SUM RESONANCES WITH SPACE CHARGE*

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Abstract

In the past years several studies, numerical and experimental, have been carried out for enlightening the effect of space charge on stored bunches. The last effort in this quest has regarded the space charge effects on the third order coupled resonance. Experimental studies have been performed at the CERN-PS and a vast simulation effort has followed to interpret the experimental findings. The interpretative base of the analysis relied on: 1) the knowledge of the mechanism of the periodic resonance crossing induced by space charge, which has been identified and confirmed in previous decade 2000-2010; 2) the new revival of the nonlinear dynamics of coupled resonances, alias the fixed-lines. The analysis of the experiment combined together both the mechanisms. However, the discussion made use of an intuitive ansatz based mainly on physics arguments. We shortly present here the re-derivation of the theory of nonlinear dynamics including space charge, and show that we retrieve the concepts used to discuss the analysis of the experiment of the 3rd order coupled resonance.

INTRODUCTION

It is here presented the effect of the space charge in the theory of resonances. The effect of space charge on the beam dynamics in coasting beams is introduced with the following two assumptions:

1) The beam is assumed in a stationary state, i.e. the beam distribution does not change during storage;
2) The effect of a resonant dynamics is assumed small so to not alter significantly the beam distribution or beam intensity so that the assumption 1) remains valid.

We next briefly discuss these two ansatzes in order to clarify the implications and limits they introduce.

The ansatz 1) means the beam is matched and not subjected to coherent effects that destabilize it. On the other hand, any coherent effect which is stationary and makes the beam envelope oscillate with regular periodic motion can be regarded as included in point 1) as far as it concerns the direct space charge. The ansatz 1) allows to discuss the effect of space charge as created by an "external force" so that in this condition is viewed as an "incoherent" force. Usually the presence of coherent effects is discussed with reference to plasma "coherent effects" such as the Debye length $\lambda_D = \sqrt{\epsilon_0 \gamma^3 m \tilde{v}_x^2 / (q^2 n)}$, where q is the particle charge, m the particle mass, and n the particle density, and $\tilde{v}_x$ is the rms "thermal" velocity component. The Debye length is a characteristic length of a collective motion of charged particles which create a shielding of local perturbations in a plasma.

If λ_D is much larger than the inter-particle average distance l_p the space charge force can be treated as a smooth applied force. If in addition λ_D is much bigger than the rms beam radius a_0 the single particle behavior dominates the dynamics (see in Ref. [1]). For a matched beam the thermal velocity is $\tilde{v}_{th}^2 = v_0^2 \tilde{\epsilon}_x / \beta_x$, and for an axi-symmetric Gaussian beam we find

$$\lambda_D^2(r) = \frac{Q_{x0}}{4|\Delta Q_x|} a_0^2 e^{\frac{1}{2} \frac{r^2}{a_0^2}}, \tag{1}$$

where Q_{x0} is the machine tune, ΔQ_x is the incoherent space charge tune-shift, and $r = \sqrt{x^2 + y^2}$. As at each r one finds a specific $\lambda_D(r)$, the most relevant for the Debye collective shielding is the smallest, which is found at $r = 0$. We attempt to capture the incoherent nature of space charge defining a "parameter of incoherence" as $\mathcal{I} = \lambda_D(0)/a_0$: the larger $\mathcal{I}$ the more "incoherent" the direct space charge is. From Eq. (1) we find

$$\mathcal{I} = \sqrt{\frac{1}{4} \frac{Q_{x0}}{|\Delta Q_x|}},$$

so if $|\Delta Q_x|/Q_{x0} = 0.25$ the collective nature of space charge may invalidate ansatzes 1), 2) as $\mathcal{I} = 1$. We instead may "safely" use the ansatzes for a space charge yielding the more conservative $\mathcal{I} = 3$, corresponding to $|\Delta Q_x|/Q_{x0} = 0.027$, a typical value for standard operational regimes in circular accelerators. This is confirmed by numerical studies for Gaussian beams [2], which have shown that space charge collective resonances are not observed. A further argument to develop the theory of resonances with space charge using a model with ansatz 1) and 2) is that long term PIC simulations still suffer of intrinsic noise heating [3–6] although the recent significant progress in creating symplectic PIC algorithms [7].

