TYPOLOGY OF SPACE CHARGE RESONANCES

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Abstract

The existence of structural space charge resonant effects in otherwise linear periodic focusing systems is well-known, but referred to in a variety of languages and contexts. We show here that for short bunched beams a "classification" in two major groups is possible, e.g. parametric resonances or instabilities on the one hand and single particle type space charge resonances on the other hand. The primary feature of distinction is that for the former the driving space charge force initially exists on the noise level (rms or higher order mismatch) only and gets amplified parametrically, hence an entirely coherent response; for the latter the driving space charge multipole is part of the initial density profile and the coherent response is weak. In the extreme limit of KV beams only parametric resonances (instabilities) exist, and in principle in all orders. For waterbag or Gaussian distributions we find half-integer parametric resonances only up to fourth order, but evidence for single particle resonances in all orders up to tenth have been identified.

INTRODUCTION

With advancing demands on the control of space charge effects for beam dynamics of both linear and circular high intensity accelerators the appearance of purely space charge driven resonances merits careful consideration. The present study considers this space charge case in an otherwise linear periodic focusing lattice. It is based on a recent analysis of the so-called ninety degree and other structural space charge driven stopbands, where it was found that a distinction between single particle type resonant effects and instabilities - here also called parametric resonances - is fundamental [1].

The analytical basis for resonant space charge phenomena in periodic focusing was given by a perturbational Vlasov analysis of structural instabilities of different order for a Kapchinskij-Vladimirskij (KV) distribution of a coasting beam [2]. Our present examples show that this earlier work - though derived for the special case of a 2d KV beam, and under the constraints of a perturbation theory, is still highly relevant and a key to differentiating types of resonant behaviour.

Experimental investigation of this stop-band was undertaken only in 2009, in a dedicated experiment at the UNILAC high intensity heavy ion linear accelerator [3]. This experiment gave evidence of a fourth order resonance as suggested already in an earlier simulation study for a periodic lattice [4], and no indication of a simultaneously occurring envelope instability was found. However, a recent study has shown that the matter is more complex and not independent of the length of the system and the initial mismatch [1]. In particular, the claim of Ref. [5] that the envelope mode is suppressed by the appearance of a fourth order resonance is not supported by our findings. Likewise, we cannot confirm a more recent interpretation that the envelope mode is induced my a mismatch induced by the fourth order resonance [6], which ignores the independence of these modes.

Note that our examples are related to short bunches, where the synchrotron period is not very different from betatron periods; in circular accelerators the synchrotron period is usually very long, which requires special consideration due to possibly different mixing effects. The suggested typology is, however, still applicable.

Resonances driven by space charge in combination with emittance exchange - so-called non-equipartitioned beams [7, 8] - are not part of the present study. They are driven by beam anisotropy rather than the periodic focusing, which leads to a related typology including single particle resonances and anisotropy driven instabilities.

RESONANT PARTICLES AND COHERENT MODES

Single Particle Resonances

The commonly considered resonances in accelerators are based on external forces periodically acting on particles. The origin of these forces usually are systematic and error multipoles of magnets, which provide the driving terms for the resonance.

In a linear lattice with non-uniform space charge density similar driving terms can be given, if one expands the space charge potential in so-called space charge "pseudo-multipoles", which particles cannot distinguish from external forces provided that the space charge terms are well-matched and follow the lattice periodicity.

Resonances in such well-matched beams will be called "single-particle" or "incoherent resonances" reflecting the fact that coherent motion does not affect the resonance condition. Using linac notation the corresponding resonance condition in the $x-x'$-plane (similar in the other planes) can be written as

$$mk_x = 360^0, \tag{1}$$

with $k_x \equiv k_{0,x} - \Delta k_{x,inc}$ the phase advance with space charge, and $k_{0,x}$ without. Note that applied to circular accelerators, these quantities would have to relate to a periodic structure cell. Here, we assume $\Delta k_{x,inc}$ (> 0) is an rms average of incoherent tune shifts and ignore possible spreads depending on amplitudes.

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Parametric Resonances

On the other hand, in beams also parametric resonances exist, which are not due to external forces from magnets or space charge, but a system parameter is periodically modulated and drives the resonance.

