GENERAL FORMULA TO DEDUCE THE SPACE CHARGE TUNE SPREAD FROM A QUADRUPOLAR PICK-UP MEASUREMENT

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Abstract

In 1966, W. Hardt derived the oscillation frequencies obtained in the presence of space charge forces and gradients errors for elliptical beams. Since then, a simple formula is usually used to relate the shift of the quadrupolar mode (obtained from the quadrupolar pick-up) and the space charge tune spread, depending only on the ratio between the two transverse equilibrium beam sizes. However, this formula is not always valid, in particular for machines running close to the coupling resonance $Q_x = Q_y$ with almost round beams. A new general formula is presented, giving the space charge tune spread as a function of i) the measured shift of the quadrupolar mode, ii) the ratio between the two transverse equilibrium beam sizes and iii) the distance between the two transverse tunes.

INTRODUCTION

The incoherent direct space charge tune spread is a fundamental parameter in the beam dynamics of high-intensity high-brightness beams but most of the time it is only computed analytically or simulated. It would be good to be able to measure it in running machines, which is possible with quadrupolar pick-ups by looking at the shift of the quadrupolar mode with intensity (note that there is no shift of the dipole mode with intensity due to the direct space charge as the latter follows the evolution of the beam centre and does not modify its motion). Since the derivation from W. Hardt of the oscillation frequencies obtained in the presence of space charge forces and gradients errors for elliptical beams [1], a simple formula is usually used to relate the (horizontal) space charge tune spread to the (horizontal) shift of the quadrupolar mode due to intensity, which depends only on the ratio between the equilibrium rms vertical beam size σ_{y0} and the equilibrium rms horizontal beam size σ_{x0} [2,3,4]

$$\Delta Q^{SC}_{x,\text{spread}} = \frac{2Q_{x0} - Q_{2x}}{\frac{1}{2}\left(3 - \frac{1}{1+\frac{\sigma_{y0}}{\sigma_{x0}}}\right)} , \qquad (1)$$

where 2 Q_{x0} is the low-intensity quadrupolar tune and Q_{2x} is the intensity-dependent quadrupolar tune.

However, Eq. (1) is not always valid and it corresponds to the case when the coupling between the two transverse planes, introduced by space charge, is neglected. This formula is in particular not valid for machines running close to the coupling resonance $Q_x = Q_y$ with almost round beams, which is the case of many machines (and in particular of the CERN LHC injectors where we plan to measure the space charge tune spread using quadrupolar pick-ups) and the purpose of this paper is to provide the more general formula which depends also on the distance between the two transverse tunes [5]. Note that the extreme cases of a small or large tune split were already discussed in Ref. [6] for the case of a round beam.

The (2D) transverse envelope equations are first reviewed in Section 1, as well as the coupled equations to be solved in the presence of small perturbations on top of equilibrium beam sizes. The usual Eq. (1) is then recovered in Section 2 in the uncoupled case. The new formula providing the space charge tune spread in the general case (i.e. also close to the coupling resonance) is finally derived and discussed in Section 4.

TRANSVERSE ENVELOPE EQUATIONS

The (2D) transverse envelope equations are now well-known and used [7,8] in particular since the work of Sacherer [9] who showed that the envelope equations derived by Kapchinsky and Vladimirsky (known as the KV equations) [10] for a continuous beam with uniform charge density and elliptical cross-section are also valid for general beam distributions if one considers the second moments only. Considering a particle in an ensemble of particles which obeys the single-particle equations, adding the space charge force to the external (linearized) force and averaging over the particle distribution, the equations of motion for the centre of mass can be obtained (note that due to Newton's third law the average of the space charge force is zero). Looking at the second moments and in particular at the position and momentum offsets of the particles from their respective averages, the 2D transverse envelope equations can finally be obtained [7,8]