In ansatz 2) the resonant dynamics is here discussed for a generic sum resonance, which can be generated by magnet errors, or by the incoherent space charge itself as the beam undergoes envelope oscillations driven by the machine optics. The requirement that a resonance does not change the beam distribution is satisfied when the number of beam particles affected by the resonance is small with respect to the total number of beam particles. This means it is assumed only a small fraction of the beam is transported around in the phase space by a resonance. In this treatment we do not consider dynamical effects such as the change of the space charge tune-spread, which would feed back on the dynamics of resonant particles. This approach has a validity when global effects induced by incoherent effects are small on the time scaled considered. A similar assumption is adopted in case of beam loss: we assume the losses to be small on the time scale considered.

RESONANCES

The theory of resonances in accelerators has got a long history. It started in the 50s with [8], and the work of Schoch [9]. The theory was based on a perturbative approach to the Hamiltonian theory. In more recent years the normal form approach has followed the advent of computers and numerical physics [10, 11]. The more popular application of the theory led to the development of driving term methods to measure and correct resonances [12].

The original theory was developed with perturbative methods, and originates a complex structure of patterns in the phase space. Without space charge, coupled resonances in phase space have a particular pattern also called fixed lines [13]. These mathematical objects created by nonlinear coupled resonances were not extensively used for controlling accelerators beams. In literature the reference to fixed lines trace back to the 80's [14, 15]. However, the recent study in Ref. [16] have shown that fixed lines play a fundamental role in determining the dynamics of halo formation in high intensity beams. A recent study of the resonant coupled dynamics is discussed in [17, 18], and is here modified to include space charge.

The approach starts from the single particle Hamiltonian, which is composed by the quadratic potential H_0 and the perturbation part H_1 which has higher order and is originated by the presence of nonlinear errors in magnets. The solution of the equations of motion for the Hamiltonian H_0 is expressed in the Courant-Snyder form $x = \sqrt{\beta_x a_x} \sin(\psi_x + \varphi_x)$, similarly for the y coordinate. For convenience the single particle invariants are re-scaled as $\hat{a}_x = a_x/\epsilon_x, \hat{a}_y = a_y/\epsilon_x$ to the ϵ_x rms beam emittance. The dynamics of particles under the full Hamiltonian $H_0 + H_1$ is derived by keeping the Courant-Snyder form, but allowing the quantities $\hat{a}_x, \varphi_x$ to vary. The treatment leads to a new set of canonical equations in the quantities $\hat{a}_x, \varphi_x, \hat{a}_y, \varphi_y$ controlled by the Hamiltonian H_1. These equations are exact, and usually are not easily solvable. However, the mathematical formulation of the canonical equations allows a representation of the problem in terms of a harmonics series. Each term of the series is an oscillating function which frequency is a combination of the machine tunes and the harmonics number m of the distribution of the machine nonlinear errors.

By setting the machine tunes Q_{x0}, Q_{y0} such that the frequency of some harmonics becomes "slow", which happens when Q_{x0}, Q_{y0} are set close $N_x Q_x + N_y Q_y = m$ for the excited resonance, the system of differential equations acquires a dynamics in which the average motion is controlled mainly by the "slow harmonics". It is then used a simplifying ansatz of neglecting all the "fast" oscillating harmonics so to obtain an "easier" set of canonical equations. This is clearly a substantial approximation as all harmonics are neglected but the slowly varying one; for weak errors this approach yields acceptable approximation for modeling the main resonance. After using a canonical transformation to remove the time

dependence from the slowly varying Hamiltonian, we find

$$\tilde{H}_1 = \frac{4}{\epsilon_x} \rho \tilde{a}_x^{n_x/2} \tilde{a}_y^{n_y/2} \cos[N_x \tilde{\varphi}_x + N_y \tilde{\varphi}_y + \alpha] +$$
$$+ (\tilde{a}_x t_x + \tilde{a}_y t_y) \frac{2\pi \Delta_{r0}}{L} + \tilde{V} \qquad (2)$$

where the $\tilde{\ }$ means that the canonical variables are the new one. A nice feature of this transformation is that $\tilde{a}_x = \hat{a}_x, \tilde{a}_y = \hat{a}_y$. The parameters t_x, t_y are defining the canonical transformation and are subject to the condition $N_x t_x + N_y t_y = 1$. This means there are infinite canonical transformations to remove the time dependence in the slowly varying Hamiltonian. The coefficients $\rho \geq 0, \alpha$ are the amplitude and phase of the driving term, which is obtained by the Fourier transform of the resonant term of the perturbing Hamiltonian H_1. The factor $4/\epsilon_x$ arises from the particular choice of normalization used to define $\hat{a}_x, \hat{a}_y$. The slowly varying potential $\tilde{V}$ is here only function of the quantities $\tilde{a}_x, \tilde{a}_y$, which have contributions from all nonlinear components in magnets. Essential in the dynamics is the distance from the resonance $\Delta_{r0} = N_x Q_{x0} + N_y Q_{y0} - m$. This expression is not an arbitrary definition, but arises directly solving the dynamics of slowly varying harmonics.

