Analysis of Envelope Perturbations in High-Intensity Beams Using Generalized Courant-Snyder Formalism

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Outline

- **Brief (and incomplete) history of envelope perturbation**
- **Overview of envelope perturbation around the matched beam**
  - KV envelope equations
  - Stability analysis
  - Krein-Gelfand-Lidskii-Morse strong stability theorem
- **A new theory for general linear coupled dynamics**
  - Generalization of the Courant-Snyder theory
  - Time-dependent canonical transformation is used
  - Phase advance determines the stability

World cup 2018

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Last night (0:1)

Tonight 9:00 PM

1	15	24	57
Mercedes-Benz BMW, Audi	?	Volvo Saab	Hyundai Kia

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GSI FAIR	?	ESS	RISP KOMAC

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  → A systematic analysis of the mismatched modes

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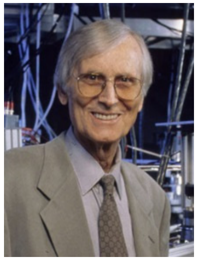

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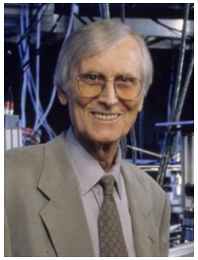

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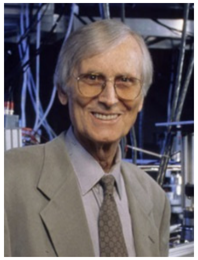

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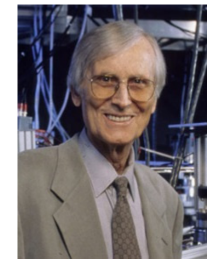

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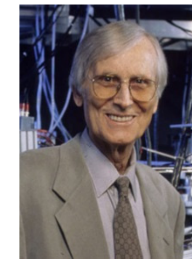

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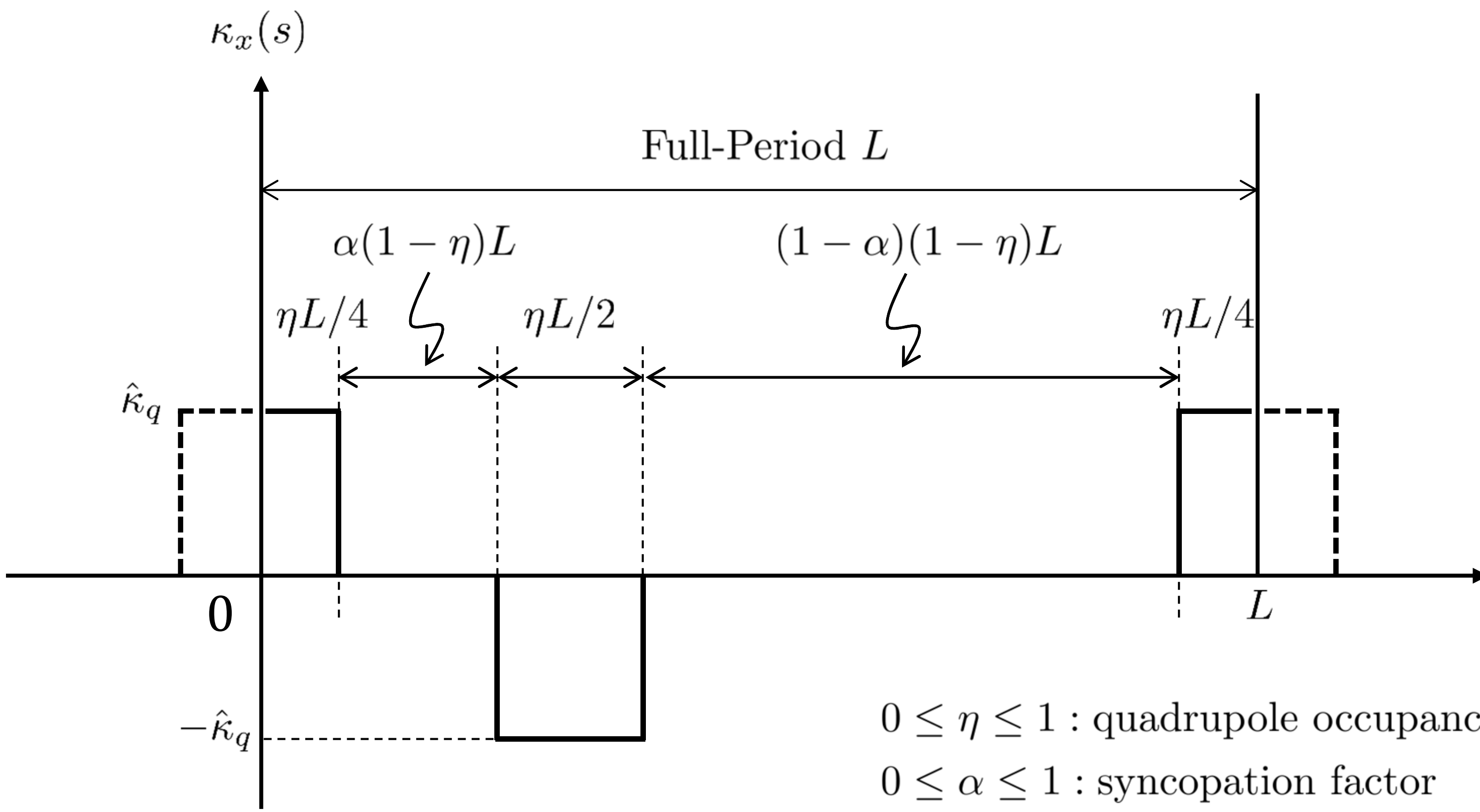
Basic setup

$\kappa_y(s) = -\chi\kappa_x(s)$



$0 \leq \eta \leq 1$: quadrupole occupancy factor

$0 \leq \alpha \leq 1$: syncopation factor

$0 < \chi$: asymmetric focusing factor

Governing equations

- KV Envelope equations:

$$\begin{aligned} a''(s) + \kappa_x a(s) - \frac{K_b}{a(s)+b(s)} - \frac{\epsilon_x^2}{a^3(s)} &= 0 \\ b''(s) + \kappa_y b(s) - \frac{K_b}{a(s)+b(s)} - \frac{\epsilon_y^2}{b^3(s)} &= 0 \end{aligned}$$

- The dimensionless parameter K_b is the self-field perveance defined either in terms of line density N_b or bunch current I_b or line charge density λ_b as

$$K_b = \frac{1}{4\pi\epsilon_0}\frac{2N_b q_b^2}{\gamma_0^2\beta_0 c p_0} = \frac{1}{2\pi\epsilon_0}\frac{q_b I_b}{\gamma_0^2 v_0^2 p_0} = \frac{1}{2\pi\epsilon_0}\frac{q_b\lambda_b}{\gamma_0^2\beta_0 c p_0}$$

- The total emittances (100% or rms edge emittances) are given by

$$\epsilon_x = 4\left[\left\langle x^2\right\rangle\left\langle x'^2\right\rangle - \left\langle xx'\right\rangle^2\right]^{1/2}, \quad \epsilon_y = 4\left[\left\langle y^2\right\rangle\left\langle y'^2\right\rangle - \left\langle yy'\right\rangle^2\right]^{1/2}$$

- Periodic matched-beam solutions (believed to have the smallest maximum excursion)

$$a_m(s) = a_m(s+L), \quad a'_m(s) = a'_m(s+L), \quad b_m(s) = b_m(s+L), \quad b'_m(s) = b'_m(s+L)$$

Linearized perturbation equations

No skew

- Small perturbation about the matched envelopes: $a(s) = a_m(s) + \delta a(s), \quad b(s) = b_m(s) + \delta b(s)$

$$\frac{d}{ds}(\delta a) = \delta a', \quad \frac{d}{ds}(\delta b) = \delta b'$$

$$\begin{aligned}
\frac{d}{ds}(\delta a') &= -\kappa_x \delta a - \frac{2K_b}{(a_m + b_m)^2}(\delta a + \delta b) - \frac{3\epsilon_x^2}{a_m^4}\delta a = -\kappa_{xm}\delta a - \kappa_{0m}\delta b \\
\frac{d}{ds}(\delta b') &= -\kappa_y \delta b - \frac{2K_b}{(a_m + b_m)^2}(\delta a + \delta b) - \frac{3\epsilon_y^2}{b_m^4}\delta b = -\kappa_{0m}\delta a - \kappa_{ym}\delta b
\end{aligned}$$

- In the matrix form:

$$\frac{d}{ds}\begin{pmatrix} \delta a \\ \delta b \\ \delta a' \\ \delta b' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_{xm} & -k_{0m} & 0 & 0 \\ -k_{0m} & -k_{ym} & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta a \\ \delta b \\ \delta a' \\ \delta b' \end{pmatrix} = \begin{pmatrix} \mathbf{0} & I \\ -\kappa_m & \mathbf{0} \end{pmatrix} z(s) = K(s)z(s)$$

- From Floquet theorem:

$$z(s + nL) = M(s + nL)z_0 = M(s)M^n(L)z_0$$

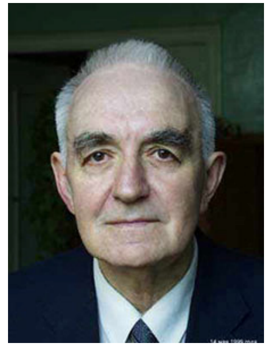
- Usually, stability is determined by the eigenvalues of the real symplectic matrix: $M(L)$

Growth factor $\quad \lambda_j = |\lambda_j| e^{i\phi_j} \quad$ Phase advance of the mode oscillations per lattice

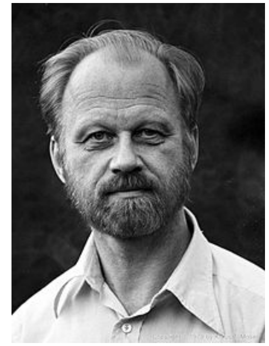
Krein-Gelfand-Lidskii-Morse strong stability theorem

Mark Krein
(1950, 1955)

Israel Gelfand
(1955)



Victor Lidskii
(1955)



Jürgen
Moser
(1958)

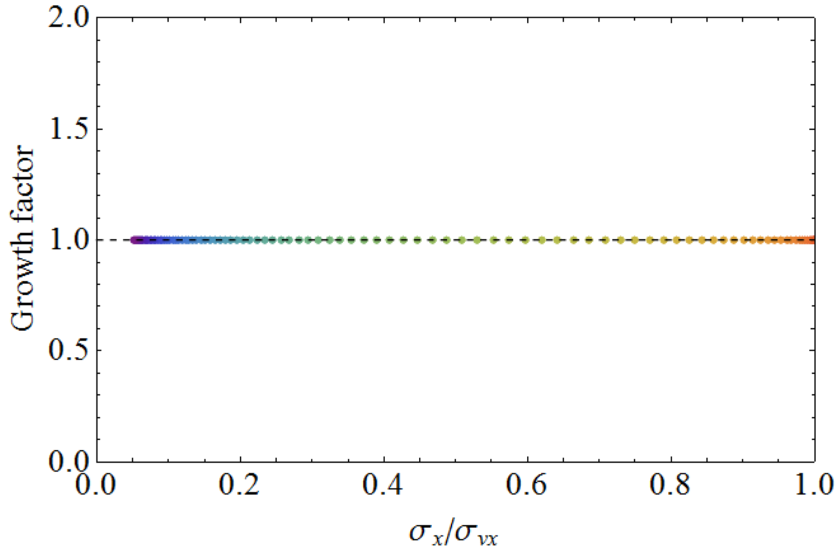
A stable symplectic matrix is strongly (structurally) unstable* iff eigenvalues collide with different Krein signatures.

