Effect of Longitudinal Beam-coupling Impedance on the Schottky Spectrum of Bunched Beams

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I. Introduction

Schottky spectra can be strongly affected by collective effects, in particular those arising from beam-coupling impedance when large bunch charges are involved. In such conditions, the direct interpretation of the measured spectra becomes difficult, which prevents the extraction of beam and machine parameters in the same way as is usually done for lower bunch charges.

Theoretical reconstructions of Schottky spectra, such as the matrix formalism [1] or the Monte Carlo approach [2, 3] are based on the following expression of the beam current [4] and assume that the synchrotron frequency distribution is known.

$$i(t) = q f_0 \sum_{i=1}^{N} \sum_{n,p=-\infty}^{\infty} J_p(n\omega_0 \hat{\tau}_i)\, e^{j(n\omega_0 t + p\Omega_{s_i} t + p\varphi_{s_i})}$$

When the particles are moving freely in the potential well of the radio frequency (RF) bucket, an analytical relation between the amplitude of the synchrotron oscillation and its frequency can be used, allowing these methods to reconstruct the Schottky spectrum from the synchrotron amplitude distribution. However, this relation has to be modified when beam-coupling impedance affects the longitudinal dynamics.

II. Theoretical description

Synchrotron oscillation without external forces

The **equation of motion** for the RF phase ϕ of a given particle is:

$$\frac{d^2\phi}{dt^2} + \Omega_0^2 \sin\phi = 0$$

With Ω_0 **the nominal synchrotron frequency** (i.e. the limit synchrotron frequency for **synchrotron amplitude** $\hat{\phi}$ approaching zero).

(non-harmonic motion)

Exact oscillation frequency

$$\Omega_s(\hat{\phi}) = \frac{\pi}{2\mathcal{K}\left[\sin\left(\frac{\hat{\phi}}{2}\right)\right]}\Omega_0$$

(K: complete elliptic integral of the first kind)

This equation is **identical to** the one of the **non-linear physical pendulum** and **solutions and approximations of this equation exist** in the literature [5].

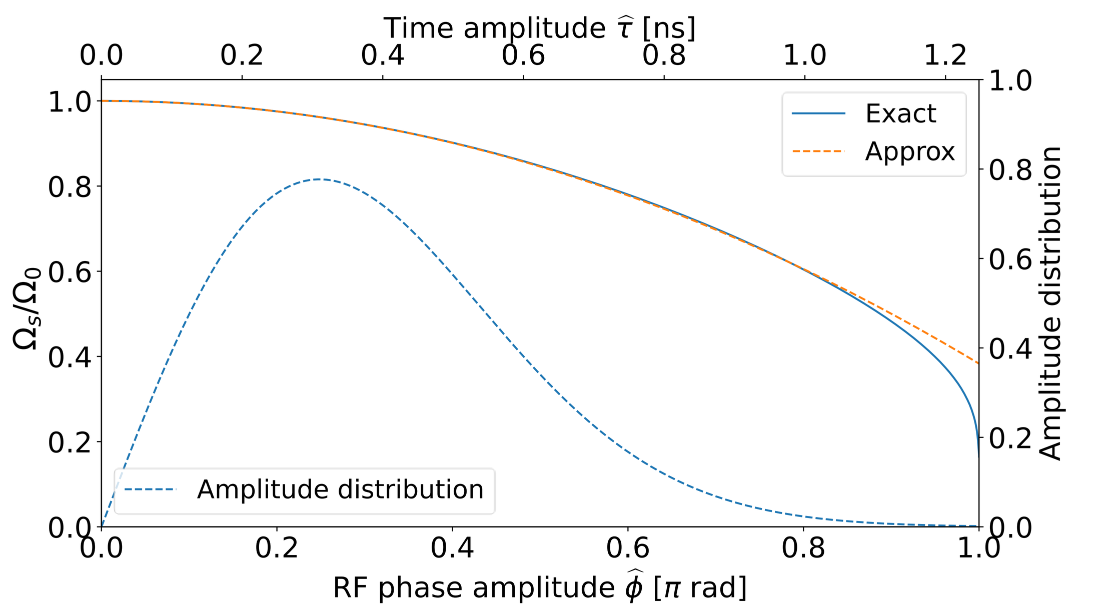
Third order approximation

$$\frac{d^2\phi}{dt^2} + \Omega_0^2\left(\phi - \frac{\phi^3}{6}\right) + O(\phi^5) = 0$$

Approx. osc. frequency

$$\Omega_s(\hat{\phi}) = \Omega_0\left(1 - \frac{\hat{\phi}^2}{16}\right)$$

Comparison between the exact and approximate expressions of the synchrotron frequency as a function of the oscillation amplitude.