The stationary solution of the canonical equations for the variables $\tilde{a}_x, \tilde{\varphi}_x, \tilde{a}_y, \tilde{\varphi}_y$ is a fixed point alias a "fixed line", which requires a specific condition on the values of the slowly varying coordinates. For example, for the third order normal resonance in absence of space charge it has already been shown that $\tilde{a}_x, \tilde{a}_y$ of any fixed line is given by

$$\tilde{a}_x = \frac{1}{16} \left[\frac{|\Delta_{r0}|\epsilon_x}{R2\rho} \right]^2 (1 - t_x)^2 \qquad (3)$$

$$\tilde{a}_y = \frac{1}{4} \left[\frac{|\Delta_{r0}|\epsilon_x}{R2\rho} \right]^2 t_x(1 - t_x) \qquad (4)$$

with $0 \leq t_x \leq 1$, and R the accelerator average radius. The projection in the physical coordinates x, x', y, y' of a fixed line leads to the shapes shown in Fig. 1. The important aspect, somewhat expected by the intuition, is that the amplitude of the fixed lines is related to the distance from the resonance. The stability of resonances, alias the fixed lines, is given by the secondary frequencies, namely the frequency of oscillations of $\tilde{a}_x, \tilde{a}_y$ around the stationary solution. For imaginary frequencies the fixed line becomes unstable.

A graphic representation of the set of fixed lines is given in Fig. 2. The curve shows the collection of stable and unstable fixed lines. Note that in this computation there is no additional source of amplitude dependent detuning.

RESONANCES AND SPACE CHARGE

Under the ansatz 1) and 2), the transverse space charge potential can be written in the analytic form

$$V_{sc} = \frac{K}{2} \int_0^\infty \frac{\exp[-0.5T(t)] - 1}{(a_0 + t)^{1/2}(b_0 + t)^{1/2}} dt \qquad (5)$$

with $T(t) = x^2/(a_0^2 + t) + y^2/(b_0^2 + t)$. The quantities a_0, b_0 are the rms sizes matched with the optics structure of the circular
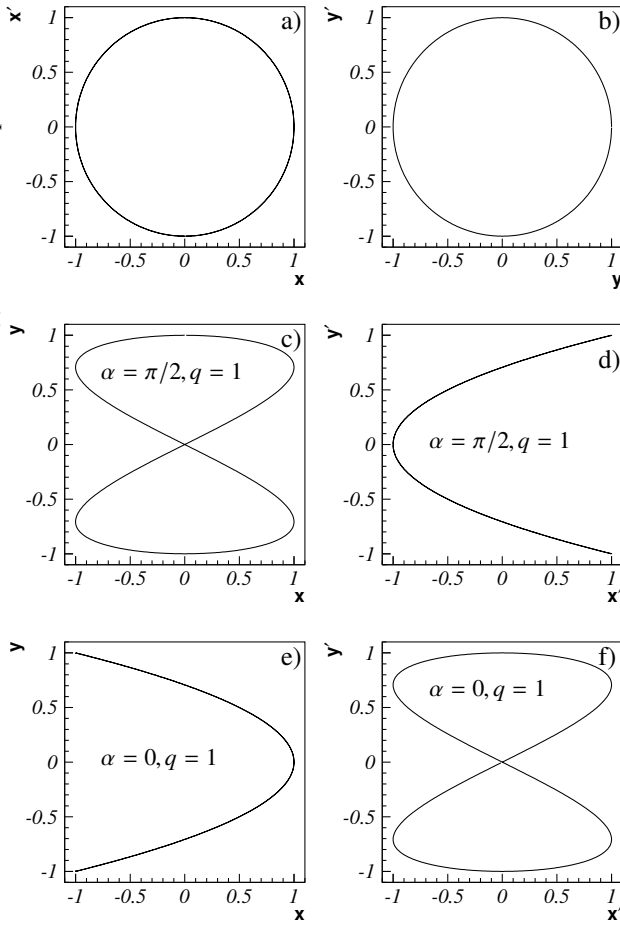
Figure 1: Fixed-line projections in normalized coordinates. The parameter q can be $0, 1$ and its value is taken to allow the existence of the fixed line.