$$a'' + K_x a - \frac{2K_{sc}}{a+b} - \frac{\varepsilon_x^2}{a^3} = 0 ,$$
$$b'' + K_y b - \frac{2K_{sc}}{a+b} - \frac{\varepsilon_y^2}{b^3} = 0 , \qquad (2)$$

with

$$a = 2\sigma_x , \qquad b = 2\sigma_y , \qquad (3)$$
$$\varepsilon_x = 4\varepsilon_{x,rms} , \qquad \varepsilon_y = 4\varepsilon_{y,rms} , \qquad (4)$$

where $'$ stands for the derivative with respect to the azimuthal coordinate s, $\sigma_{x,y}$ are the transverse rms beam sizes, $K_{x,y}$ describe the transverse external forces, $\varepsilon_{x,y,rms}$ are the transverse rms beam emittances and K_{sc} is a coef-

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ficient proportional to the horizontal space charge tune spread (considering that the space charge tune spread extends from the low intensity tune to the tune with maximum space charge tune shift) through (in the smooth approximation) [11]

$$\Delta Q^{SC}_{x,\,\mathrm{spread}} = -\,\Delta Q^{SC}_{x,\,\mathrm{linear\ shift}} = \frac{K_{sc}\,R^2}{Q_{x0}\,a_0\left(a_0+b_0\right)}\,, \qquad (5)$$

where R is the average machine radius, Q_{x0} is the low-intensity horizontal tune and a_0 and b_0 are the horizontal and vertical equilibrium beam sizes (obtained from Eq. (2) when the terms with the derivative are zero). Both transverse planes have thus to be treated jointly for high-intensity beams due to the space-charge coupling.

The beam may execute some collective motion on top of the equilibrium beam sizes

$$a(s) = a_0 - \Delta a(s) \quad \text{and} \quad b(s) = b_0 + \Delta b(s)\,, \qquad (6)$$

where the perturbations Δa and Δb are considered small with respect to the equilibrium beam sizes. Linearizing the equations yields

$$\begin{aligned} \Delta a'' + K_a\,\Delta a &= K\,\Delta b\,, \\ \Delta b'' + K_b\,\Delta b &= K\,\Delta a\,, \end{aligned} \qquad (7)$$

with

$$K_a = 4K_x - \frac{2K_{sc}\left(2a_0+3b_0\right)}{a_0\left(a_0+b_0\right)^2}\,, \qquad (8)$$

$$K_b = 4K_y - \frac{2K_{sc}\left(2b_0+3a_0\right)}{b_0\left(a_0+b_0\right)^2}\,, \qquad (9)$$

$$K = \frac{2K_{sc}}{\left(a_0+b_0\right)^2}\,. \qquad (10)$$

Using the smooth approximation

$$K_x = \left(Q_{x0}/R\right)^2\,, \qquad K_a = \left(Q_a/R\right)^2\,, \qquad (11)$$

$$K_y = \left(Q_{y0}/R\right)^2\,, \qquad K_b = \left(Q_b/R\right)^2\,, \qquad (12)$$

and assuming small tune shifts, yields

$$\begin{aligned} Q_a &= 2Q_{x0} + \Delta Q_a = 2Q_{x0} - \frac{K_{sc}\,R^2\left(2a_0+3b_0\right)}{2Q_{x0}\,a_0\left(a_0+b_0\right)^2}\,, \\ Q_b &= 2Q_{y0} + \Delta Q_b = 2Q_{y0} - \frac{K_{sc}\,R^2\left(2b_0+3a_0\right)}{2Q_{y0}\,b_0\left(a_0+b_0\right)^2}\,. \end{aligned} \qquad (13)$$

The coupled equations can finally be re-written

$$\begin{aligned} \frac{d^2\Delta a}{d\phi^2} + Q_a^2\,\Delta a &= K\,R^2\,\Delta b\,, \\ \frac{d^2\Delta b}{d\phi^2} + Q_b^2\,\Delta b &= K\,R^2\,\Delta a\,, \end{aligned} \qquad (14)$$

with $\phi = \Omega_0\,t$, where Ω_0 is the angular revolution frequency and t the time.