Equation of motion with impedance

Additional external forces, such as the one coming from **beam-coupling impedance**, will influence the longitudinal dynamics of the particle [6].

➔ With additional forces, the previous equation of motion becomes:

$$\ddot{\phi} + \Omega_0^2 \sin\phi = \frac{\eta h \omega_0}{p_0} \sum_i F_i(t)$$

Beam-coupling impedance

$$F_{Imp}(t) = e\left[\vec{E} + \vec{\beta}c \times \vec{B}\right]_{\parallel}(t, z = c\tau(t))$$

$$= \frac{-Ne^2}{2\pi C}\int_{-\infty}^{\infty} Z_{\parallel}(\omega)\hat{\Lambda}(\omega) e^{j\frac{\omega z}{c}} d\omega$$

$$= \frac{-Ie}{C}\sum_{p=-\infty}^{\infty} Z_{\parallel}(p)\hat{\lambda}(p) e^{jp\omega_0\tau(t)}$$

Beam spectrum:

$$\hat{\Lambda}(\omega) = \mathcal{F}\{\lambda(t) * \text{Ш}_{T_0}(t)\} = \mathcal{F}\{\lambda(t)\}\,\mathcal{F}\{\text{Ш}_{T_0}(t)\} = \hat{\lambda}(\omega)\omega_0 \text{Ш}_{\omega_0}(\omega)$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\phi^{2n+1} \qquad \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{jp}{h}\right)^n \phi^n$$

$$\ddot{\phi} + \Omega_0^2 \sin\phi = \Omega_0^2 \frac{I}{\hat{V}\cos\phi_s}\sum_{p=-\infty}^{\infty} Z_{\parallel}(p)\hat{\lambda}(p) e^{j\frac{p}{h}\phi}$$

Expanding the sine and exponential function with their Maclaurin series.

The idea is that, for small oscillation amplitudes, only the first order terms can be kept, while for larger amplitudes, higher order terms can be taken into account.

(1)
$$\ddot{\phi} + \Omega_0^2 \sum_{n=0}^{\infty} S_n \phi^n = 0$$

General equation of motion with impedance

With the coefficients:

$$S_n = \begin{cases} -Z_n & : n \text{ even} \\ \frac{j^{n-1}}{n!} - Z_n & : n \text{ odd} \end{cases}$$

$$Z_n = \frac{I}{\hat{V}\cos\phi_s}\sum_{p=-\infty}^{\infty} Z_{\parallel}(p)\hat{\lambda}(p)\frac{1}{n!}\left(\frac{jp}{h}\right)^n$$

With the notations:

- $\mathcal{F}$: Fourier transform.
- η : slippage factor.
- p_0 : reference momentum.
- h : RF harmonic number.
- $\lambda(t)$: bunch profile.
- Ш_{T_0} : Dirac comb of period T_0.
- $T_0 = 2\pi/\omega_0$: revolution period.
- C : accelerator circumference.
- $Z_{\parallel}(\omega)$: longitudinal impedance.
- N : number of particle in the bunch.
- e : elementary charge.
- $I = Ne/T_0$: bunch current.
- τ : time arrival difference between a given particle and the synchronous particle.

Longitudinal broad-band resonator impedance

- The developed theory will be applied to the particular case of a longitudinal **broad-band resonator**.
- The **even terms in Eq. (1)** are responsible for the synchronous phase shift and it can be shown that, in the particular case of a broad-band resonator, their contribution **can be neglected**.

