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CLOSED FORM FORMULAS FOR THE INDIRECT SPACE CHARGE WAKE FUNCTION OF AXISYMMETRIC STRUCTURES

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Abstract

Indirect space charge contributes significantly to the impedance of non-ultrarelativistic machines such as the LEIR, PSB and PS at CERN. While general expressions exist in frequency domain for the beam coupling impedance, the time domain wake function is typically obtained numerically, thanks to an inverse Fourier transform. An analytical expression for the indirect space charge wake function, including the time dependence as a function of particle velocity, is nevertheless highly desirable to improve the accuracy of time domain beam dynamics simulations of coherent instabilities. In this work, a general formula for the indirect space charge wake function is derived from the residue theorem. Moreover, simple approximated expressions reproducing the time and velocity dependence are also provided, which can even be corrected to recover an exact formula, thanks to a numerical factor computed once for all. The expressions obtained are successfully benchmarked with a purely numerical approach based on the Fourier transform.

INTRODUCTION

In high intensity synchrotrons, electromagnetic fields created by the beam passage through its interaction with its surroundings, is one of the main collective effect that can limit the performance of the machine, potentially leading to detrimental consequences such as heat load, emittance growth or, in the most dramatic cases, beam instabilities. Its role was pointed out as early as in 1965 in the seminal work of Laslett et al [1], and the concept of beam coupling impedance introduced slightly later by Sessler and Vaccaro [2]. The impedance, defined as the Fourier transform of the integrated force, felt by a witness (trailing) charge as a consequence of the interaction of a source (leading) charge with a certain accelerator equipment, normalised to the excitation, proved to be a very useful descriptor of the magnitude of such effects in a given machine. In particular, it can be efficiently used in Vlasov equation solvers to compute and predict instabilities.

Over the past fifty years many analytical formulas have been derived to provide such impedances, in particular in the case of a smooth, axisymmetric structure such as a vacuum pipe, made of one or several layers of materials in the radial direction [3–15]. On the other hand, the equivalent time domain quantity, namely the wake function [16][1], remains elusive, at least in the form of analytic formulas: for cylindrical resistive geometries one can mention the famous thick-wall formula [5], its extension to non-ultrarelativistic beams [17], as well as a thin-wall formula [18].

As a matter of fact, even the simplest case of a perfectly conductive, cylindrical beam pipe, often called the indirect space charge (ISC) impedance, although well-known in frequency domain [8] still lacks a formula for its wake function—one can mention the ultrarelativistic expression by Chao [5], which is of limited practical interest as it is expressed as a Dirac delta function in the longitudinal coordinate along the bunch z. A more general wake function formula would nevertheless be useful when performing macroparticle simulations. In particular, in low-energy machines, the ISC is strong and its broad-band nature makes it very peaked close to $z = 0$ (the position of the source particle creating the wake), which in turn requires a very fine discretization along z, leading to time-consuming simulations. A simple analytical formula would therefore be highly beneficial to macroparticle simulations.

The ISC impedance is traditionally separated from the direct space charge (i.e. the impedance in free space), the latter being fundamentally of non-linear nature. In an axisymmetric structure, the longitudinal and transverse dipolar ISC impedances can be expressed respectively as [8]:

$$Z_\parallel(\omega) = \frac{i\omega\mu_0 L}{2\pi\beta^2\gamma^2}\frac{K_0\left(\frac{kb}{\gamma}\right)}{I_0\left(\frac{kb}{\gamma}\right)}, Z_\perp^{dip}(\omega) = \frac{ik^2 Z_0 L}{4\pi\beta\gamma^4}\frac{K_1\left(\frac{kb}{\gamma}\right)}{I_1\left(\frac{kb}{\gamma}\right)}, \tag{1}$$

with i the imaginary unit, b the radius of the pipe, L its length, $\omega > 0$ the angular frequency (in rad/s), μ_0 the vacuum permeability, $Z_0 = \mu_0 c$ the free space impedance, c the speed of light in vacuum, $v = \beta c$ the beam velocity, γ the relativistic mass factor, $k \equiv \omega/v$ the wave number, and I_0, I_1, K_0 and K_1 modified Bessel functions of the first and second kinds. Note that SI units are used throughout these proceedings.