machine, namely $a_0^2 = \beta_x(s)\epsilon_x, b_0^2 = \beta_y(s)\epsilon_y$. The rms sizes a_0, b_0 exhibit the s dependence deriving from the accelerator structure and so does V_{sc}. This potential generates a Gaussian transverse distribution $\propto \exp[-0.5(x^2/a_0^2 + y^2/b_0^2)]$. It can be proven that the "slowly varying potential" generated by space charge can be retrieved from Eq. (5) and has a form $\tilde{V}_{sc}(\tilde{a}_x, \tilde{a}_y)$ function of the normalized action $\tilde{a}_x, \tilde{a}_y$ only.

This potential enters in Eq. (2) and affects the solution of the stationary values of the canonical variables $\tilde{a}_x, \tilde{\varphi}_x, \tilde{a}_y, \tilde{\varphi}_y$.

The Fourier transform of V_{sc} allows to retrieve the driving terms excited by space charge ρ_{sc}, α_{sc}. So one can find the situation that space charge excites a resonance and also creates the amplitude dependent detuning fixing the position of the fixed lines. We do not discuss here this particular case.

The resonances are characterized by the condition $N_x\tilde{\varphi}_x + N_y\tilde{\varphi}_y + \alpha = \pi q$, with q and integer and by the conditions

$$t_x\frac{\Delta_{r0}}{R} + \frac{\partial\tilde{V}}{\partial\tilde{a}_x} \simeq 0 \qquad (6)$$

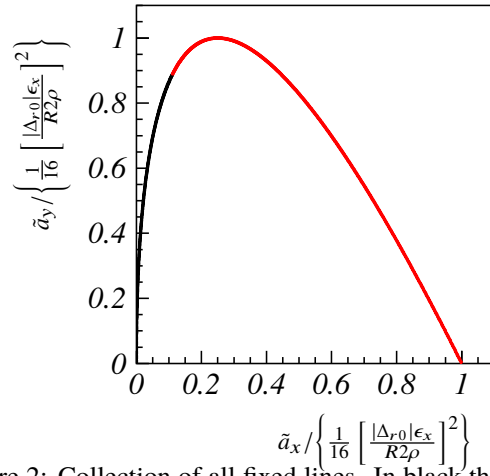$$t_y\frac{\Delta_{r0}}{R} + \frac{\partial\tilde{V}}{\partial\tilde{a}_y} \simeq 0 \qquad (7)$$

Figure 2: Collection of all fixed lines. In black the stable fixed lines, in red the unstable fixed lines.

It can be proven that these two conditions are necessary and sufficient to the condition

$$\frac{\Delta_{r0}}{R} + N_x\frac{\partial\tilde{V}}{\partial\tilde{a}_x} + N_y\frac{\partial\tilde{V}}{\partial\tilde{a}_y} \simeq 0. \qquad (8)$$

In addition it can be proven that $R\frac{\partial\tilde{V}}{\partial\tilde{a}_x} = \Delta Q_x(\tilde{a}_x, \tilde{a}_y)$ is the space charge induced amplitude dependent incoherent detuning. (similarly for the other plane). We therefore find that the condition for reaching a resonance, namely a stationary solution of the canonical equations in the time-independent set of coordinates, is given by

$$\Delta_{r0} + N_x\Delta Q_x(\tilde{a}_x, \tilde{a}_y) + N_y\Delta Q_y(\tilde{a}_x, \tilde{a}_y) \simeq 0. \qquad (9)$$

