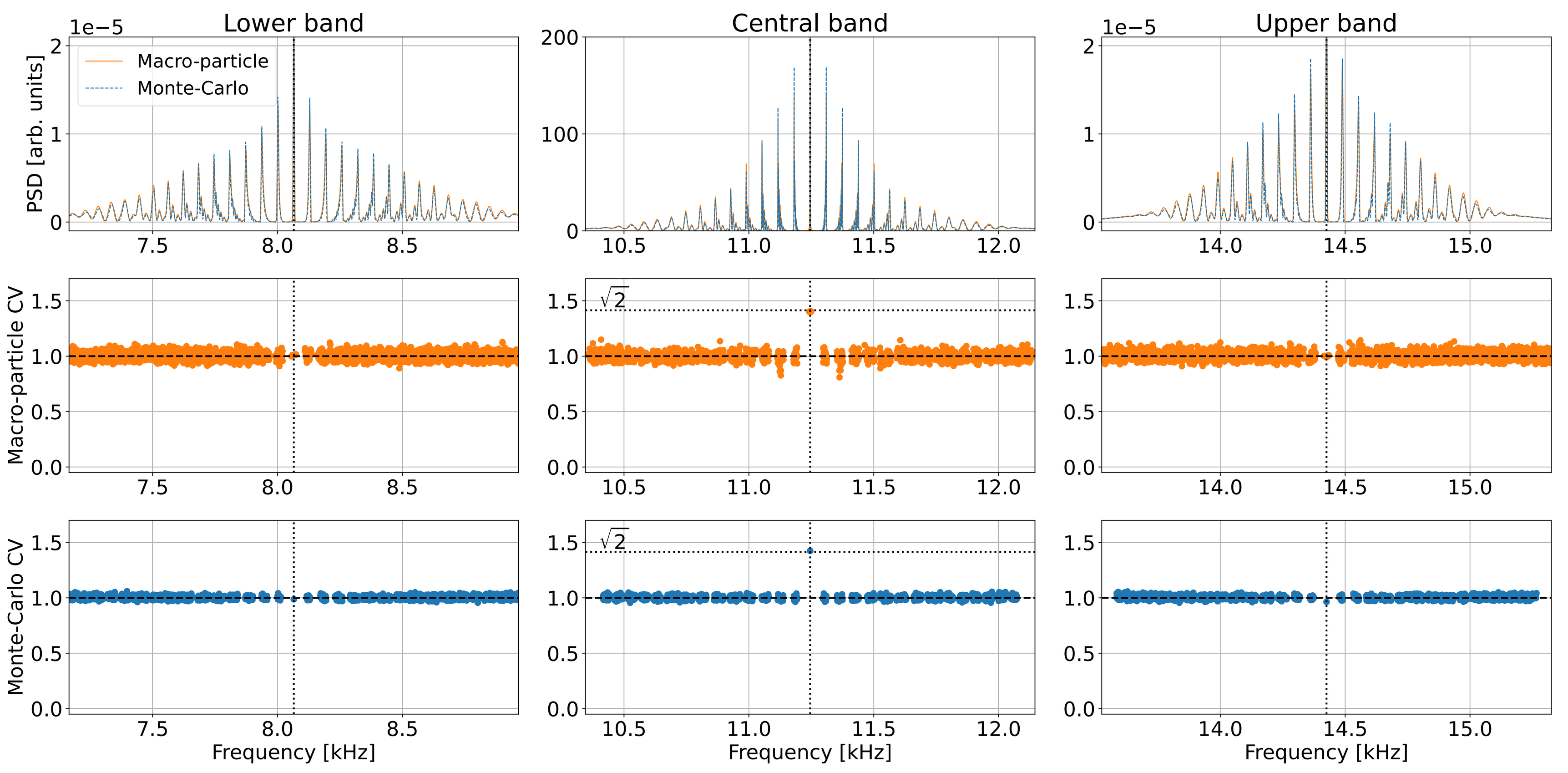
Harmonic scan of the expected powers of the transverse Bessel satellites for a Gaussian bunch profile (σ = 0.31 ns). The solid lines correspond to a zero chromaticity $Q\xi$ = 0 while the dashed lines correspond to $Q\xi$ = 20.



III. Simulations

- The theoretical equations derived in this study are compared against macro-particle [2-4] and Monte-Carlo [4, 5] simulations of a realistic LHC ion fill.
- Both simulation techniques show a good agreement with the theoretical predictions and a clear convergence to a CV of $\sqrt{2}$ and **1** can be observed.

Macro-particle and Monte-Carlo simulations of a Schottky spectrum. The average spectrum and CV are based on 1000 random draws of the bunch for the macro-particle simulation and 5000 random draws for the Monte-Carlo simulation.