In the following sections we will provide expressions for the ISC wake functions, defined as Fourier integrals:

$$W_\parallel(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega e^{i\omega\frac{z}{v}} Z_\parallel(\omega), \tag{2}$$

$$W_\perp^{dip}(z) = -\frac{i}{2\pi}\int_{-\infty}^{+\infty} d\omega e^{i\omega\frac{z}{v}} Z_\perp^{dip}(\omega). \tag{3}$$

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[1] The wake function should not to be confused with the so-called wake potential that represents the convolution of the wake function by the longitudinal bunch distribution. In early works (and in particular in Ref. [16]), the wake function was actually often called "wake potential".

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We will provide first an exact formalism to compute the ISC wake functions in the longitudinal and transverse planes. Then, a numerical method will be briefly described and used to benchmark the obtained formulas. This will be followed by the description of a simplified approach in the transverse case, providing a more handy formula that can even be corrected to recover exact results. Finally, we will give our concluding remarks.

EXACT ANALYTICAL APPROACH

In this section we first outline the formalism used to compute Fourier integrals, before applying it to the specific case of the ISC.

Computation of Fourier Integrals With the Residue Theorem

Given a function $f(\omega)$ defined in the complex plane, its Fourier integral at a time $\tau > 0$ can be expressed as

$$\int_{-\infty}^{\infty} e^{i\omega\tau} f(\omega)\mathrm{d}\omega = \lim_{R\to\infty} \int_{-R}^{R} e^{i\omega\tau} f(\omega)\mathrm{d}\omega$$
$$= \lim_{R\to\infty} \left(\oint_{C_1+C_2} e^{i\omega\tau} f(\omega)\mathrm{d}\omega - \int_{C_2} e^{i\omega\tau} f(\omega)\mathrm{d}\omega \right), \quad (4)$$

with C_1 the path along the real axis going from $-R$ to R, and C_2 the counter-clockwise half-circle from R to $-R$, in the positive imaginary part of the complex plane, as illustrated in Fig. 1. By virtue of Jordan's lemma, the second integral vanishes when R goes to infinity, as long as the maximum of f on the half-circle C_2 goes to zero at the limit in R, in other words when

$$\lim_{R\to\infty} \left[\max_{0\leq\theta\leq\pi} f\left(Re^{i\theta}\right) \right] = 0. \quad (5)$$

In Eq. (4), the integral on the closed contour $C_1 + C_2$ can be

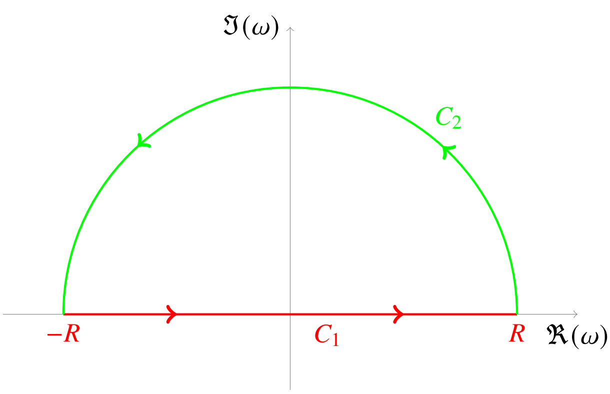

Figure 1: Sketch of the integration path in the complex plane. Arrows indicate the direction of integration.

expressed using Cauchy's residue theorem when f is meromorphic on the upper complex half plane (i.e. holomorphic except for a set of isolated points). Assuming also that all the singularities of f are simple poles, we get

$$\int_{-\infty}^{\infty} e^{i\omega\tau} f(\omega)\mathrm{d}\omega = 2\pi i \sum_k e^{i\omega_k\tau} \mathrm{Res}\left[f, \omega_k\right], \quad (6)$$

where $\mathrm{Res}\left[f, \omega_k\right]$ is the value of the residue of f at the pole ω_k, and the sum runs over all poles in the upper half-plane.

Note that when $\tau < 0$, the same reasoning can be performed using instead a half-circle contour lying in the lower half-plane of $\mathbb{C}$, but since the direction of integration of $C_1 + C_2$ becomes clockwise, a minus sign appears in front of the residue sum in Eq. (6).