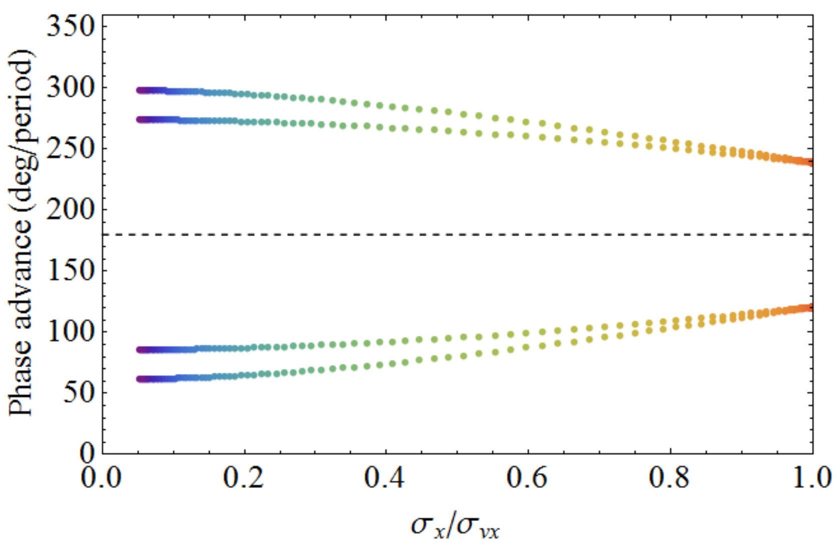
$$\text{Krein signature} \equiv \text{Sign}\left(\Psi^{\dagger} i J \Psi\right) = +1, 0, -1$$

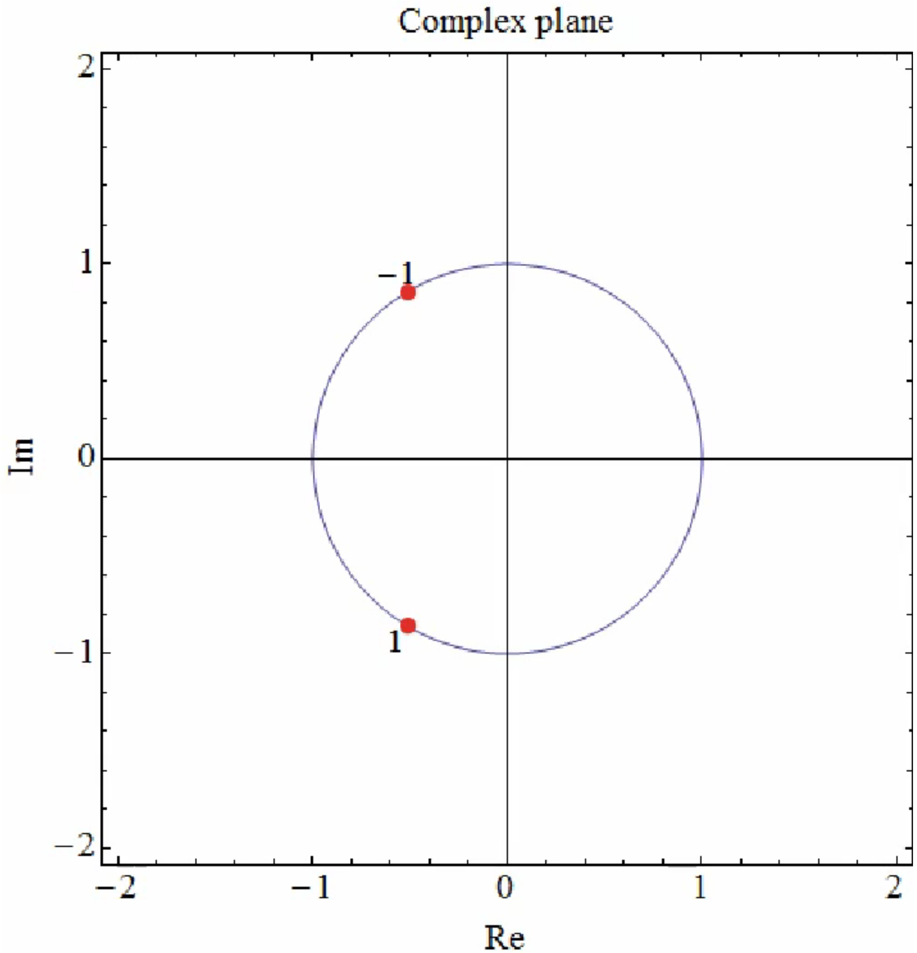
Eigenvector

* A small perturbation in parameter alters the topological character of the trajectories

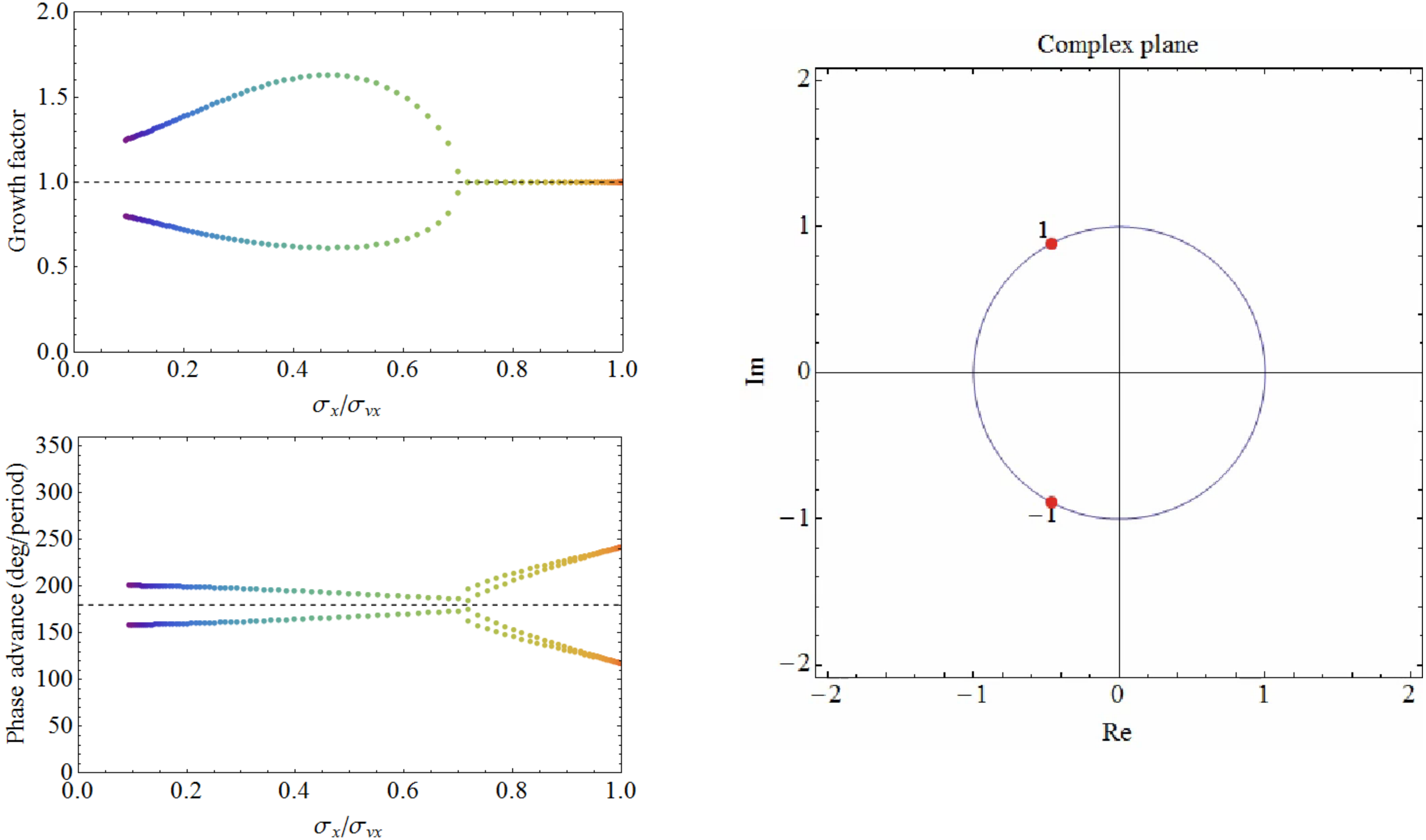
Case 1: $\hat{\kappa}_q = 15,\ \eta = 0.3,\ \alpha = 0.5,\ \chi = 1,\ \sigma_{vx} = 60.38°$







Case 2: $\hat{\kappa}_q = 26,\ \eta = 0.3,\ \alpha = 0.5,\ \chi = 1,\ \sigma_{vx} = 121.1^\circ$



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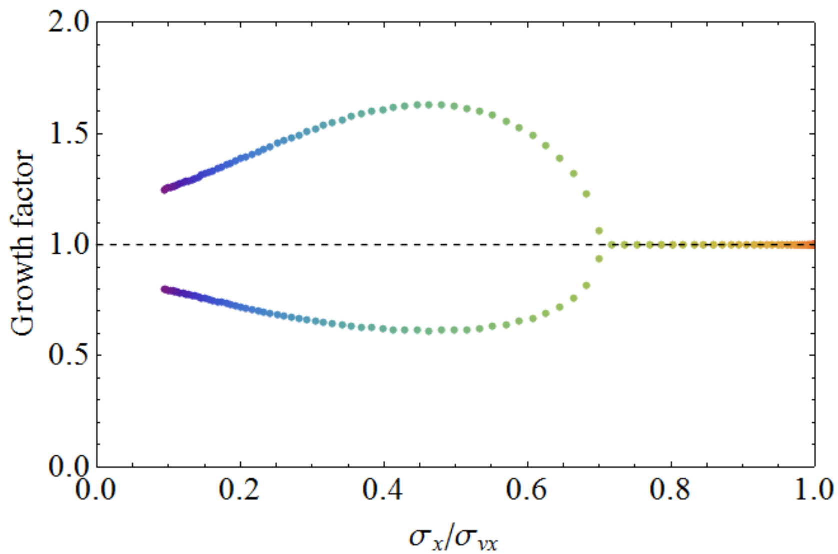