An example of parametric resonance in accelerators is the single-particle "Mathieu stopband", which prevents single particle transport in periodic focusing and leads to a runaway effect, if the phase advance is 180^0 per focusing period. This example is a special case of the well-known behaviour of a linear differential equation of Hill's type, $d^2x/dt^2 + (\omega^2 + f(t))x = 0$, where ω is the eigenfrequency of the free oscillation and $f(t)$ a parameter of the oscillating system varying periodically with ω_0. The parametric excitation leads to exponential instability of an arbitrarily small initial perturbation, if

$$\omega = \frac{n}{2}\omega_0, \tag{2}$$

where n is a positive integer. The widest stopband is the "half-integer" case, $n = 1$, often called "parametric instability" or "sub-harmonic instability" [9].

Here we are interested in the case, where not a single particle is driven parametrically, but a *coherent eigenmode* of a beam. The lowest order phenomenon is the envelope instability [4, 10], where a quadrupolar or envelope eigenmode is driven parametrically.

The corresponding coherent resonance condition for a parametric resonance is written with ω relating to the frequency of a coherent eigenmode of the beam. The order of the motion in the $x - x'$plane is described by m as in Eq. (4), but - in principle - an additional parameter n describing the parametric order according to Eq. (2) is needed. Thus, the resonance condition for a coherent eigenmode in the $x - x'$-plane can be written as:

$$\omega \equiv mk_{0,x} - \Delta k_{m,coh} = \frac{n}{2}360^0. \tag{3}$$

Note that $\Delta k_{m,coh}$ stands symbolically for a *coherent* space charge tune shift to express the fact that there is an underlying coherent motion, which leads to a shift of this resonance with respect to the incoherent one. In most cases it is positive, but negative values cannot be excluded (Ref. [2] and in smooth approximation Ref. [11]). The shift is comparable in size with the incoherent space charge tune shift and depends on the specific eigenmode, hence on m, and possibly on other parameters as well. The shift reflects the fact that coherent and incoherent resonances differ in nature and can be displaced. According to Ref. [2] one needs to take into account that in practice there is also a finite width of stopbands in addition to a shift.

A well-known example of half-integer parametric resonance is the transverse envelope instability of an ellipsoidal bunch in a symmetric periodic FODO array of quadrupoles. We assume a longitudinal focusing provided by two thin rf gaps in the center of both drift spaces in each cell and employ the (3d) KV-envelope equation option of the TRACEWIN code [12]. Parameters are set within a stop-band of instability by assuming $k_{0,x,y} = 100^0$, a moderate space charge leading to $k_{x,y} = 82^0$, and the longitudinal focusing arbitrarily set to $k_{0,z} = 50^0$.

In the above nomenclature the envelope parametric instability case is written as $\omega = 2k_{0,x,y} - \Delta k_{2,coh} = \frac{1}{2}360^0 = 180^0$ and shown in Fig. 1, where the initial exponential phase is followed by a more chaotic pattern of saturation, damping and growth. Details of the lattice and initial envelope are shown in the insert. Note that the three rms emittances are initially chosen equal, which results in bunches slightly elongated from spherical. A necessary condition for instability

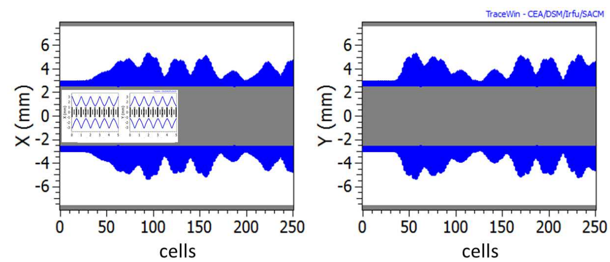

Figure 1: Evolution of KV-envelopes versus cell number for $k_{0,x,y} = 100^0$, $k_{x,y} = 82^0$.

of this mode is $k_{0,x} > 90^0$, while k_x is below 90^0, and similar for y. This enclosure of 90^0 justifies the nomenclature 90^0-stopband.

Higher Order Parametric Instabilities

Here we examine the possibility of higher than $m = 2$ parametric resonances. A third order parametric instability with $m = 3$ and $n = 1$, hence $3k_{0,x,y} - \Delta k_{3,coh} = 180^0$ is expected for a 60^0 stopband in full agreement with the analytical theory of Ref. [2]. Note that a beam symmetric in x and y has no space charge multipole to drive a third order resonance phenomenon, except by parametric instability.