Case of the ISC

The longitudinal and transverse ISC impedances are defined by Eq. (1) for $\omega \geq 0$. For $\omega < 0$, the impedances are obtained from the following symmetry relations (which ensure that the wake functions remain real quantities)

$$Z_{\parallel}(\omega) = Z_{\parallel}(-\omega)^*, \quad Z_{\perp}^{dip}(\omega) = -Z_{\perp}^{dip}(-\omega)^*, \quad (7)$$

where Z^* indicates the complex conjugate of Z.

These relations do not give the same results as Eq. (1), had the latter be applied to a negative ω, because these expressions do not obey to the same symmetry relations. In particular, in the left half-plane the modified Bessel functions K_m ($m = 0$ or 1) are obtained from their values on the right half-plane by analytic continuation [19], and a branch cut appears, which has to remain out of the integration contour—we choose it along the negative imaginary semi-axis when $z > 0$ and the positive one for $z < 0$. From these considerations we can write, for $\Re(x) < 0$:

$$K_0(x) = K_0(-x) - \mathrm{sgn}(z) i\pi I_0(-x), \qquad I_0(x) = I_0(-x),$$
$$K_1(x) = -K_1(-x) - \mathrm{sgn}(z) i\pi I_1(-x), \qquad I_1(x) = -I_1(-x), \quad (8)$$

where sgn(z) denotes the sign of z. Therefore, for $\omega < 0$ the longitudinal and transverse ISC can be expressed respectively as

$$Z_{\parallel}(\omega) = iA\omega \frac{K_0\left(\frac{kb}{\gamma}\right)}{I_0\left(\frac{kb}{\gamma}\right)} - \mathrm{sgn}(z)\pi A\omega, \quad (9)$$

$$Z_{\perp}^{dip}(\omega) = iB\omega^2 \frac{K_1\left(\frac{kb}{\gamma}\right)}{I_1\left(\frac{kb}{\gamma}\right)} + \mathrm{sgn}(z)\pi B\omega^2, \quad (10)$$

with $A \equiv \frac{\mu_0 L}{2\pi\beta^2\gamma^2}$ and $B \equiv \frac{Z_0 L}{4\pi\beta^3 c^2\gamma^4}$. Hence, for the longitudinal wake function we get, with the additional term of Eq. (9):

$$W_{\parallel}(z) = \frac{iA}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega e^{i\omega\frac{z}{v}} \omega \frac{K_0\left(\frac{kb}{\gamma}\right)}{I_0\left(\frac{kb}{\gamma}\right)} - \frac{\mathrm{sgn}(z)A}{2} \int_{-\infty}^{0} \mathrm{d}\omega e^{i\omega\frac{z}{v}} \omega.$$

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The function f defined by $f(\omega)=\omega\frac{K_0\left(\frac{kb}{\gamma}\right)}{I_0\left(\frac{kb}{\gamma}\right)}$ fulfils Jordan's condition from Eq. (5) in the upper-right quadrant (from the asymptotic expansion of the Bessel functions [19]), but not in the upper-left quadrant. Using again Eq. (8), the corresponding first term of f does fulfil Jordan's condition but not the term proportional to ω, leading to an integral over a quarter circle along C_2 which remains non-zero. The latter can be combined with the semi-infinite integral above:

$$\lim_{R\to\infty}\left(\int_{\mathrm{sgn}(z)iR}^{-R}\mathrm{d}\omega e^{i\omega\frac{z}{v}}\omega\right)+\int_{-\infty}^{0}\mathrm{d}\omega e^{i\omega\frac{z}{v}}\omega = \int_{\mathrm{sgn}(z)i\infty}^{0}\mathrm{d}\omega e^{i\omega\frac{z}{v}}\omega=\frac{v^2}{z^2},$$

applying the residue theorem to the function $\omega\longrightarrow\omega e^{i\omega\frac{z}{v}}$ which exhibits no pole. Using then Eq. (6) we finally obtain

$$W_{\parallel}(z)=-A\mathrm{sgn}(z)\sum_k e^{i\omega_k\frac{z}{v}}\mathrm{Res}\left[f,\omega_k\right]-\frac{\mathrm{sgn}(z)Av^2}{2z^2}. \tag{11}$$

The function f is meromorphic except for the branch cut of K_0, which touches the contour of integration only at $\omega=0$, but can be bypassed by modifying slightly the contour around the origin thanks to a half-circle of vanishing radius. The corresponding contribution to the integral vanishes as well, as $\omega=0$ is not a singularity of f, which can be readily seen by noticing that [19] $f(\omega)\sim-\omega\ln(\omega)\longrightarrow_{\omega\to0}0$.