Expanding Eq. (1) up to the third order gives:

$$\ddot{\phi} + \Omega_0^2\left(S_1\phi + S_3\phi^3\right) + O(\phi^5) = 0$$

Approx. osc. frequency

$$\Omega_s(\hat{\phi}) = \Omega_0\sqrt{S_1}\left(1 + \frac{3S_3}{8S_1}\hat{\phi}^2\right) \quad (2)$$

Broad-band resonator:

$$Z_{\parallel}^{BB}(\omega) = \frac{R_{\parallel}}{1 - jQ\left(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r}\right)}$$

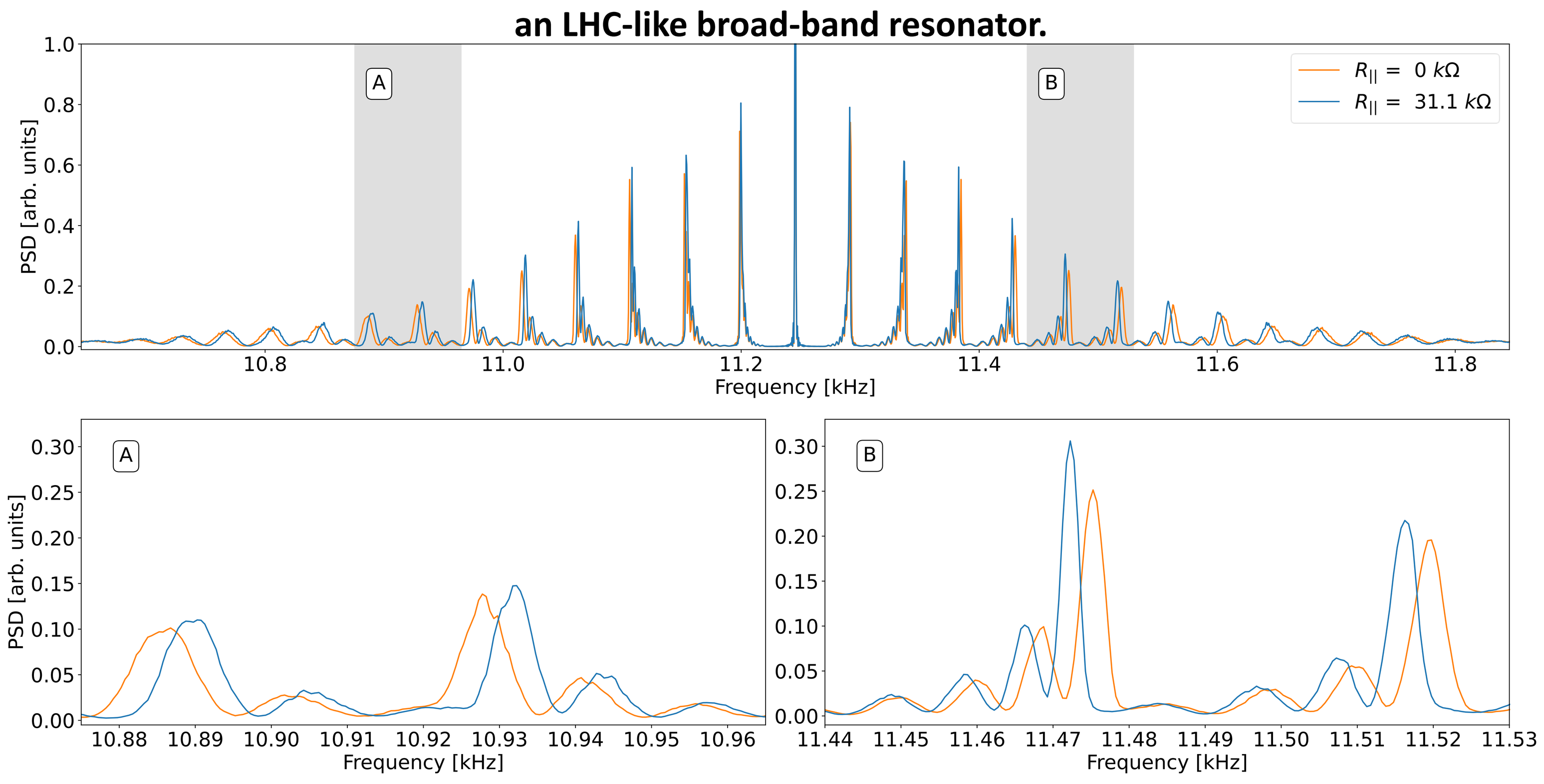
With:

- $R_{\parallel}$: shunt impedance.
- ω_r : cut-off frequency.
- Q : quality factor.