The case is retrieved by our 3d simulations for the example $k_{0,x,y} = 90^0$, and found to exist over the same range of $k_{x,y}$ as predicted in Ref. [2]. For a waterbag distribution and $k_{x,y} = 41.5^0$ the result is shown in Fig. 2. The roughly doubling of rms emittances is accompanied by a three-fold phase space structure as shown on the cell 45 phase space plot. The "triangles" repeat their orientation every second lattice cell, which confirms the 180^0 condition. The growth of the rms emittance in the exponential phase is preceded by a weak sixth order single particle (incoherent) resonance - following the notation $6k_{x,y} = 360^0$ as indicated by the phase space symmetry insert at cell 5 - but quickly overtaken by the lower order parametric case.

This combined third and sixth order phenomenon is similarly encountered in an analogous interplay of second and fourth order as will be discussed in Section .

We also confirm the additional existence of the *half-integer* parametric instability $4k_{0,x,y} - \Delta k_{4,coh} = 180^0$ - also predicted in Ref. [2] - with a stopband near 45^0. We choose $k_{0,x,y} = 70^0$ and take an intensity corresponding to

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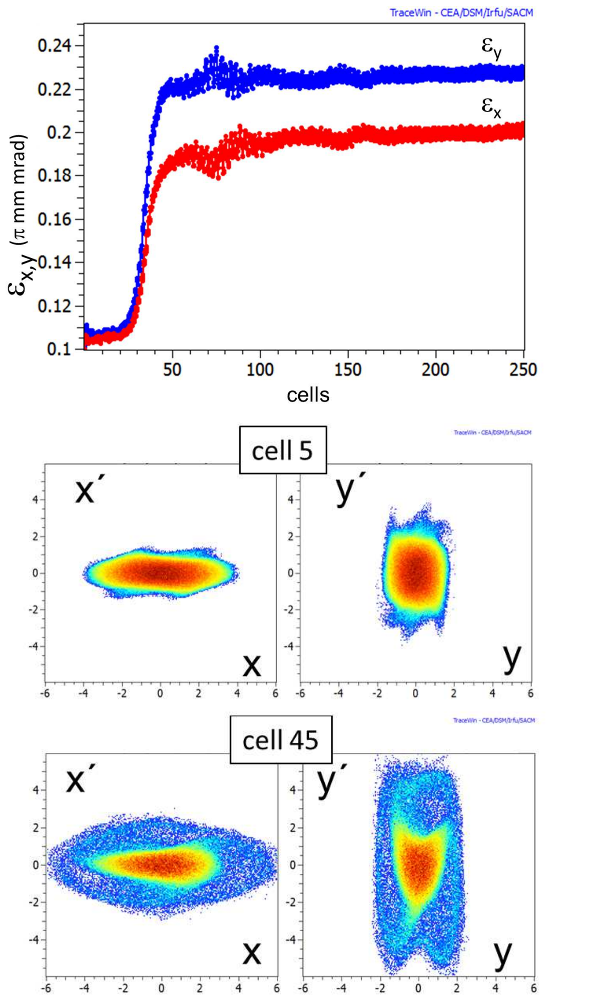

Figure 2: Top graph: rms emittances versus cell number for $k_{0,x,y} = 90^0$, $k_{x,y} = 41.5^0$ for a waterbag distribution. Bottom graph: phase space plots.

$k_{x,y} = 35^0$. For $k_{0,z} = 50^0$ ($k_z = 17^0$) as before we find a weak evidence of this mode, with only 4% emittance growth. However, by raising $k_{0,z}$ to 120^0 we obtain a 40% emittance growth as is shown in Fig. 3, with the phase space insert at cell 20 confirming the fourth order structure. We assume that the roughly 5 times faster effective synchrotron oscillation plays a role and possibly reduces the transverse Landau damping effect, which needs additional study.

Using KV-beams we have also found higher order parametric cases with relatively small emittance effects, but no such evidence was found for waterbag beams. We thus conclude that parametric resonances are insignificant beyond fourth order - at least in the range of parameters studied here.