The poles of f are given by the zeros of I_0, which are all simple [19] and correspond to the real zeros $j_{0,k}>0$ of J_0 (Bessel function of the first kind) through

$$\omega_k=\frac{ij_{0,k}\gamma v}{b}\quad\text{for } z>0 \text{ (poles in upper half-plane)},$$
$$\omega_k=\frac{-ij_{0,k}\gamma v}{b}\quad\text{for } z<0 \text{ (poles in lower half-plane)}, \tag{12}$$

from $I_0(x)=J_0(ix)$. Since the poles are simple, the residue at ω_k can be obtained from L'Hôpital's rule:

$$\mathrm{Res}[f,\omega_k]=\frac{\mathrm{sgn}(z)ij_{0,k}\gamma v}{b}\frac{K_0\left(\mathrm{sgn}(z)ij_{0,k}\right)}{\frac{b}{\gamma v}I_0'\left(\mathrm{sgn}(z)ij_{0,k}\right)} = \frac{j_{0,k}\gamma^2v^2}{b^2}\frac{K_0\left(\mathrm{sgn}(z)ij_{0,k}\right)}{J_1\left(j_{0,k}\right)}, \tag{13}$$

where we have used that $I_0'(\pm ix)=I_1(\pm ix)=\pm iJ_1(x)$ [19]. Combining Eqs. (11), (12) and (13) we get the total longitudinal wake:

$$W_{\parallel}(z)=-\frac{\mathrm{sgn}(z)L}{4\pi\varepsilon_0\gamma^2}\frac{1}{z^2} -\frac{\mathrm{sgn}(z)L}{2\pi\varepsilon_0b^2}\sum_{k=1}^{\infty}e^{-\frac{j_{0,k}\gamma|z|}{b}}\frac{j_{0,k}K_0\left(\mathrm{sgn}(z)ij_{0,k}\right)}{J_1\left(j_{0,k}\right)}. \tag{14}$$

with $\varepsilon_0\equiv\frac{1}{\mu_0c^2}$ the vacuum permittivity.

In transverse, a very similar reasoning (in particular regarding the additional integral from the partly unfulfilled Jordan's condition) gives

$$W_{\perp}^{dip}(z)=iB\sum_k e^{i\omega_k\frac{z}{v}}\mathrm{Res}\left[g,\omega_k\right]+\frac{\mathrm{sgn}(z)Bv^3}{z^3}, \tag{15}$$

with g the function defined by $g(\omega)=\omega^2\frac{K_1\left(\frac{kb}{\gamma}\right)}{I_1\left(\frac{kb}{\gamma}\right)}$. As f defined above, g does not have a singularity in 0 but a well-defined limit $\frac{2v^2\gamma^2}{b^2}$, and exhibits a set of simple poles obtained from the zeros $j_{1,k}>0$ of J_1 (using the relation $I_1(x)=-iJ_1(ix)$):

$$\omega_k=\frac{i\mathrm{sgn}(z)j_{1,k}\gamma v}{b}. \tag{16}$$

The residue sum is then obtained again from L'Hôpital's rule, and the total transverse wake reads

$$W_{\perp}^{dip}(z)=\frac{\mathrm{sgn}(z)L}{4\pi\varepsilon_0\gamma^4}\frac{1}{z^3} -\frac{i\mathrm{sgn}(z)L}{2\pi\varepsilon_0\gamma b^3}\sum_{k=1}^{\infty}e^{-\frac{j_{1,k}\gamma|z|}{b}}\frac{j_{1,k}^2K_1\left(\mathrm{sgn}(z)ij_{1,k}\right)}{J_0\left(j_{1,k}\right)-J_2\left(j_{1,k}\right)}, \tag{17}$$

where we used $2I_1'(\pm ix)=J_0(x)-J_2(x)$.