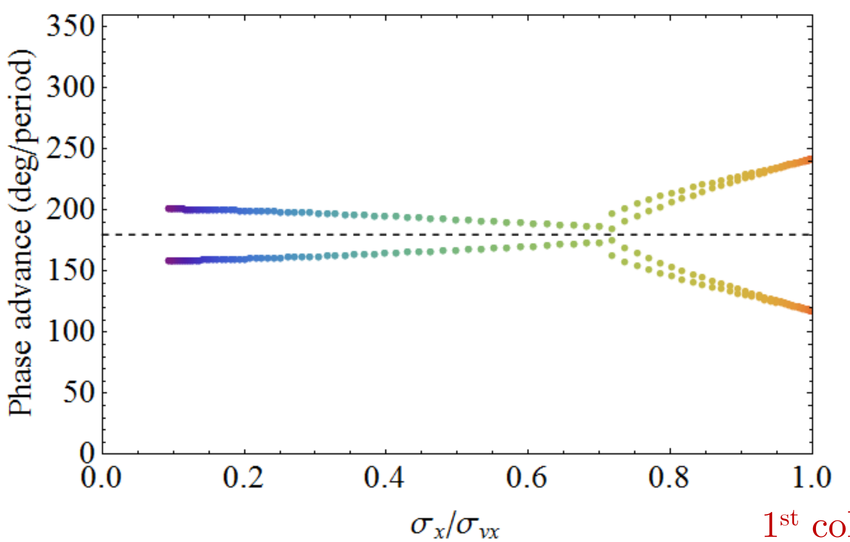

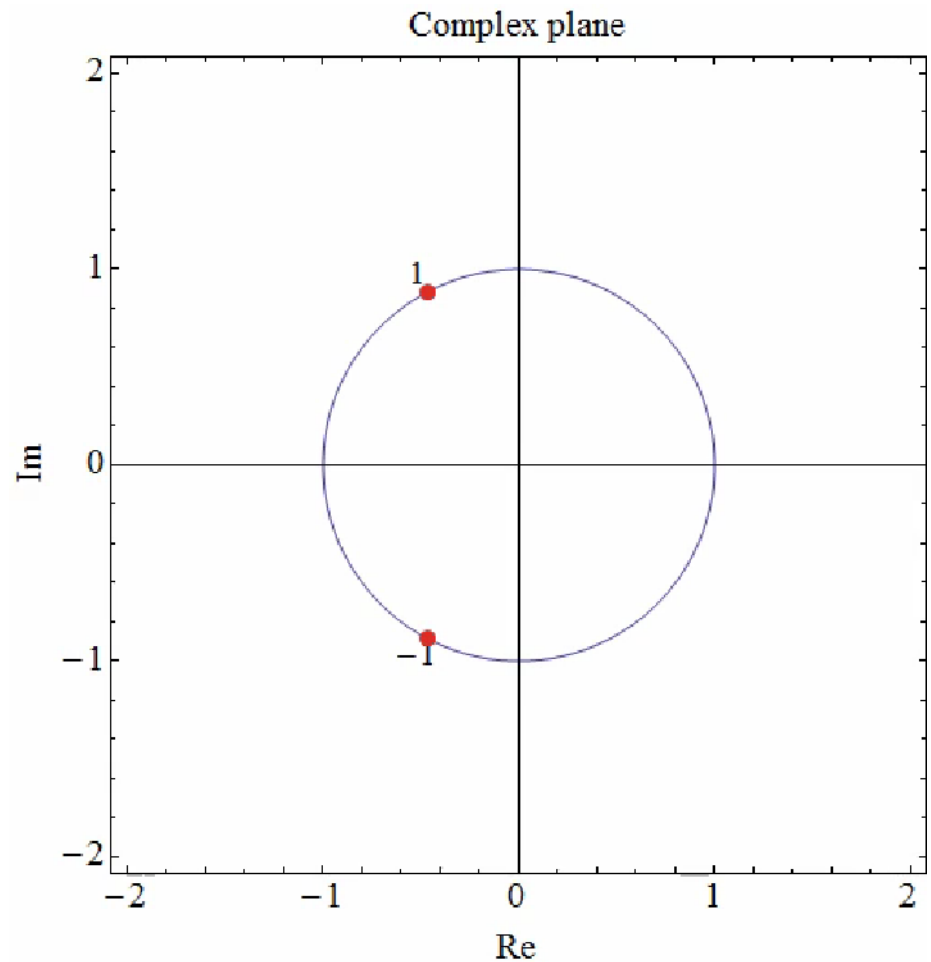

1st collision → No immediate break up (bypassing each other)
→ Very sensitive → Cannot be measured by eigenvalue

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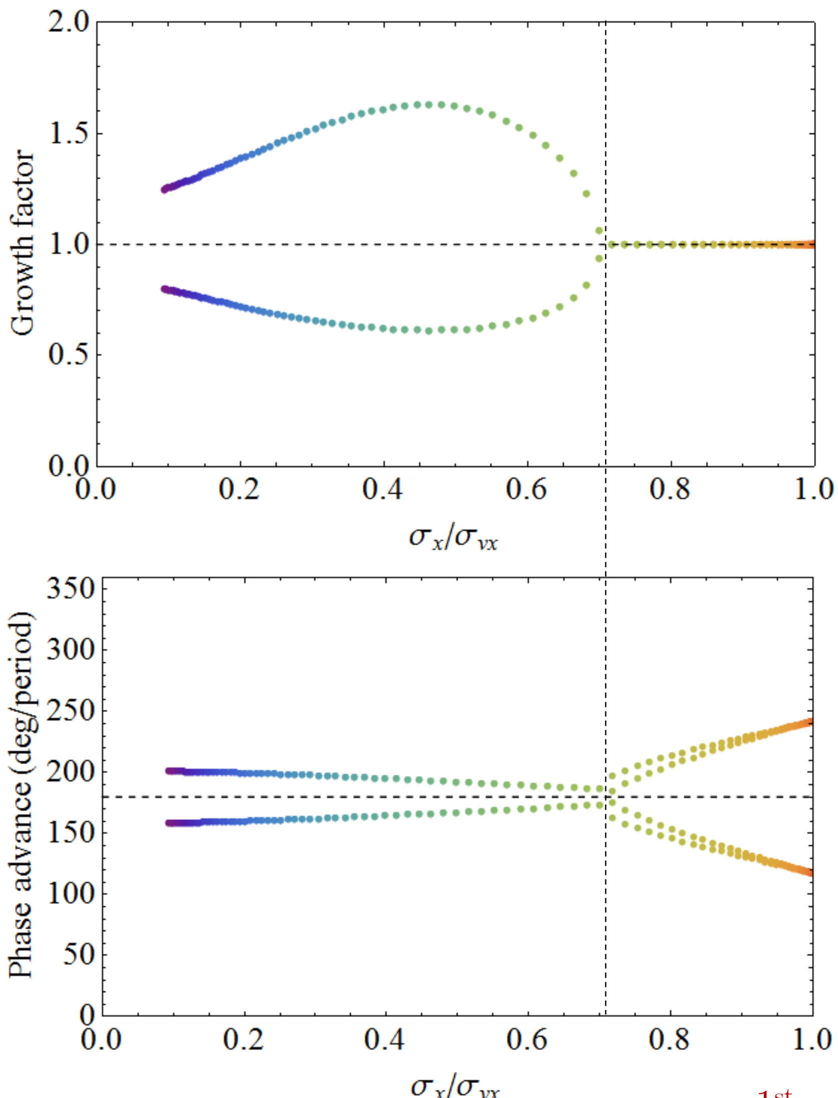