FAR FROM THE COUPLING RESONANCE INDUCED BY SPACE CHARGE

Far from the coupling resonance $Q_a = Q_b$, the two equations of Eq. (14) can be considered uncoupled and the solutions of the homogeneous equations are given by

$$\Delta a = \Delta a_0\,e^{jQ_a\phi} \quad \text{and} \quad \Delta b = \Delta b_0\,e^{jQ_b\phi}\,. \qquad (15)$$

Starting from the definition of Q_a in Eq. (13) and expressing K_{sc} with respect to the space charge tune spread (using Eq. (5)), Eq. (1) can be recovered, where Q_a is noted there Q_{2x}. Figure 1 shows how the relation between the space charge tune spread and the measured quadrupolar tune shift varies with the ratio of the transverse equilibrium beam sizes.

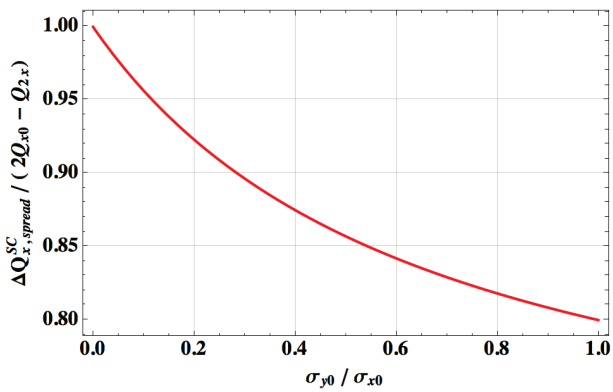

Figure 1: Ratio between the space charge tune spread and the measured quadrupolar tune shift with respect to the ratio of the transverse equilibrium beam sizes.

CLOSE TO THE COUPLING RESONANCE INDUCED BY SPACE CHARGE

Close to the coupling resonance $Q_a = Q_b$, the solutions of the coupled equations of Eq. (14) are a bit more involved. The coupled oscillations can be solved by searching the normal (i.e. decoupled) modes (u,v) linked by a simple rotation (see also Fig. 2)

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$$\begin{pmatrix} \Delta a \\ \Delta b \end{pmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} , \quad (16)$$

where α is the coupling angle (equal to 0 in the absence if coupling and to ± 45 deg in the presence of full coupling).

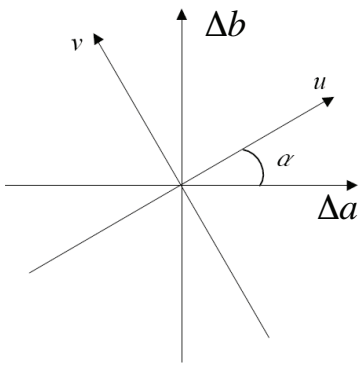

Figure 2: Tilted normal modes due to coupling.

The equations of the two normal modes are given by

$$\frac{d^2u}{d\phi^2} + Q_u^2 u = 0 , \qquad \frac{d^2v}{d\phi^2} + Q_v^2 v = 0 , \quad (17)$$

with

$$Q_u = Q_a - \frac{|C|}{2}\tan\alpha , \qquad Q_v = Q_b + \frac{|C|}{2}\tan\alpha , \quad (18)$$

$$\tan(2\alpha) = \frac{|C|}{\Delta} , \qquad |C| = \frac{R^2 K}{Q_0} , \quad (19)$$

$$\Delta = Q_b - Q_a , \qquad Q_0 = (Q_{x0} + Q_{y0})/2 . \quad (20)$$

Using the fact that

$$\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha} , \quad (21)$$

$$\frac{|C|}{2}\tan\alpha = \frac{1}{2}\left(-\Delta \mp \sqrt{\Delta^2 + C^2}\right) , \quad (22)$$

and therefore Eq. (18) can be re-written

$$\begin{aligned} Q_u &= Q_a - \frac{1}{2}\left(-\Delta \mp \sqrt{\Delta^2 + C^2}\right) , \\ Q_v &= Q_b + \frac{1}{2}\left(-\Delta \mp \sqrt{\Delta^2 + C^2}\right) , \end{aligned} \quad (23)$$

where the ± sign depends on the sign of Δ (it should be the same sign as the one of Δ).