III. Simulations

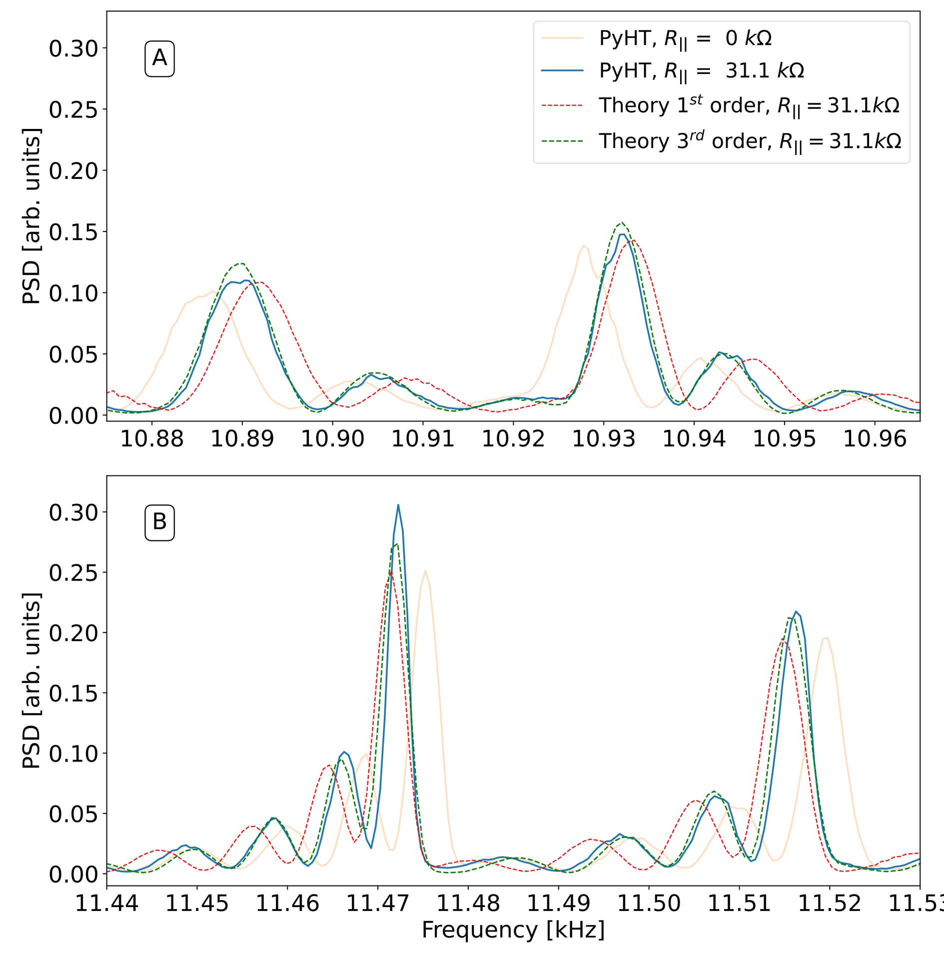
- The simulation is conducted with PyHEADTAIL [7] and aims to reproduce the typical conditions of an **LHC proton fill at injection**.
- The value of the parameters chosen for the broad-band resonator correspond to a **significant part of the impedance in the LHC** that can be modelled as a **broad-band** resonator**.**
- The following effects of the broad-band resonator can be observed:
  - **Shift of the nominal synchrotron frequency**. All the satellites converge toward the central one. This shift is due to the term S_1 in Eq. (2) and the new nominal synchrotron frequency is $\Omega_0\sqrt{S_1}$.
  - The broad-band resonator will reduce the nominal synchrotron frequency for a machine operating above transition.
  - **Amplitude dependent synchrotron frequency shift** due to the higher order terms S_{2n+1}, $n \geq 1$.