This result is not an ansatz but a direct consequence of the canonical equations of the slow variables. We can use this result to define an extension of the concept of amplitude dependent detuning with the expression

$$\Delta_r(\tilde{a}_x, \tilde{a}_y) = \Delta_{r0} +$$
$$+N_x\Delta Q_x(\tilde{a}_x, \tilde{a}_y) + N_y\Delta Q_y(\tilde{a}_x, \tilde{a}_y) \qquad (10)$$

which is an *"amplitude dependent resonance detuning"*, now defined for resonances of any order: the resonance condition is met for $\tilde{a}_x, \tilde{a}_y$ such that $\Delta_r \simeq 0$. This result confirms and gives a theoretical ground to the discussion carried out in Ref. [16]. From Eq. (10) it is evident that there is a maximum *resonance detuning* $\mathscr{D}_{r,sc} = N_x\Delta Q_x(0,0)+N_y\Delta Q_y(0,0)$ due to space charge. The quantities $\Delta_r(\tilde{a}_x, \tilde{a}_y)$, $\mathscr{D}_{r,sc}$ determine the location of the resonance in presence of space charge. The symbol $\simeq$ in Eqs. (6), (7) and (8) is used because we have neglected the contribution of the resonant term in the Hamiltonian. This is equivalent to assume that the driving term ρ is small.

Note that Eq. (10) depends on the two normalized actions $\tilde{a}_x, \tilde{a}_y$, and it may seem strange. However this is not the case, in fact, there are infinite fixed lines, each identified by a parameter: for example for the case of the 3rd order
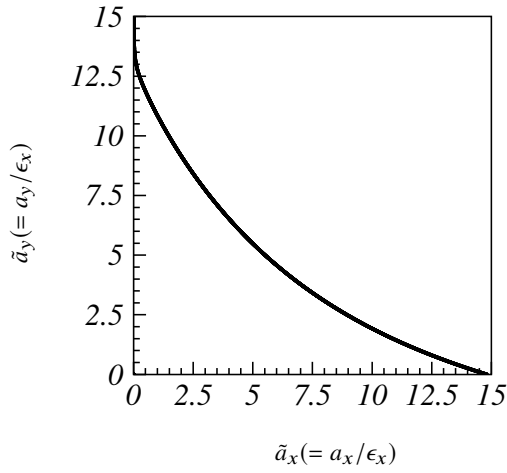
Figure 3: Collection of all fixed lines for a high intensity beam with parameters $\Delta_{r0} = 0.056$, $\mathscr{D}_{r,sc} = -0.174$.

resonance with no space charge the parameter is t_x in the parameterization given by Eqs. (3) and (4). A more general parameterization makes use of the invariant $N_y \tilde{a}_x = N_x \tilde{a}_y + C$ characterizing the dynamics of the slowly varying Hamiltonian. For the third order resonance, fixing t_x is equivalent to fix C. When space charge is included, the parametrization of the fixed lines is more complex and not expressible in a simple analytic formula as in Eqs. (3) and (4). Nevertheless, once C is fixed, it means a specific correlation is chosen in the space of the actions a_x, a_y and the resonance detuning Δ_r defined in Eq. (10) allows to determine the fixed line associated to C.

Therefore for any C the condition $\Delta_r = 0$ allows to estimate the location of the associated fixed line. As an example we show the effect introduced by the space charge on the collection of all the fixed lines shown in Fig. 2 for the case of the third order resonance. The result is shown in Fig. 3 for $\Delta_{r0} = 0.056$, $\mathscr{D}_{r,sc} = -0.174$ corresponding to the settings, beam sizes, and space charge tune-shifts of the CERN-PS measurements in Ref. [16].

A comparison of Fig. 3 with Fig. 2 shows that space charge stabilized all unstable fixed lines, and changed dramatically the shape of the curve in the neighborhood of $\tilde{a}_x = 0$. A direct comparison of the values of $\tilde{a}_x, \tilde{a}_y$ is not possible as the sizes of Fig. 2 scale with Δ_{r0} and go to zero for $\Delta_{r0} \to 0$. Instead in presence of space charge, if $\Delta_{r0} \to 0$ the collection of fixed lines in Fig. 3 becomes larger and larger. However, qualitatively, from the pattern in Fig. 3 we learn that space charge changes the direction of the black curve in Fig. 2 and brings it to the point $\tilde{a}_x = 0$, $\tilde{a}_y \sim [|\Delta_{r0}|\epsilon_x/(R2\rho)]^2/16$.

HIGH ORDER RESONANCES WITH SPACE CHARGE

The theoretical approach here discussed allows the computation of the fixed lines in presence of space charge for resonances of any order and arbitrary strength of space charge (in the range of circular machines). We consider for convenience of demonstration the scenario of Ref. [16]. We

keep the space charge tune-shift so $\mathscr{D}_{r,sc} = -0.174$, and $\Delta_{r0} = 0.056$. This is reached by changing the machine tunes Q_{x0}, Q_{y0} and setting it above the resonance of choice, that is we require $0 \le -\Delta_{r0}/\mathscr{D}_{r,sc} \le 1$.