NUMERICAL BENCHMARK

To compute wake functions when analytical formulas cannot be found, one typically resorts to a numerical integration, using discrete, fast Fourier transforms. In the case of smooth, slowly decaying beam coupling impedances, this can lead to the need of a large number of points, difficult or even impossible to handle [15], related to the number of decades to be meshed with evenly-spaced frequencies.

Another approach is preferred here: we use an uneven frequency sampling, combined with an exact integration of a linear interpolation of the impedance, over each sub-interval. The latter aspect was inspired by a method first found by Filon in 1928 [20], and extended later in various works [21–25]. The full method we use here is extensively described in Ref. [15].

We consider a Fourier integral of the form

$$I(\tau)=\int_{\omega_{min}}^{\infty}\mathrm{d}\omega e^{i\omega\tau}f(\omega). \tag{18}$$

Wake functions given by Eqs. (2) or (3) can be easily cast into this semi-infinite form (with $\omega_{min}=0$), using the symmetry properties of the impedances given in Eq. (7).

We first set ω_{max} high enough for the f function to be small and decaying for $\omega>\omega_{max}$, and we cut the interval $[\omega_{min},\omega_{max}]$ into several sub-intervals, not necessarily equidistant, delimited by the angular frequencies ω_j with j from 0 to N ($\omega_0=\omega_{min}$ and $\omega_N=\omega_{max}$). Then

$$f(\omega)\approx p_j(\omega)\quad\text{for } \omega_j\le\omega\le\omega_{j+1}, \tag{19}$$

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where p_j is the interpolating (linear) polynomial on the interval $[\omega_j, \omega_{j+1}]$. The integral $I(\tau)$ is then obtained from

$$I(\tau) = \sum_{j=0}^{N-1} \int_{\omega_j}^{\omega_{j+1}} \mathrm{d}\omega e^{i\omega\tau} f(\omega) + \int_{\omega_{max}}^{\infty} \mathrm{d}\omega e^{i\omega\tau} f(\omega). \tag{20}$$

The second integral is approximated by assuming that $f(\omega) \approx f(\omega_{max})$ for $\omega > \omega_{max}$, while the first one is computed using Filon's method on each sub-interval:

$$\begin{aligned} I(\tau) &\approx \sum_{j=0}^{N-1} \int_{\omega_j}^{\omega_{j+1}} \mathrm{d}\omega e^{i\omega\tau} p_j(\omega) + e^{i\omega_{max}\tau} \frac{i f(\omega_{max})}{\tau} \\ &\approx \sum_{j=0}^{N-1} \Delta_j \left[f(\omega_j) e^{i\omega_{j+1}\tau} \Lambda(-\Delta_j \tau) \right. \\ &\left. + f(\omega_{j+1}) e^{i\omega_j \tau} \Lambda(\Delta_j \tau) \right] + e^{i\omega_{max}\tau} \frac{i f(\omega_{max})}{\tau}, \end{aligned} \tag{21}$$

with $\Delta_j \equiv \omega_{j+1} - \omega_j$ and Λ defined by

$$\Lambda(x) = -\frac{i e^{ix}}{x} + \frac{e^{ix} - 1}{x^2}. \tag{22}$$

Note that the Taylor expansion of Λ is often useful to compute values for small arguments:

$$\Lambda(x) = \sum_{n=0}^{\infty} \frac{1}{n+2} \frac{(ix)^n}{n!}. \tag{23}$$

Extensions of the method with cubic interpolation, or using a higher order Taylor expansion of f for $\omega > \omega_{max}$, are described in Ref. [15]. An automatic refinement procedure to achieve convergence vs. frequency sampling, was also implemented and used for various kinds of impedances.

The numerical approach just outlined is compared to the exact formulas derived above in Figs. 2 and 3, which reveals the excellent agreement between the two methods.