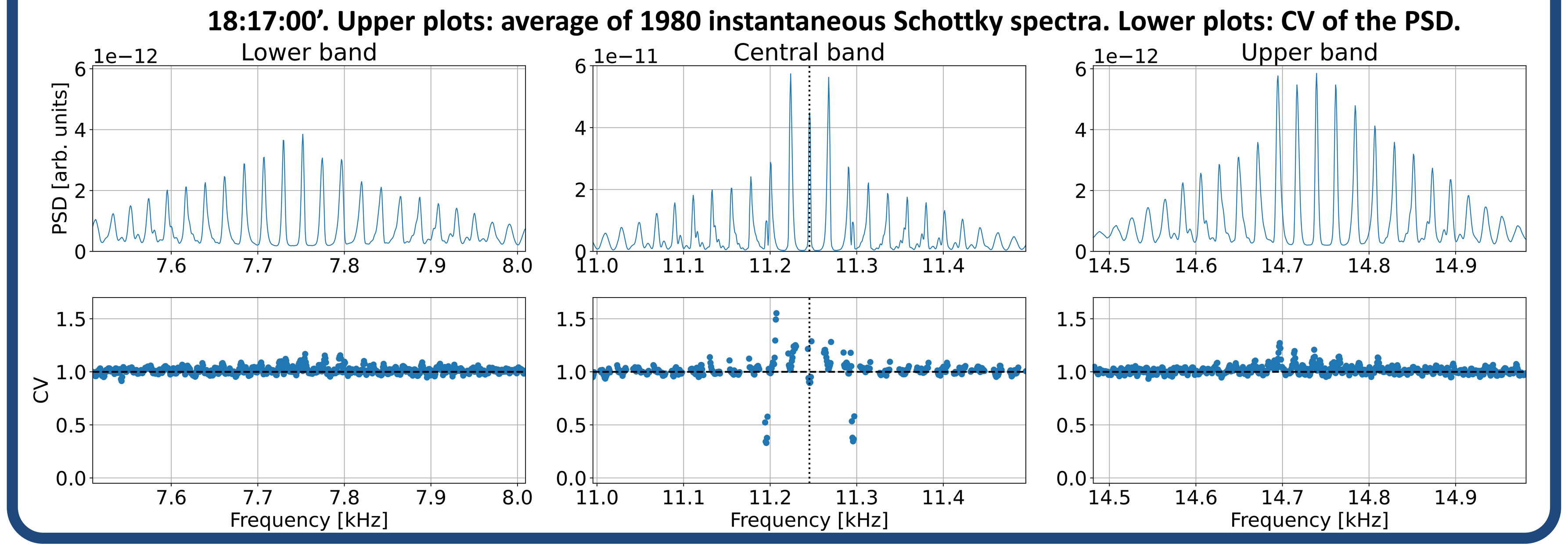


Frequencies on all the plots have been shifted from the LHC Schottky harmonic, h = 427725, to the first harmonic.

IV. Measurements

- The previously developed **theory** is studying the mean and variance **over different random draws of the bunches** (phases and amplitudes of synchrotron and betatron motions).
- In the **real machine**, there is a **unique instance of the bunch**.
- However, the **relative initial phase differences are drifting over time** due the non-linearities of the machine.
- After a given period, this unpredictable drift is equivalent to drawing new random phases, making the averaging over successive instantaneous spectra equivalent to the averaging over different random draws of the bunch.
- The same conclusion can not be drawn for the longitudinal central satellite, since the synchrotron phases do not appear in their power.
- The CV for the transverse sidebands and the non-central ($|p| > 2$) longitudinal satellites agree well with the theory.
- The **CV for $p = \pm 1$ and $p = \pm 2$ longitudinal satellites deviate from the predictions**, which suggest that these satellites are subject to effect beyond the adopted theory of Schottky spectra and **should not be used for diagnostic.**

Experimental LHC Schottky spectrum taking during fill 8412 from '2022-11-18 17:44:00' to '2022-11-18 18:17:00'. Upper plots: average of 1980 instantaneous Schottky spectra. Lower plots: CV of the PSD.



V. Conclusion

- The aim of this study was to quantify the statistical properties of Schottky spectra and reveal the dependencies on particle count, harmonic number, and oscillation amplitude distribution.
- Our analysis demonstrated the convergence of the coefficient of variation to specific, non-zero values, for different Bessel satellites.
- Our findings were validated through simulations and data from the LHC Schottky monitors and while transverse and non-central longitudinal satellites exhibit consistent convergence in the experimental data, the behaviour of the central longitudinal satellites still requires further investigation.

References

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[4] C. Lannoy, D. Alves, K. Łasocha, N. Mounet, and T. Pieloni, "LHC Schottky Spectrum from Macro-Particle Simulations," *JACoW IBIC*, vol. 2022, pp. 308–312, 2022. doi:10.18429/JACoW-IBIC2022-TUP34

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