Ninety Degrees Stopband

We start with a Gaussian distribution truncated at 3.4 σ, transverse tune of $k_{0x,y} = 120^0$ and a space charge depressed tune $k_{x,y} = 70^0$. As shown in Fig. 4 a fourth order phase space structure evolves quickly with 90% of rms emittance growth in less than 10 cells. This can be identified as a single

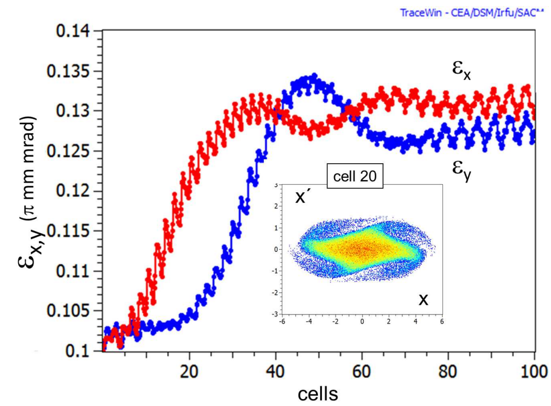

Figure 3: Rms emittances versus cell number for paramteric fourth order mode at $k_{0,x,y} = 70^0$, $k_{x,y} = 35^0$ and waterbag distribution.

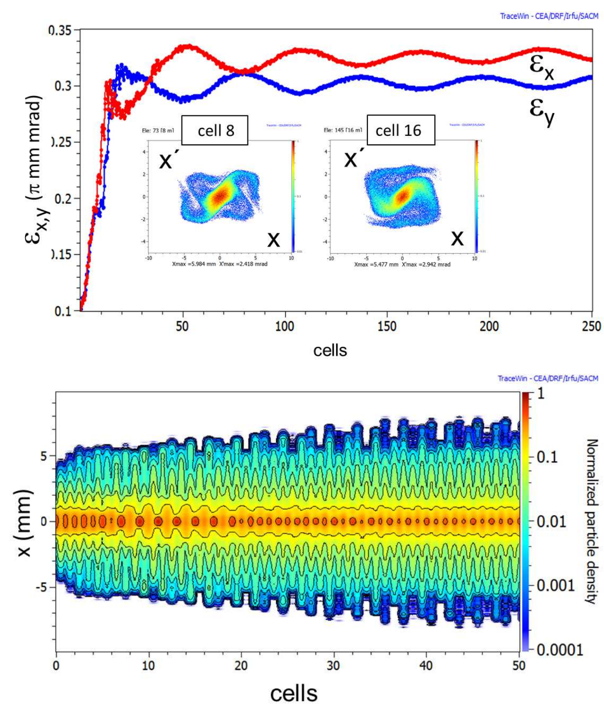

Figure 4: Top graph: rms emittances versus cell number for $k_{0,x,y} = 120^0$, $k_{x,y} = 70^0$ for Gaussian distribution with inserts showing phase space plots. Bottom graph: evolution of density (with contour lines) in x along the lattice.

particle resonance owed to the presence of a strong space charge octupole in the initial Gaussian distribution.

The subsequent evolution to a total growth of over 200% is characterized by a sudden switch to the envelope instability with a dominating two-fold symmetry as shown in the insert at cell 16. The evidence of a growing envelope instability is owed to the fact that the envelope instability is an independent eigenmode, which grows exponentially from the rms mismatch of the initial beam - however small it is. The recently presented argument of Ref. [6] that it grows from a mismatch induced by the preceding fourth

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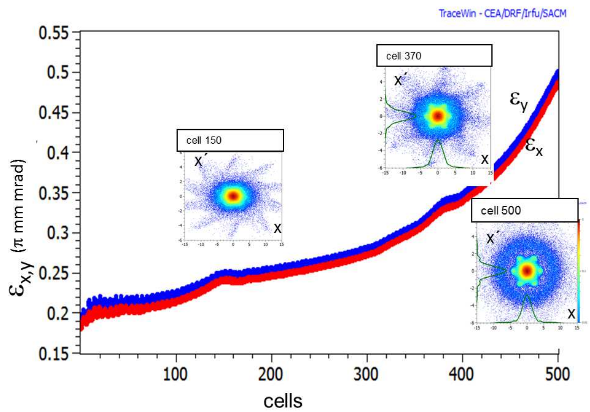

Figure 5: Rrms emittances versus cell number for dynamical ramp of $k_{0,x,y} = 150^0 \rightarrow 130^0$ for Gaussian distribution with inserts showing phase space plots.

order resonance contradicts the fact that the envelope equations - where fourth order effects are entirely absent - already reveal the 90 degree stopband. The extent and shape of it is fully consistent with the PIC simulation result as was shown in Ref. [1].