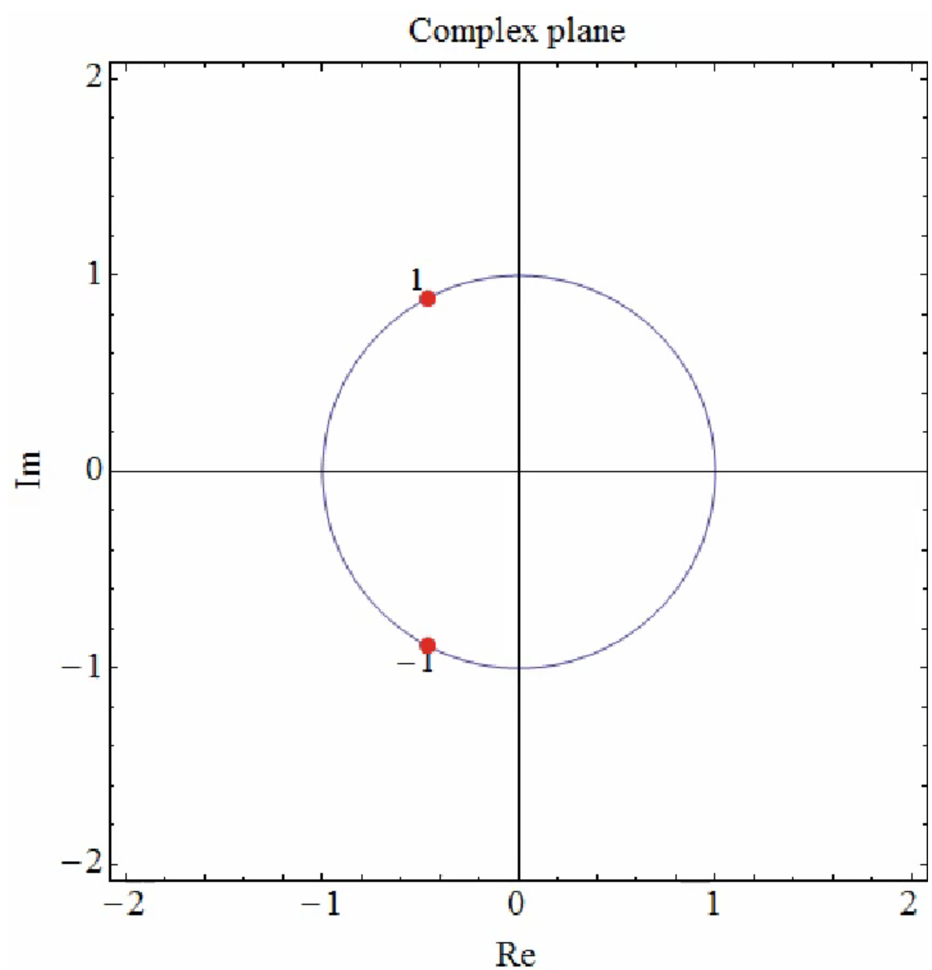

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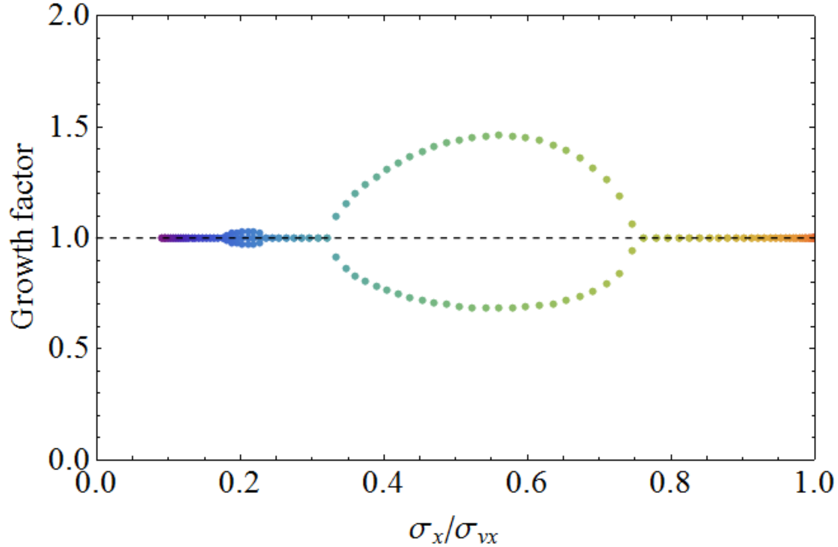
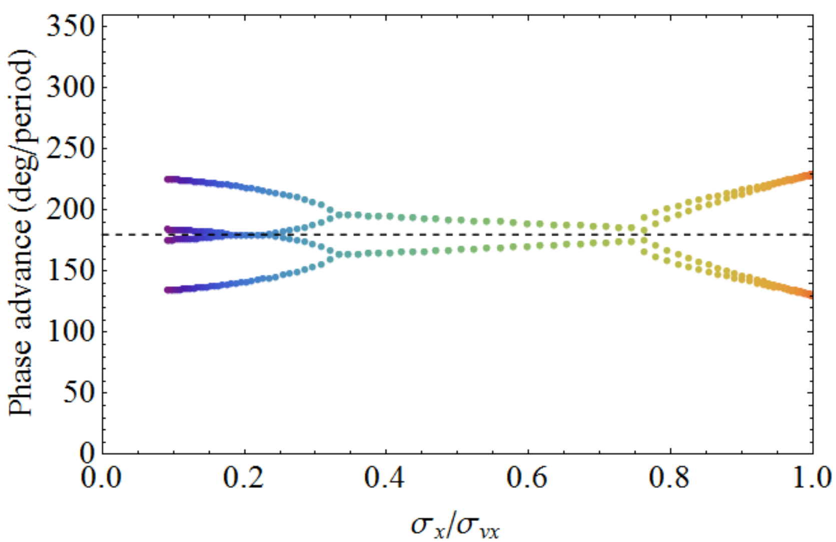
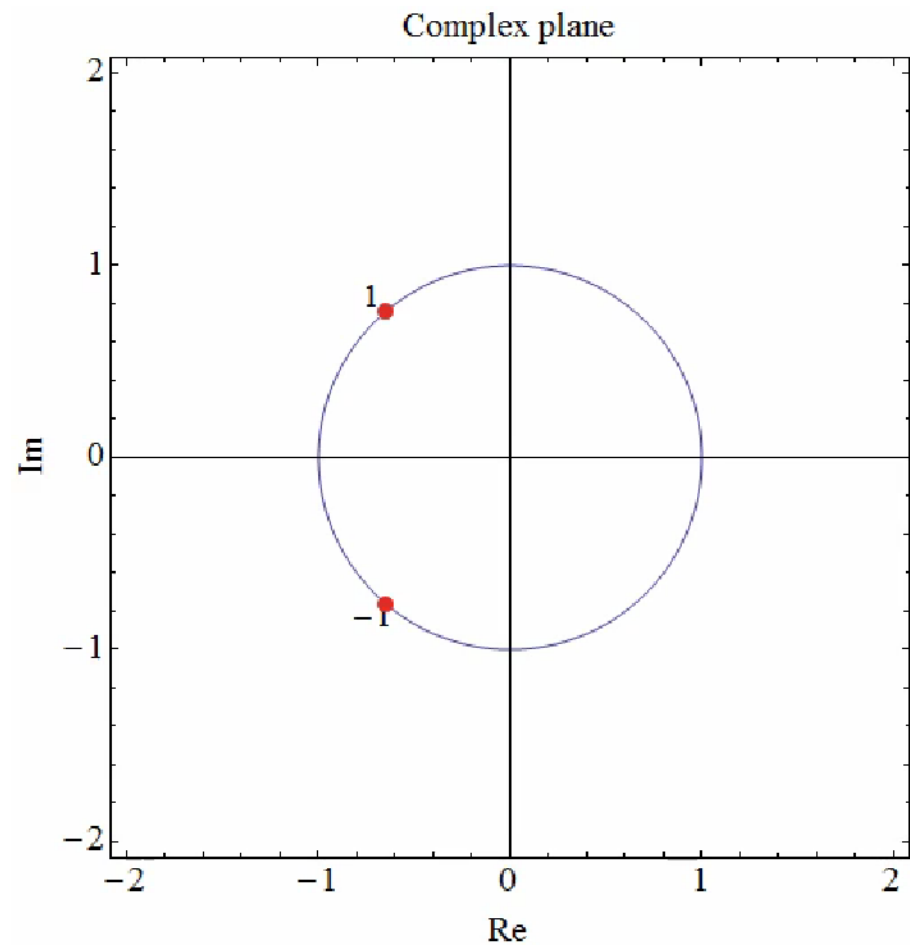
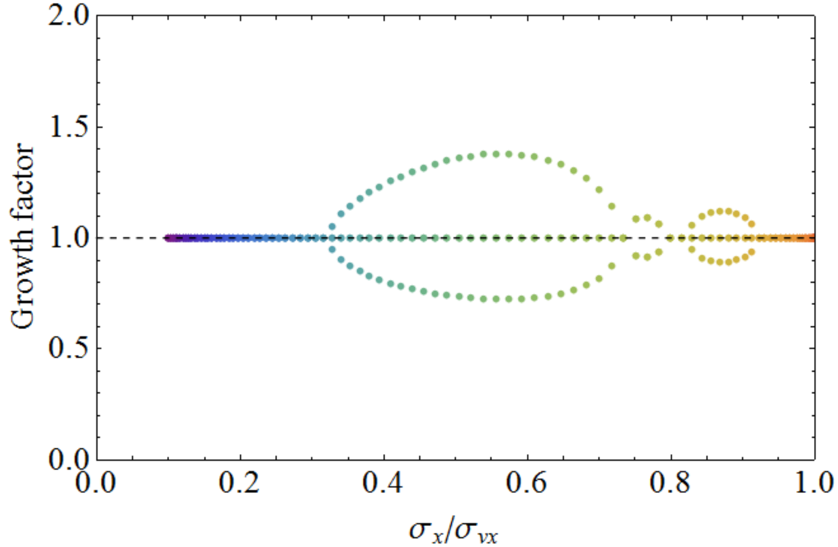
Case 3: $\hat{\kappa}_q = 16,\ \eta = 0.6,\ \alpha = 0.1,\ \chi = 1,\ \sigma_{vx} = 114.14^\circ$