Simulated longitudinal Schottky spectra with (blue) and without (orange) an LHC-like broad-band resonator.



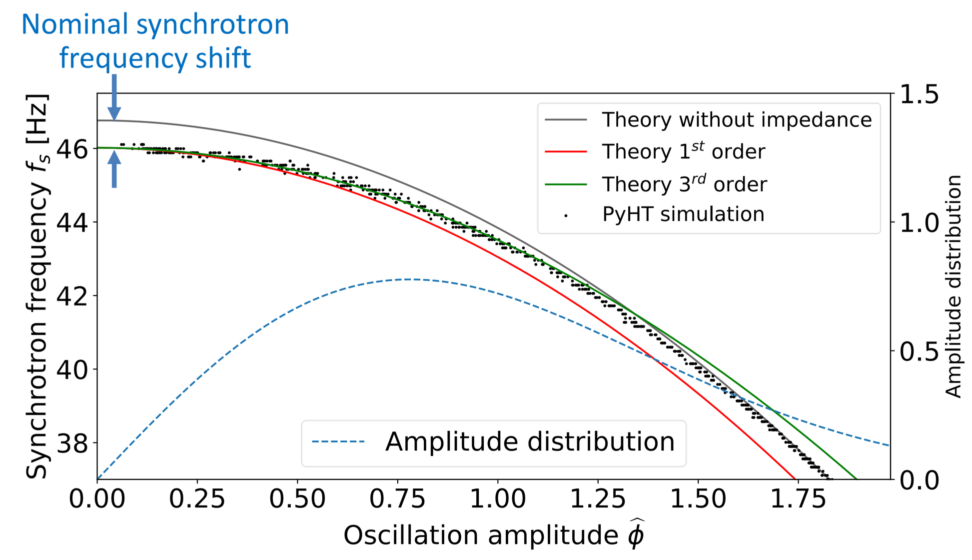
Frequencies on all the plots have been shifted from the LHC Schottky harmonic, $h = 427725$, to the first harmonic.

Comparison of the macro-particle simulation (blue) against the adapted matrix formalism, with Eq. (2) including impedance terms Z_n up to the first (red) and third (green) order.



- The macro-particle simulation is compared against the theoretical matrix formalism, where the relation between synchrotron amplitudes and frequencies has been generalized with Eq. (2) to include impedance effects.
- The nominal synchrotron frequency shift is well reproduced by the 1st order theory, while the shift for non-zero amplitude particles, requires higher order terms (Z_n).
- With the third order impedance term, the theory is in good agreement agreement with the simulation, as it also includes the amplitude dependent synchrotron frequency shift.

Comparison of Eq. (2) including impedance terms Z_n up to the first (red) and third (green) order, against macro-particle simulation (black dots).



- The validity of Eq. (2) can also be probed by extracting the relation $\Omega s(\hat{\phi})$ from the macro-particle simulation.
- On the above figure, each black dot corresponds to the synchrotron amplitude and frequency of a given macro-particle.
- The nominal synchrotron frequency shift is well reproduced by the first order term Z_1
- The shift for larger amplitude particles is correct with the third order theory, up to a certain amplitude $\hat{\phi} \sim 1.25$ rad.
- In order to extend the region where Eq. (2) is valid, one would need to take into account the fifth order term S_5, in the equation of motion.
- However, this is not crucial since the majority of the particles are well described, as can be seen from the amplitude distribution.

V. Conclusion

- The aim of this study was to explore the effects of impedance on the Schottky spectrum.
- The longitudinal equation of motion was generalized to include the forces coming from any impedance $Z(\omega)$, and the case of a broad-band resonator was studied in more detail.
- The relation between synchrotron amplitude and frequency was generalized to include the effect of a broad-band resonator, allowing existing theoretical reconstruction methods of Schottky spectra to include impedance effects.
- The developed theory was shown to be in good agreement with macro-particle simulations, by correctly reproducing the amplitude dependent synchrotron tune shift, and the sub-structure of the Schottky spectrum satellites.

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