Results for the evolution of the density in x along the lattice are shown in Fig. 4. They indicate an initial incoherent growth due to the fourth order single particle resonance. A coherent motion takes over between cells 8-10, and a gradual return to more incoherent motion again after 20 cells. The contour lines of peak density in the centre emphasize this transfer of incoherent motion well matched to the lattice to a coherent one, where two cells are needed to perform one period of the strongly excited envelope oscillation.

This coherence is a characteristic of the half-integer parametric resonance as opposed to the incoherent behaviour at single particle resonances.

Higher Order Lattice Harmonics

Above the 90^0 stopband we have not found any evidence of parametric resonances. Theoretically, a 120^0 integer parametric stopband of a third order resonance $3k_{0,x,y} - \Delta k_{3,coh} = 360^0$ could exist, but it must be assumed to be much weaker than the half-integer parametric resonance found above at 60^0.

Instead, a slow ramp of $k_{0,x,y}$ from $150^0 \rightarrow 130^0$ shows that a number of space charge driven single particle resonances exist as shown in Fig. 5. They require, however, higher harmonics h of the lattice function (with space charge) according to

$$mk_{x,y} = 360^0 h. \tag{4}$$

Thus, resonances of 10th, 8th and 6th order are crossed sequentially for $k_{x,y} = 144^0, 135^0, 120^0$ pertaining to $h = 2, 3, 4$. For better identification of the resonant structure and order we have seeded the initial Gaussian distribution with an initial halo seed of 10% of the particles spread out to an rms emittance enhanced by a factor of nine.

	Single particle	Parametric
Driving term	Incoherent (matched) space charge distribution	Coherent space charge
Source	Non-uniform density	Eigenmode
Linac resonance condition	$mk_{xy}=360^0$ h h: lattice harmonic	$mk_{0xy}-\Delta k_{m,coh}$ $=n/2\ 360^0$
Ring resonance condition	$mQ_{xy}=N$ h N: superperiods	$mQ_{0xy}-\Delta Q_{m,coh}$ $=n/2$ N
Range of values identified	m=2...10 h=1...4	m=2,3,4 n=1 (half-integer) h=1

Figure 6: Typology of structural space charge resonances.

SUMMARY ON TYPOLOGY

We have identified two groups of structural resonant space charge effects:

- *Single particle resonances:* The driving space charge term (pseudo-multipole) stems from the matched initial distribution. In the course of the resonance the driving term may change and/or de-tuning may occur. Higher order harmonics of the focusing play a role above the 90^0 stopband. In an initial KV-type distribution space charge multipoles are absent beyond second order, hence the corresponding single particle resonances are also absent.
- *Parametric resonances:* Here the driving space charge term is pumped parametrically from initial noise, if the resonance condition is fulfilled. For non-KV beams we practically only find *half-integer* cases of parametric resonances, and up to fourth order. For initial KV-type distributions, instead, all orders of parametric resonances exist in theory [2] - in simulation limited by resolution.

The distinction single particle versus parametric resonances is not always sharp. Assume a distribution close to a KV, but with weakly non-uniform density. In such a case one can still expect single particle resonances, but a parametric enhancement of the correlated pseudo-multipole as well.

Three parameters characterize the different types of structure space charge resonances (see Fig. 6): m stands for the order of the resonance (space charge potential term order); n for the order in the parametric driving mechanism; h the lattice harmonic, which is theoretically relevant for both, single particle and parametric resonance, but we have not found values different from unity for the latter.

CONCLUSION

This work shows that in high intensity beams two major groups of structural space charge resonant effects exist. They are distinguished by whether the space charge driving term is present in the initial beam, or built up from noise by a parametric process. The latter has no counterpart in externally driven resonances and is of the so-called half-integer parametric resonance or instability type.

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A common feature at different orders is the joint appearance of a parametric resonance and a single particle resonance of twice the order. As an example, the envelope instability and fourth order single particle resonance are independent mechanisms - but not without influencing each other. The associated ninety degree stopband is of particular concern in high current linacs.

The role of the third - here longitudinal - dimension with the synchrotron oscillation as additional mechanism possibly influencing Landau damping must be addressed carefully in future work.

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