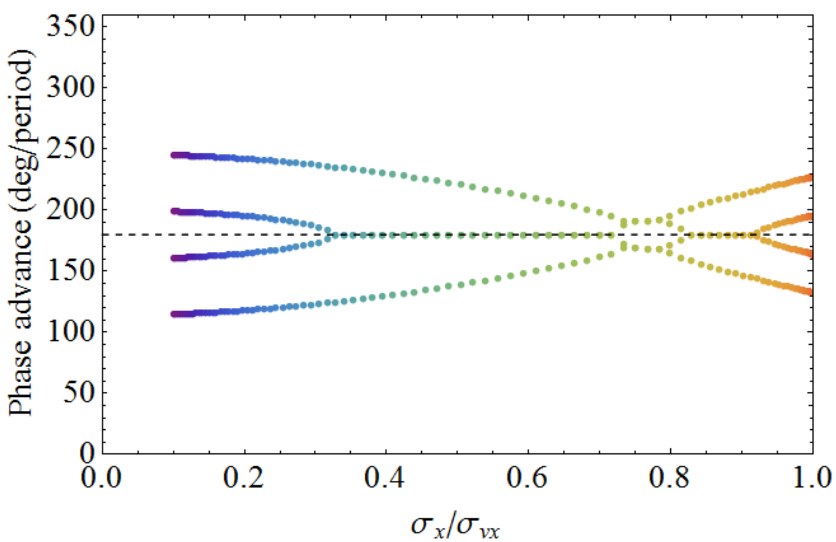


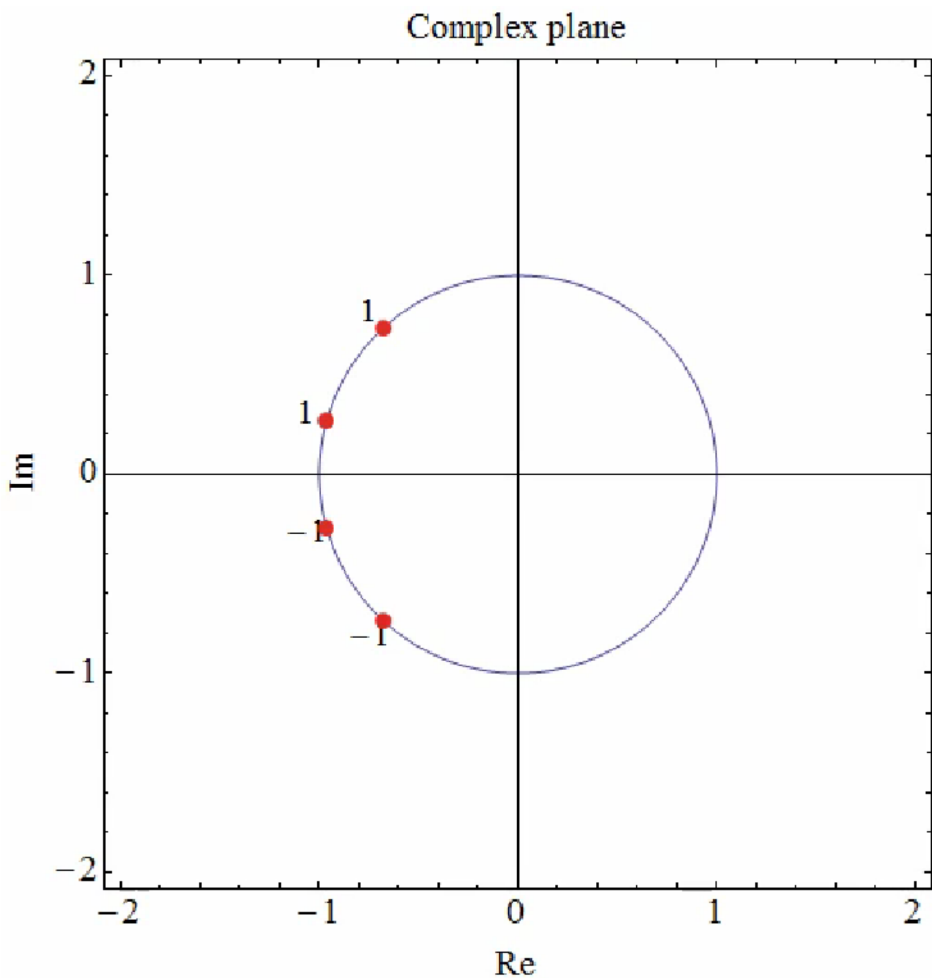




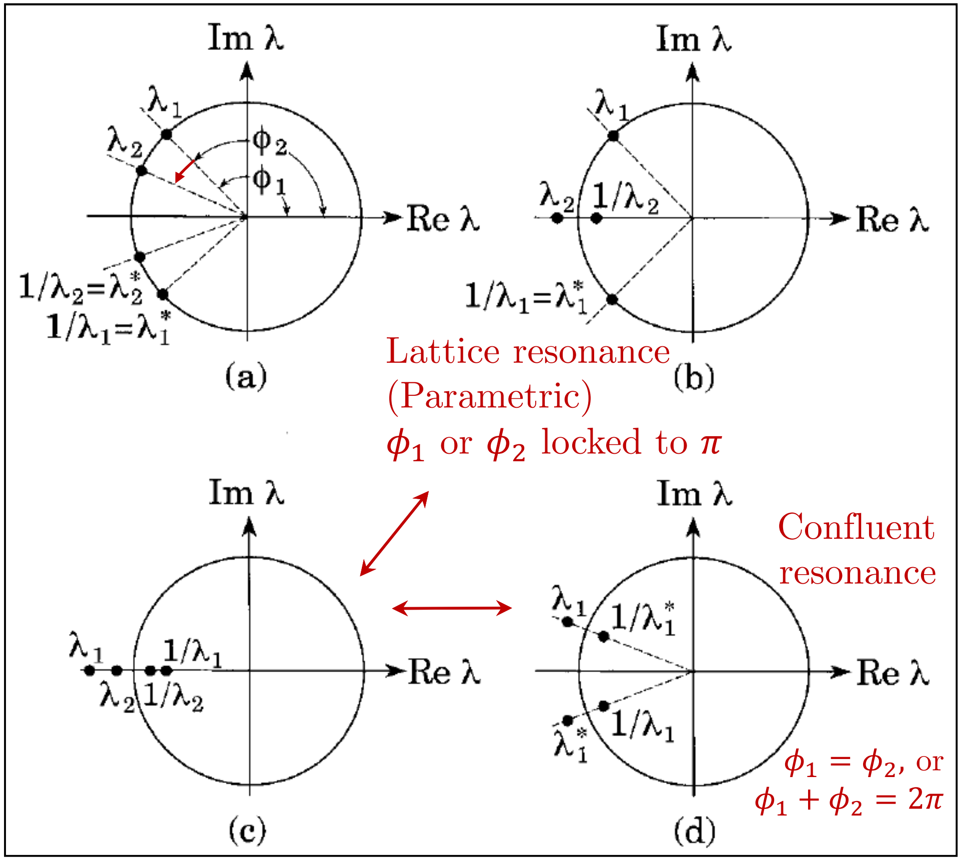
Case 4: $\hat{\kappa}_q = 25,\ \eta = 0.3,\ \alpha = 0.5,\ \chi = 0.9,\ \sigma_{vx} = 113.73°,\ \sigma_{vy} = 97.85°$







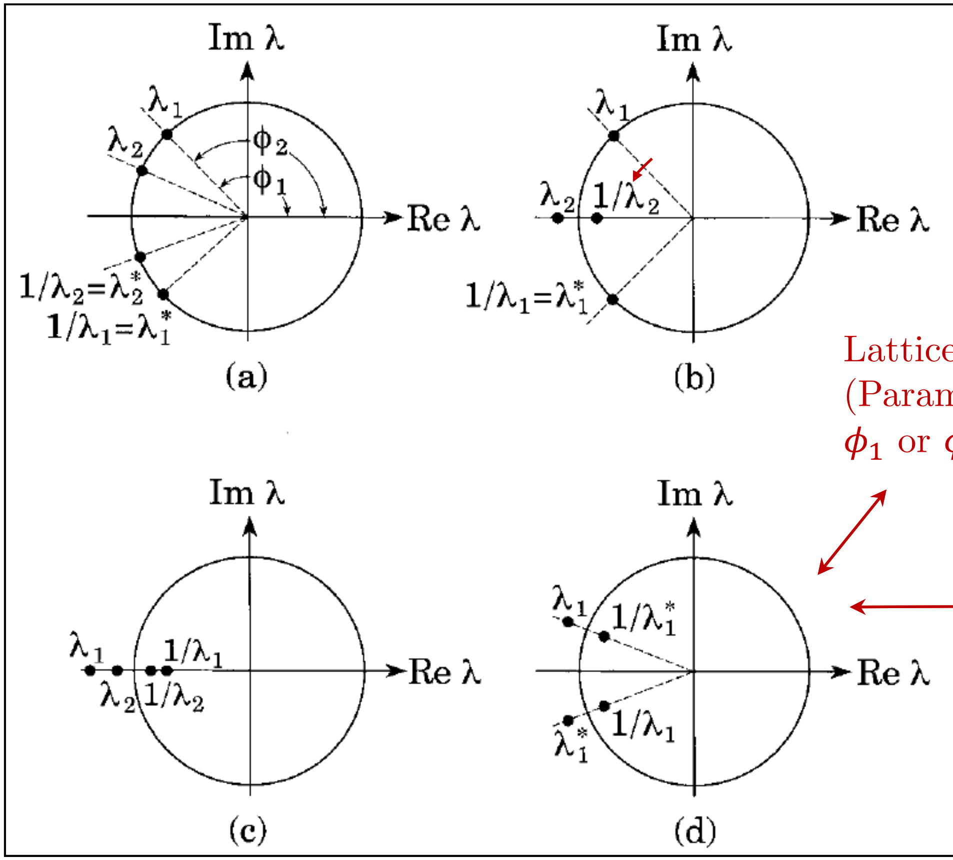
Location of eigenvalues



[From M. Reiser's textbook]

- Mathematically, this problem is identical to the two-dimensional linear oscillator without space charge treated by Courant and Snyder.

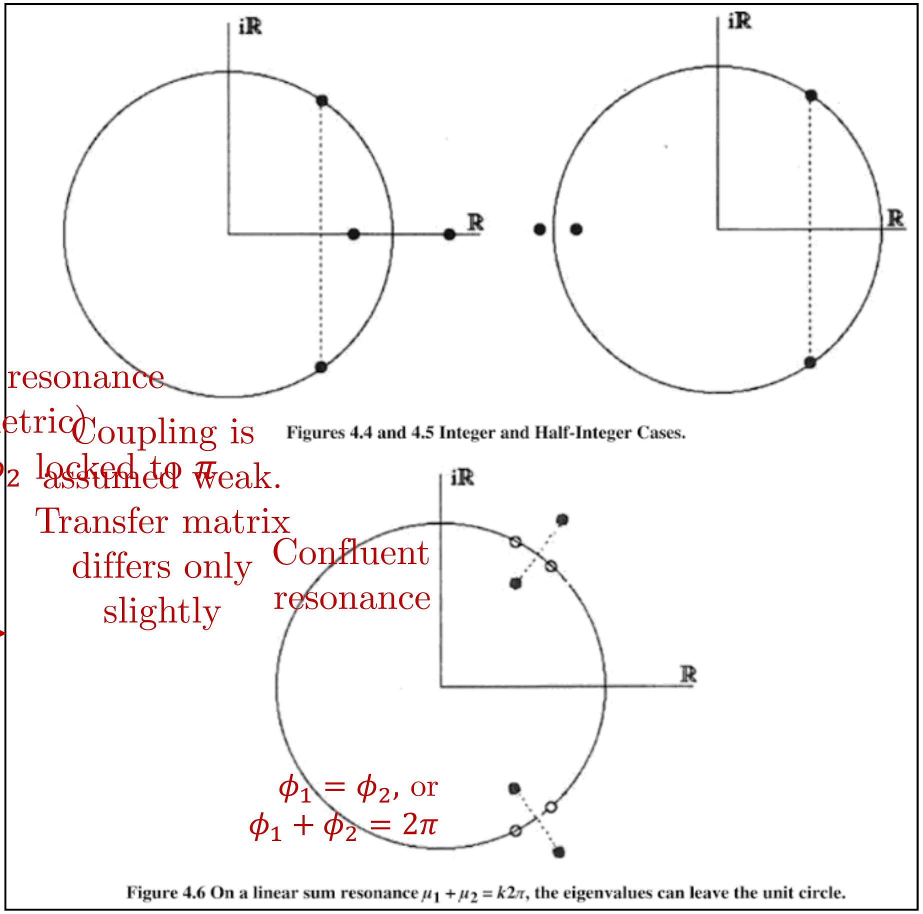
Location of eigenvalues



[From M. Reiser's textbook]

Lattice resonance
(Parametric)
ϕ_1 or ϕ_2 locked to π

Coupling is assumed weak.
Transfer matrix differs only slightly



Figures 4.4 and 4.5 Integer and Half-Integer Cases.

Figure 4.6 On a linear sum resonance $\mu_1 + \mu_2 = k2\pi$, the eigenvalues can leave the unit circle.

[Based on Courant-Snyder's original paper and E. Forest's textbook]

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Courant-Snyder theory for 1D uncoupled dynamics

$$q''(s) + \kappa_q(s) q(s) = 0$$

$$q(s) = A w(s) \cos[\phi(s) + \delta_0]$$

Courant-Snyder invariant

Ernest Courant (1958)

$$A^2 = \frac{q^2}{w^2} + (wq' - w'q)^2 = const. = I_{CS}$$

$$w''(s) + \kappa_q(s) w(s) = \underline{w^{-3}(s)}$$

Envelope Eq.

$$\begin{bmatrix} q \\ q' \end{bmatrix} = M(s) \begin{bmatrix} q_0 \\ q_0' \end{bmatrix} \qquad \phi'(s) = w^{-2}(s)$$