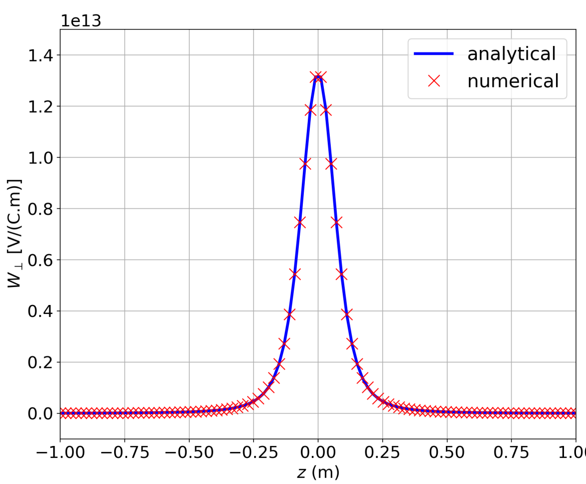

Figure 2: Comparison between the exact transverse ISC wake function from Eq. (17) (solid blue line) and the numerical approach (red crosses) ($L = 1$ m, $b = 0.08$ m, $\gamma = 1.05$).

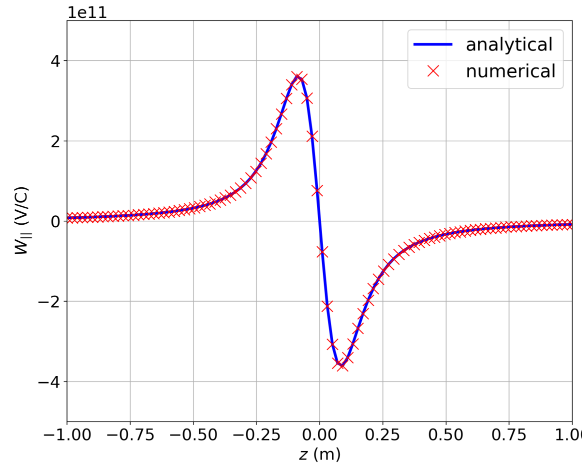

Figure 3: Comparison between the exact longitudinal ISC wake function from Eq. (14) (solid blue line) and the numerical approach (red crosses) ($L = 1$ m, $b = 0.08$ m, $\gamma = 1.05$).

SIMPLIFIED EXPRESSION FOR THE TRANSVERSE WAKE

In this section an approximated formula for the transverse ISC is presented. We will then recover the exact result with a simple correction scheme.

Approximate Formula

To approximate the transverse ISC impedance given in Eq. (1), one could use the approximation for small arguments of both modified Bessel function, namely $K_1(x) \approx 1/x$ and $I_1(x) \approx x/2$ [19]. This would lead to an expression without dependency on frequency, which would be useless to compute the ISC wake function. A useful expression can rather be obtained using solely the approximation for a small argument of the modified Bessel function I_1. In this case the transverse impedance can be rewritten as follows:

$$Z_\perp^{dip}(\omega) \approx \frac{ikZ_0 L\, K_1\left(\frac{kb}{\gamma}\right)}{2\pi b\beta\gamma^3}. \tag{24}$$

The Fourier transform of Eq. (24) can be performed analytically, which gives a useful approximation of the transverse ISC wake function:

$$W_\perp^{dip}(z) \approx \frac{Z_0 L}{4\pi\beta\gamma^4 \left(z^2 + \frac{b^2}{\gamma^2}\right)^{3/2}}. \tag{25}$$

In Fig. 4, a comparison is performed between the approximated wake function and the exact analytical formula from the previous section, showing that the approximated formula describes, with a reasonable degree of accuracy, the dependency of the transverse wake function with the relativistic factor.

Note that an improved formula can be obtained by combining the approximation for a small argument of the modified Bessel function I_1 with the approximation $I_1(x) \approx \frac{1}{2K_1(x)}$ [26].

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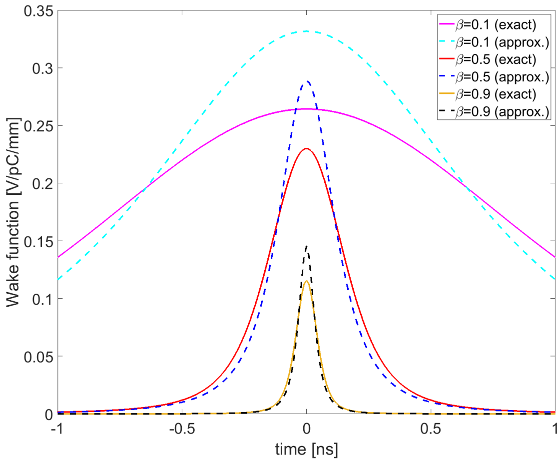

Figure 4: Comparison between the approximated wake function formula from Eq. (25) (dashed lines) and the exact one from Eq. (17) (full lines), for a chamber with radius $b = 0.03$ m and length $L = 1$ m, for various values of the relativistic factor β.