In Fig. 4 we show two examples of high order resonances as Poincaré surface of section for fixed lines defined by the parameter $C = 0$. In Fig. 4 top we show $x - y$ projection of the fixed line for the 4th order skew resonance $Q_x + 3Q_y = N$, the strength of the octupole has been artificially enhanced so to enable to the nonlinear tracking the resolution of the resonance. Black dots are the result from the analytic theory; red the dots are the particle positions obtained from tracking turn after turn. The bottom picture shows the 7th order normal resonance $5Q_x + 2Q_y = N$. For sake of comparison we also show in Fig. 5 the fixed line for $Q_x + 3Q_y = N$, but $C = 20$. Comparing Fig. 4 top with Fig. 5 it is visible the change in aspect ratio of the fixed-line according to the value of C. However note: the topology of the two curves remain the same.

SUMMARY

In this proceeding we have shortly presented the theory of resonances with space charge. We have shown that the
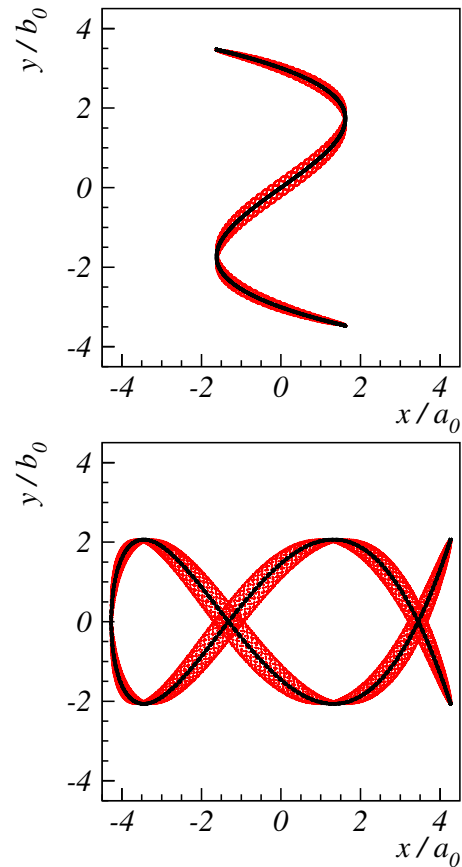

Figure 4: Fixed lines type $C = 0$ in a high intensity beam with $\Delta_{r0} = 0.056$, $\mathscr{D}_{r,sc} = -0.174$. On the top picture the resonance is $Q_x + 3Q_y = N$, bottom picture for $5Q_x + 2Q_y = N$.
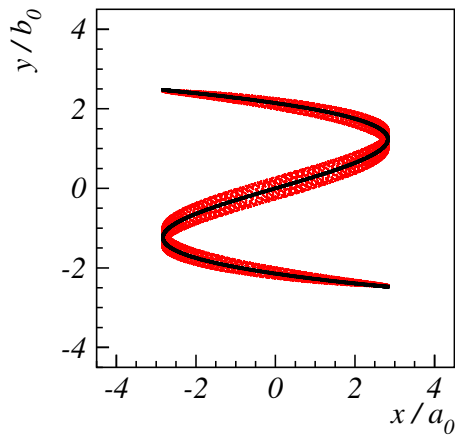
Figure 5: Fixed line for $C = 20$ for a high intensity beam with parameters $\Delta_{r0} = 0.056$, $\mathscr{D}_{r,sc} = -0.174$.

prediction of resonant structures in phase space is possible under a general ansatz. We confirm from a more solid theoretical ground the intuitive approach used in previous work making use of the *"resonance detuning"*. The complexity of the dynamics remains considerable, but in spite of this an analogy with the one dimensional treatment of resonances and space charge is possible. The amplitude dependent detuning can be generalized with the *amplitude dependent "resonance detuning"* $\Delta_r(\tilde{a}_x, \tilde{a}_y)$ and used as a tool for predicting the location of fixed lines

The comparison of particle tracking with the predictions of the theory shows good agreement (Fig. 4, Fig. 5), and this encourages to consider this approach useful to reach quick results to the problem of the periodic resonance crossing induced by high intensity bunched beams stored for long term. The material here presented does not allow a complete discussion of the consequences of the theory and its limits. This subject will be part of a future publication.

ACKNOWLEDGMENT

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