Phase advance rate

Transfer matrix

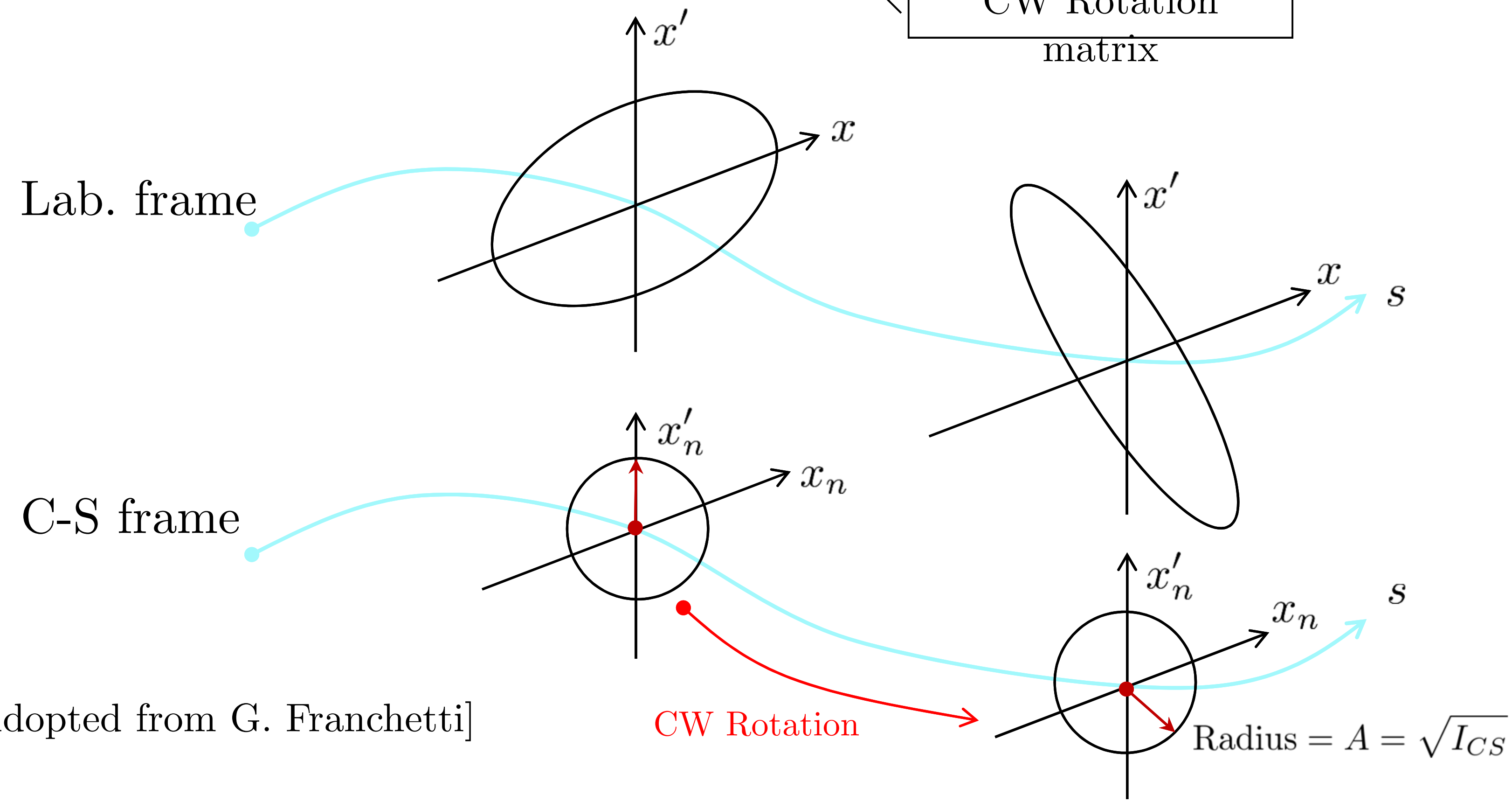
$$M(s) = \begin{bmatrix} \sqrt{\frac{\beta}{\beta_0}}(\cos\phi + \alpha_0 \sin\phi) & \sqrt{\beta\beta_0}\sin\phi \\ -\frac{1+\alpha\alpha_0}{\sqrt{\beta\beta_0}}\sin\phi + \frac{\alpha_0 - \alpha}{\sqrt{\beta\beta_0}}\cos\phi & \sqrt{\frac{\beta_0}{\beta}}(\cos\phi - \alpha\sin\phi) \end{bmatrix}$$

$$\beta = w^2, \ \alpha = -\frac{1}{2}\beta' = -ww', \ \phi = \int_0^s \frac{ds}{w^2}$$

Rotation in nomalized phase space coordinates

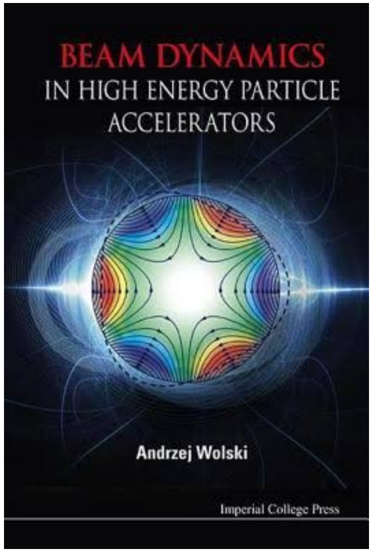
$$
\begin{aligned}
M(s) &= \begin{bmatrix} \sqrt{\frac{\beta}{\beta_0}}(\cos\phi + \alpha_0 \sin\phi) & \sqrt{\beta\beta_0}\sin\phi \\ -\frac{1+\alpha\alpha_0}{\sqrt{\beta\beta_0}}\sin\phi + \frac{\alpha_0-\alpha}{\sqrt{\beta\beta_0}}\cos\phi & \sqrt{\frac{\beta_0}{\beta}}(\cos\phi - \alpha\sin\phi) \end{bmatrix} \\
&= \begin{bmatrix} w & 0 \\ w' & w^{-1} \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} w_0^{-1} & 0 \\ -w_0' & w_0 \end{bmatrix}
\end{aligned}
$$

CW Rotation matrix



[Adopted from G. Franchetti]

What about 2D coupled dynamics case?



Coupling between horizontal and vertical motion can occur in a beam line either by design (for example, because of the inclusion of skew quadrupole or solenoid magnets), or as a result of alignment errors on the magnets (such as the tilt of a quadrupole around its magnetic axis). It is important to be able to describe coupling and its effects on the beam,

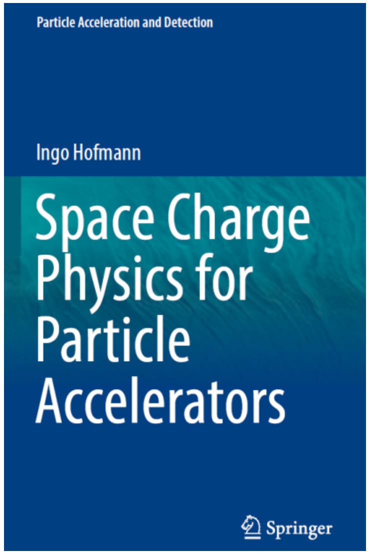
and there are several methods that have been developed to do this in a convenient way. Unfortunately, no single method has been adopted as a universal standard, and it would not be practical to try to cover here all (or even several) of the methods that are in use. Therefore, we restrict our

Lee Teng

[Ripken, 70; Mais-Ripken, 87; Wiedemann, 99]
[Teng, 71; Edward-Teng, 73]
[Sagan, 99]
[Wolski, 06 &14]
[Lebedev-Bogacz, 10]

Motivated by the great success of C-S theory

Space-charge is even more difficut to handle



3.2.3 *Chernin's Equations*

An extension of the second order rms envelope approach to include the linear coupling from skew quadrupole components, combined with space charge, has been derived by Chernin in [11]. The resulting equations are considerably more complex due to the additional coupled moments.

This may at least in part explain why these important equations have found relatively little attention so far, and space charge is hardly considered in linear coupling.[3] An example demonstrating the importance of this interplay is discussed in Sect. 8.2.2.

David Chernin (SAIC)

[3] More recently, similar equations with linear coupling and skewed space charge terms have been derived in [12, 13] and, with application to a twisted quadrupole channel, in [14].

11. D. Chernin, Part. Accel. **24**, 29 (1988)
12. M. Chung, H. Qin, E.P. Gilson, R.C. Davidson, Phys. Plasmas **20**, 083121 (2013)
13. H. Qin, R.C. Davidson, Phys. Rev. Lett. **110**, 064803 (2013)
14. A. Goswami, P. Sing Babu, V.S. Panditc, Eur. Phys. J. Plus **131**, 393 (2016)

Today's talk

How did we get normalized coordinates? → Time-dependent canonical transformation $S(s)$

H. Qin

RCD

$$\bar{z} = S(s)z$$

$$\bar{H} = \frac{1}{2}\bar{z}^T \bar{A}_c(s)\bar{z}$$

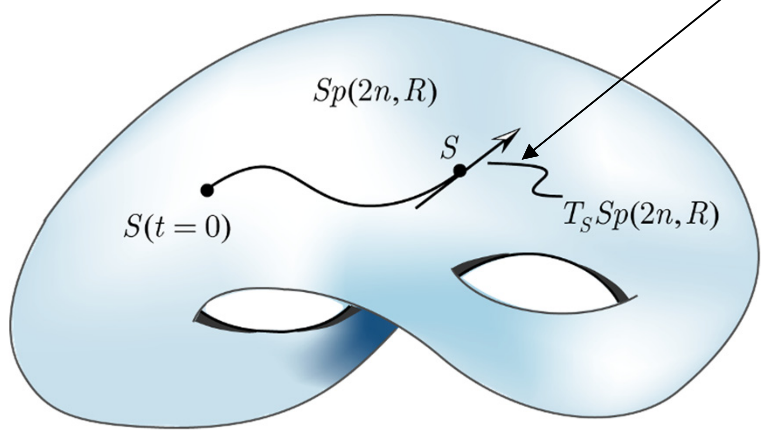
Target Hamiltonian

$$\begin{aligned} z' &= J\nabla H = JA_c z \\ \bar{z}' &= J\nabla\bar{H} = J\bar{A}_c\bar{z} = J\bar{A}_c S z \\ \bar{z}' &= [S(s)z]' = S'z + Sz' = (S' + SJA_c)\,z \end{aligned}$$

$$\boxed{S' = J\bar{A}_c S - SJA_c}$$

: describing flow of S

$$z = \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix}$$



S' belongs to tangent space of $Sp(2n,R)$

Symplectic group: $SJS^T = J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

H. Qin et al., PRL **111** (2014), PRAB **17** (2014)

Generalized theory has the similar structures

Generalized theory has the similar structures

Envelope function

$$w(s)$$

Envelope matrix

$$W(s) = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix}$$

Generalized theory has the similar structures

Envelope function

$$w(s)$$

Envelope matrix

$$W(s) = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix}$$

Envelope equation

$$w'' + \kappa_q w = w^{-3}$$

Matrix envelope equation

$$W'' + \kappa W = (W^T W W^T)^{-1}$$

Generalized theory has the similar structures

Envelope function: $w(s)$	Envelope matrix: $W(s) = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix}$
Envelope equation: $w'' + \kappa_q w = w^{-3}$	Matrix envelope equation: $W'' + \kappa W = (W^T W W^T)^{-1}$
Phase advance rate: $\phi'(s) = w^{-2}$	Phase advance rate: $\mu(s) = \left(W^T W\right)^{-1}$

Generalized theory has the similar structures

Envelope function	$w(s)$	$W(s) = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix}$	Envelope matrix
Envelope equation	$w'' + \kappa_q w = w^{-3}$	$W'' + \kappa W = (W^T W W^T)^{-1}$	Matrix envelope equation
Phase advance rate	$\phi'(s) = w^{-2}$	$\mu(s) = \left(W^T W\right)^{-1}$	Phase advance rate
Phase advance	$R^{-1} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$	$P^{-1} = \begin{bmatrix} C_o^T & S_i^T \\ -S_i^T & C_o^T \end{bmatrix}$	4D symplectic rotation