Correcting Factor

The ratio between the exact and the approximated formulas (which we will call hereafter the "discrepancy function") turns out to depend only on τa with $a = b/(v\gamma)$ and $\tau = z/v$. A graphical representation of this function for two different values of a is displayed in Fig. 5.

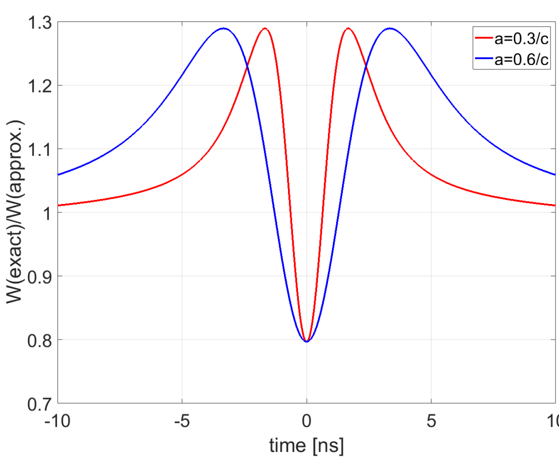

Figure 5: Discrepancy function between the exact and approximated formulas (Eqs. (17) and (25) respectively), for two different values of $a = b/(v\gamma)$.

Using this peculiarity we can obtain an exact formula for the ISC wake function, which will require the computation of a single reference discrepancy function. The generalized, corrected formula using the discrepancy function can be written as follows:

$$W_{\perp}^{dip}(\tau) = F(\tau_n)\frac{Z_0 L}{2\pi c^2 \beta^3 \gamma^4 (\tau^2 + a^2)^{3/2}}, \tag{26}$$

where $F(\tau_n)$ is the discrepancy function with $\tau_n = \tau a_{ref}/a$ and $a_{ref} = b_{ref}/(c\beta_{ref}\gamma_{ref})$. Therefore, computing once for all $F(\tau_n)$, it is possible to scale appropriately $W_{\perp}^{dip}$ for any value of a. The equation has been successfully benchmarked with the exact approach for various values of the relativistic factor β (see Fig. 6).

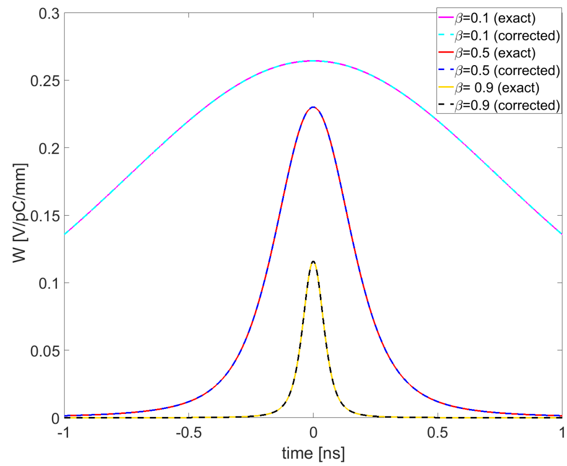

Figure 6: Comparison between the corrected wake function formula from Eq. (26) (dashed lines) and the exact one from Eq. (17) (full lines) for various values of the relativistic factor β.

CONCLUSION

General formulas for the indirect longitudinal and transverse space charge wake functions have been obtained from an exact analytical approach. Moreover, a simpler expression reproducing the time and velocity dependence has been found for the transverse wake, and can even be corrected to recover an exact formula, thanks to a numerical function that can be calculated once for all. The expressions obtained have been successfully benchmarked with a purely numerical approach based on an uneven frequency sampling and a Filon-like method to compute the Fourier integral.

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