Generalized theory has the similar structures

Envelope function: $w(s)$	→	Envelope matrix: $W(s) = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix}$
Envelope equation: $w'' + \kappa_q w = w^{-3}$	→	Matrix envelope equation: $W'' + \kappa W = (W^T W W^T)^{-1}$
Phase advance rate: $\phi'(s) = w^{-2}$	→	Phase advance rate: $\mu(s) = \left(W^T W\right)^{-1}$
Phase advance: $R^{-1} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$	→	4D symplectic rotation: $P^{-1} = \begin{bmatrix} C_o^T & S_i^T \\ -S_i^T & C_o^T \end{bmatrix}$
Transfer matrix: $M(s) = \begin{bmatrix} w & 0 \\ w' & w^{-1} \end{bmatrix} R^{-1} \begin{bmatrix} w_0^{-1} & 0 \\ -w_0' & w_0 \end{bmatrix}$	→	Transfer matrix: $M(s) = \begin{bmatrix} W & 0 \\ W' & W^{-T} \end{bmatrix} P^{-1} \begin{bmatrix} W^{-1} & 0 \\ -(W')^T & W^T \end{bmatrix}_0$

Generalized theory has the similar structures

Envelope function: $w(s)$	Envelope matrix: $W(s) = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix}$
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Phase advance: $R^{-1} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$	4D symplectic rotation: $P^{-1} = \begin{bmatrix} C_o^T & S_i^T \\ -S_i^T & C_o^T \end{bmatrix}$
Transfer matrix: $M(s) = \begin{bmatrix} w & 0 \\ w' & w^{-1} \end{bmatrix} R^{-1} \begin{bmatrix} w_0^{-1} & 0 \\ -w_0' & w_0 \end{bmatrix}$	Transfer matrix: $M(s) = \begin{bmatrix} W & 0 \\ W' & W^{-T} \end{bmatrix} P^{-1} \begin{bmatrix} W^{-1} & 0 \\ -(W')^T & W^T \end{bmatrix}_0$
Invariant: $I_{\mathrm{CS}} = \begin{bmatrix} q \\ q' \end{bmatrix}^T \begin{bmatrix} w^{-2} + w'^2 & -w'w \\ -ww' & w^2 \end{bmatrix} \begin{bmatrix} q \\ q' \end{bmatrix}$	Invariant: $I_\xi = z^T \begin{bmatrix} (WW^T)^{-1} + W'W'^T & -W'W^T \\ -WW'^T & WW^T \end{bmatrix} z$

Coupled lattice stability condition

- Envelope equation has no matched solution → the lattice is unstable.

- Envelope equation has a matched solution → the symplectic rotation phase advance $P(L)$ determines the spectral and structural stabilities.

$$M(L) = S_L^{-1} P^T(L) S_0 = S_0^{-1} P^T(L) S_0$$

$$M^n(L) = S_0^{-1} P^{Tn}(L) S_0$$

- A spectrally stable lattice is strongly (structurally) unstable iff eigenvalues of $P(L)$ collide with different Krein signatures.

$$\text{Krein signature} \equiv \text{Sign}\left(\Psi^\dagger iJ\Psi\right) = \text{Sign}\left(\Psi^\dagger S_0^T iJS_0\Psi\right) = \text{Sign}\left[(S_0\Psi)^\dagger iJ(S_0\Psi)\right]$$

Eigenvector of $M(L)$

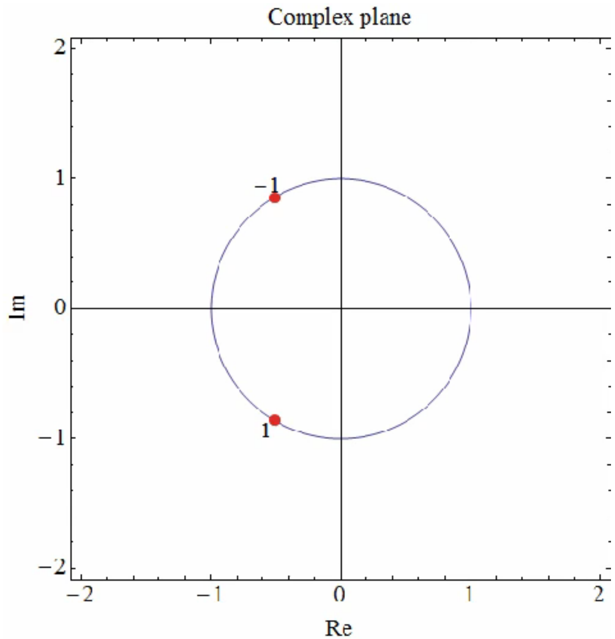
Eigenvector of $P^T(L)$

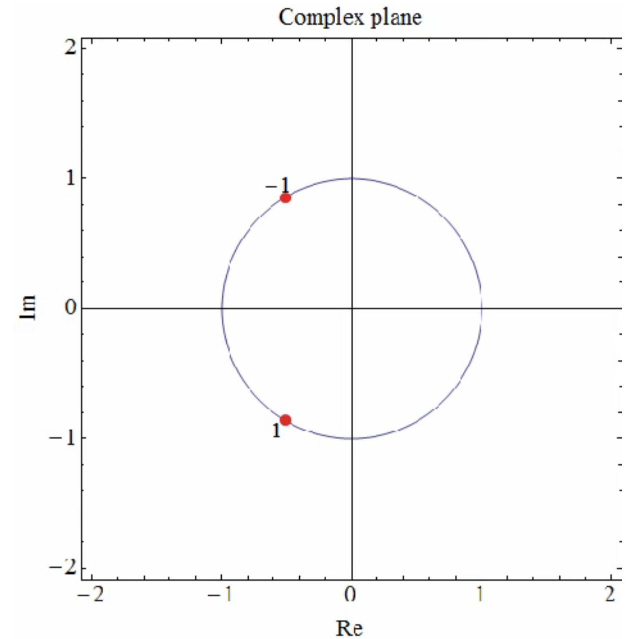
[H. Qin and M. Chung et al, PoP **22** (2015)]

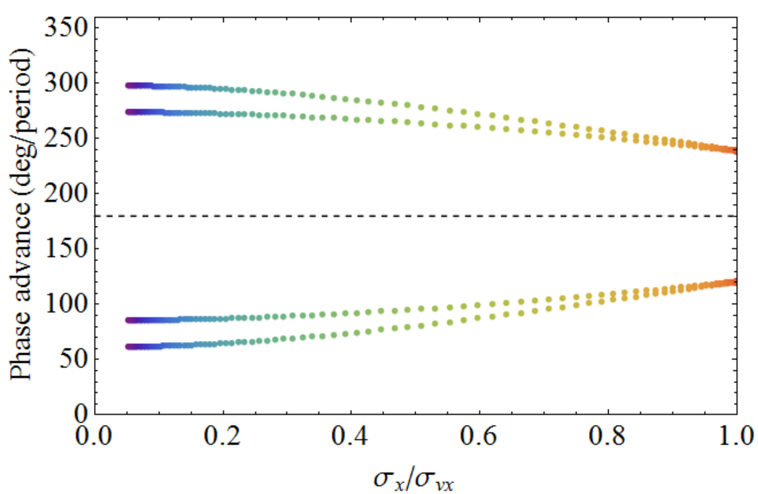
Comparison: $M(L)$ vs. $P^T(L)$

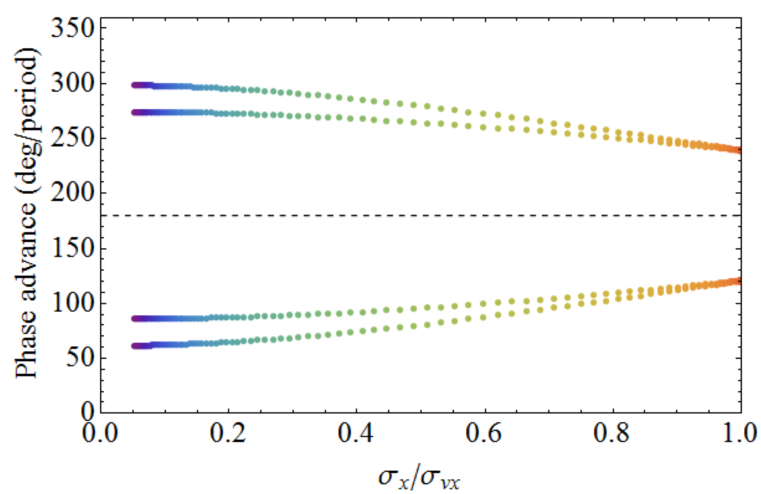
Based on $M(L)$

Based on $P^T(L)$









Courant-Snyder stability conditions for weakly coupled lattice

One-turn (lattice period) phase advance

$$P^T(L) = \begin{bmatrix} C_o^T & S_i^T \\ -S_i^T & C_o^T \end{bmatrix} \simeq \begin{bmatrix} \cos\phi_x & 0 & \sin\phi_x & 0 \\ 0 & \cos\phi_y & 0 & \sin\phi_y \\ -\sin\phi_x & 0 & \cos\phi_x & 0 \\ 0 & -\sin\phi_y & 0 & \cos\phi_y \end{bmatrix}$$

Ernest Courant (1958)

- Its four sets of eigenvalues, eigenvectors, signatures:

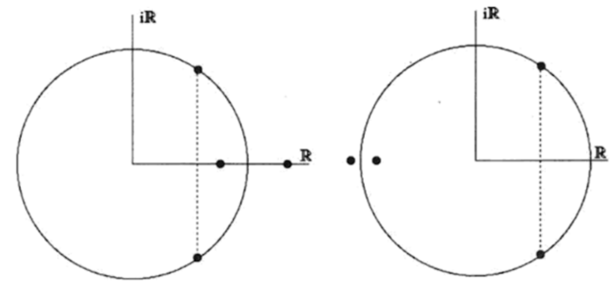
$$\lambda_{x+} = \cos\phi_x + i\sin\phi_x, \quad \Psi_{x+} = (1,0,i,0)^T, \quad \text{Krein signature} = -1$$

$$\lambda_{x-} = \cos\phi_x - i\sin\phi_x, \quad \Psi_{x-} = (1,0,-i,0)^T, \quad \text{Krein signature} = +1$$

$$\lambda_{y+} = \cos\phi_y + i\sin\phi_y, \quad \Psi_{y+} = (0,1,0,i)^T, \quad \text{Krein signature} = -1$$

$$\lambda_{y-} = \cos\phi_y - i\sin\phi_y, \quad \Psi_{y-} = (0,1,0,-i)^T, \quad \text{Krein signature} = +1$$

Four possibilities of resonances (Krein collisions)

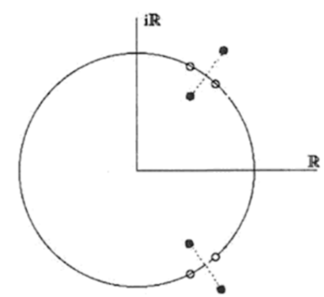


1. Self-resonance in the x-direction:

   $\lambda_{x+} = \lambda_{x-} = \pm 1$ for $\phi_x = n\pi$ → Different signature → Half-integer/Integer Resonance

2. Self-resonance in the y-direction:

   $\lambda_{y+} = \lambda_{y-} = \pm 1$ for $\phi_y = n\pi$ → Different signature → Half-integer/Integer Resonance



3. Sum resonance:

   $\lambda_{x+} = \lambda_{y-}$ and $\lambda_{x-} = \lambda_{y+}$ for $\phi_x + \phi_y = 2n\pi$ → Different signature → Sum resonances

4. Difference resonance:

   $\lambda_{x+} = \lambda_{y+}$ and $\lambda_{x-} = \lambda_{y-}$ for $\phi_x - \phi_y = 2n\pi$ → Same signature → Difference resonance

Envelope perturbation in terms of linear coupled dynamics

- First, we put the perturbation equation into "Hamiltonian form"

$$\frac{dz(s)}{ds} = \begin{pmatrix} \mathbf{0} & I \\ -\kappa_m & \mathbf{0} \end{pmatrix} z(s) = K(s)z(s)$$

$$\longrightarrow z' = JA_c z, \quad \text{where} \quad A_c = J^{-1}K = \begin{pmatrix} \kappa_m & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix}$$

$$\longrightarrow H = \frac{1}{2} z^T A_c(s) z$$

- Then, we can obtain quadratic "envelope mode Courant-Snyder invariant":

$$I_\xi = z^T \begin{bmatrix} (WW^T)^{-1} + W'W'^T & -W'W^T \\ -WW'^T & WW^T \end{bmatrix} z = \begin{bmatrix} \delta a \\ \delta b \\ \delta a' \\ \delta b' \end{bmatrix}^T \begin{bmatrix} \gamma & \alpha^T \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} \delta a \\ \delta b \\ \delta a' \\ \delta b' \end{bmatrix}$$ → 4D Hyper-ellipsoid

Matrix version of Twiss parameters

- Also, we can obtain "4D symplectic rotation":

$$P^{-1} = P^T = \begin{bmatrix} C_o^T & S_i^T \\ -S_i^T & C_o^T \end{bmatrix} \in U(2n, \mathsf{R}) := Sp(2n, \mathsf{R}) \cap O(2n, \mathsf{R}) \simeq U(n)$$

Parametrization of the symplectic rotation

Overall phase

Euler rotations

Pauli matrices

$$
\begin{aligned}
U(2) &= e^{i\lambda} R(\alpha,\beta,\gamma) \\
&= e^{i\lambda} \exp(-i\sigma_3\alpha)\exp(-i\sigma_2\beta)\exp(-i\sigma_3\gamma) \\
&= e^{i\lambda} \begin{pmatrix} e^{-i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & 0 \\ 0 & e^{i\gamma} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mapsto & \begin{pmatrix} \mathrm{Cos}[\lambda] & 0 & -\mathrm{Sin}[\lambda] & 0 \\ 0 & \mathrm{Cos}[\lambda] & 0 & -\mathrm{Sin}[\lambda] \\ \mathrm{Sin}[\lambda] & 0 & \mathrm{Cos}[\lambda] & 0 \\ 0 & \mathrm{Sin}[\lambda] & 0 & \mathrm{Cos}[\lambda] \end{pmatrix} \begin{pmatrix} \mathrm{Cos}[\alpha] & 0 & \mathrm{Sin}[\alpha] & 0 \\ 0 & \mathrm{Cos}[\alpha] & 0 & -\mathrm{Sin}[\alpha] \\ -\mathrm{Sin}[\alpha] & 0 & \mathrm{Cos}[\alpha] & 0 \\ 0 & \mathrm{Sin}[\alpha] & 0 & \mathrm{Cos}[\alpha] \end{pmatrix} \\
\times & \begin{pmatrix} \mathrm{Cos}[\beta] & -\mathrm{Sin}[\beta] & 0 & 0 \\ \mathrm{Sin}[\beta] & \mathrm{Cos}[\beta] & 0 & 0 \\ 0 & 0 & \mathrm{Cos}[\beta] & -\mathrm{Sin}[\beta] \\ 0 & 0 & \mathrm{Sin}[\beta] & \mathrm{Cos}[\beta] \end{pmatrix} \begin{pmatrix} \mathrm{Cos}[\gamma] & 0 & \mathrm{Sin}[\gamma] & 0 \\ 0 & \mathrm{Cos}[\gamma] & 0 & -\mathrm{Sin}[\gamma] \\ -\mathrm{Sin}[\gamma] & 0 & \mathrm{Cos}[\gamma] & 0 \\ 0 & \mathrm{Sin}[\gamma] & 0 & \mathrm{Cos}[\gamma] \end{pmatrix}
\end{aligned}
$$

If β is 0, then there is no coupling

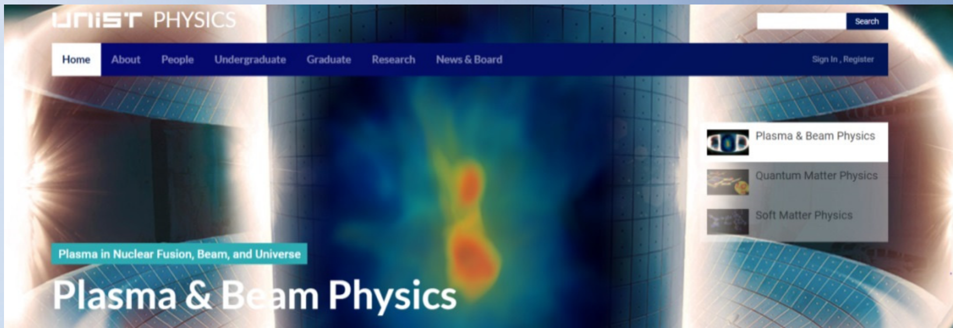
- Interpretation of the coupled dynamics in terms of the 4D rotation is under way, and will be presented elsewhere.

Conclusions

- **Stability of high intensity beam transport in a periodic focusing lattice is of significant importance.**

- **Stability analysis in terms of Krein collision and linear coupled dynamics (phase advance matrix) is under way.**

- **Courant-Snyder theory has been generalized to linear coupled transverse dynamics:**
  - Envelope function → Envelope matrix
  - Envelope equation → Matrix envelope equation
  - Phase advance →4D symplectic rotation
  - CS invariant → Generalized CS invariant
  - Transfer matrix → (Back transform) × 4D rotation × (normal form)



Thank you for your attention !

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2009

Back Up Slides

Step I: Envelope Matrix and Matrix Envelope Equation

$$H = \frac{1}{2} z^T A_c(s) z$$

$$A_c(s) = \begin{bmatrix} \kappa(s) & 0 \\ 0 & I \end{bmatrix}$$

$$\xrightarrow{\bar{z} = S(s) z}$$

$$\bar{H} = \frac{1}{2} \bar{z}^T \bar{A}_c(s) \bar{z}$$

$$\bar{A}_c(s) = \begin{bmatrix} \mu(s) & 0 \\ 0 & \mu(s) \end{bmatrix}$$

$$S = \begin{bmatrix} W^{-1} & 0 \\ -\left(W^T\right)' & W^T \end{bmatrix}, \quad W = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix}$$: Envelope matrix

scalar

$$\mu = \left(W^T W\right)^{-1}$$: phase advance rate

$$W'' + \kappa W = (W^T W W^T)^{-1}$$

: matrix envelope equation

H. Qin et al., PRL **111** (2014), PRAB **17** (2014)

Step II: Phase Advance Matrix

$$\bar{H} = \frac{1}{2}\bar{z}^T \bar{A}_c(s)\bar{z}$$

$$\bar{A}_c(s) = \left[\begin{array}{cc} \mu(s) & 0 \\ 0 & \mu(s) \end{array}\right]$$

$$\bar{\bar{z}} = P(s)\bar{z}$$

$$\bar{\bar{H}} = 0$$

$$\bar{\bar{A}}_c = 0$$

Dynamics is trivial:

$$\bar{\bar{z}} = \bar{\bar{z}}_0 = const.$$

$$P = \left[\begin{array}{cc} C_o & -S_i \\ S_i & C_o \end{array}\right]$$: Phase advance matrix

$$PP^T = \left[\begin{array}{cc} I & 0 \\ 0 & I \end{array}\right], \quad \text{and} \quad \det(P) = 1 \quad \in Sp(4) \cap O(4) \simeq U(2)$$

: P is not only a simplectic, but also a rotaion matrix

$$C_o' = -S_i \left(W^T W\right)^{-1}, \quad S_i' = +C_o \left(W^T W\right)^{-1}$$

$$S_i C_o^T = C_o S_i^T, \quad S_i S_i^T + C_o C_o^T = I$$

: Cosine/Sine-like matrices

H. Qin et al., PRL **111** (2014), PRAB **17** (2014)

Step III: Transfer Matrix

$$\bar{\bar{H}} = 0$$
$$\bar{\bar{A}}_c = 0$$

Dynamics is trivial:

$$\bar{\bar{z}} = \bar{\bar{z}}_0 = const.$$

$$\xrightarrow{\bar{\bar{z}} = P(s)S(s)z}$$

$$\begin{aligned} \bar{\bar{z}} &= P(s)S(s)z(s) \\ &= P_0 S_0 z_0 = \bar{\bar{z}}_0 \end{aligned}$$

$$z(s) = S^{-1}P^{-1}P_0S_0z_0 = M(s)z_0$$

$$\begin{aligned} M(s) &= S^{-1}P^T S_0 \\ &= \begin{bmatrix} W & 0 \\ W' & W^{-T} \end{bmatrix} \underbrace{\begin{bmatrix} C_o^T & S_i^T \\ -S_i^T & C_o^T \end{bmatrix}}_{\in Sp(4) \cap O(4) \simeq U(2)} \begin{bmatrix} W^{-1} & 0 \\ -(W')^T & W^T \end{bmatrix}_0 \end{aligned}$$

$P^{-1} = P^T$

$P_0 = I$

: without loss of generality

H. Qin et al., PRL **111** (2014), PRAB **17** (2014)

Step IV: Invariant

$$\bar{\bar{H}} = 0$$

$$\bar{\bar{A}}_c = 0$$

Dynamics is trivial:

$$\bar{\bar{z}} = \bar{\bar{z}}_0 = const.$$

$$\bar{\bar{z}} = P(s)S(s)z$$

Invariant of the dynamics that is quadratic in phase-spece coordinate:

$$\begin{aligned} I_\xi &= \bar{\bar{z}}_0^T \xi \bar{\bar{z}}_0 \\ &= \bar{\bar{z}}^T \xi \bar{\bar{z}} = const. \end{aligned}$$

Arbitrary constant positive-definite matrix

$\xi = I$: A special case of equal eigen-emittances

$$\begin{aligned} I_\xi &= z^T S^T P^T P S z \\ &= z^T \begin{bmatrix} W^{-T} & -W' \\ 0 & W \end{bmatrix} \begin{bmatrix} W^{-1} & 0 \\ -(W')^T & W^T \end{bmatrix} z \\ &= z^T \begin{bmatrix} (WW^T)^{-1} + W'W'^T & -W'W^T \\ -WW'^T & WW^T \end{bmatrix} z \\ &= z^T \begin{bmatrix} \gamma & \alpha^T \\ \alpha & \beta \end{bmatrix} z \end{aligned}$$

Matrix version of Twiss parameters

H. Qin et al., PRL **111** (2014), PRAB **17** (2014)