The new formula can then be deduced and it is given by

$$\Delta Q_{x,\mathrm{spread}}^{SC,new} = \frac{q}{\frac{1}{2}\left(3 - \frac{1}{1+x}\right)} - \frac{-\Delta \mp \sqrt{\Delta^2 + C^2}}{3 - \frac{1}{1+x}} , \quad (24)$$

with

$$q = 2Q_{x0} - Q_u , \quad x = \frac{\sigma_{y0}}{\sigma_{x0}} \quad \text{and} \quad y = Q_{y0} - Q_{x0} , \quad (25)$$

$$\Delta = 2y + \frac{3\,\Delta Q_{x,\mathrm{spread}}^{SC,new}}{2}\left(1 - \frac{1}{x}\right) , \quad (26)$$

$$|C| = \frac{2\,\Delta Q_{x,\mathrm{spread}}^{SC,new}}{1+x} . \quad (27)$$

The observable from the quadrupolar pick-up (the quadrupolar tune shift) is q and the first part of Eq. (24) is the usual formula (see Eq. (1)). Solving Eq. (24) yields the new general formula giving the horizontal space charge tune spread as a function of the ratio between the vertical and horizontal equilibrium beam sizes (x), the distance between the transverse tunes (y) and the measured quadrupolar tune shift (q) [5]

$$\Delta Q_{x,\mathrm{spread}}^{SC,new}(x,y,q) = \frac{1}{6+9x+6x^2}\Big[q\left(3+7x+7x^2+3x^3\right) + 4xy + 10x^2y + 6x^3y \mp (1+x)\sqrt{q^2\left(9-2x^2+9x^4\right) + 4qx\left(-6-x+6x^2+9x^3\right)y + 4x^2\left(2+3x\right)^2y^2}\Big] \quad (28)$$

with the – sign when Δ > 0 (and the + sign when Δ < 0). The ratio between the new formula from Eq. (28) and the usual formula from Eq. (1) is plotted in Fig. 3 for the example case q = 0.4, where it can be seen that the usual formula should not work for the machines running close to the coupling resonance with almost round beams, as the CERN LHC injectors. More detailed (2D) plots are shown in Fig. 4 for different parameters, revealing that a difference up to a factor ~ 1.5-2 can be reached in some cases.

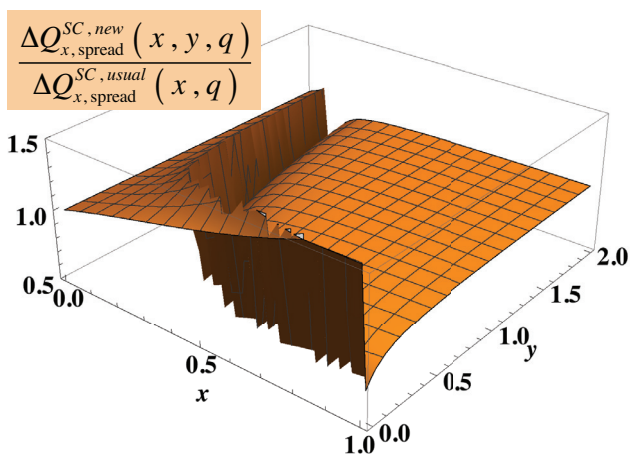

Figure 3: Ratio between the new formula from Eq. (28) and the usual formula from Eq. (1) as a function of x and y, for the example case q = 0.4 [5].

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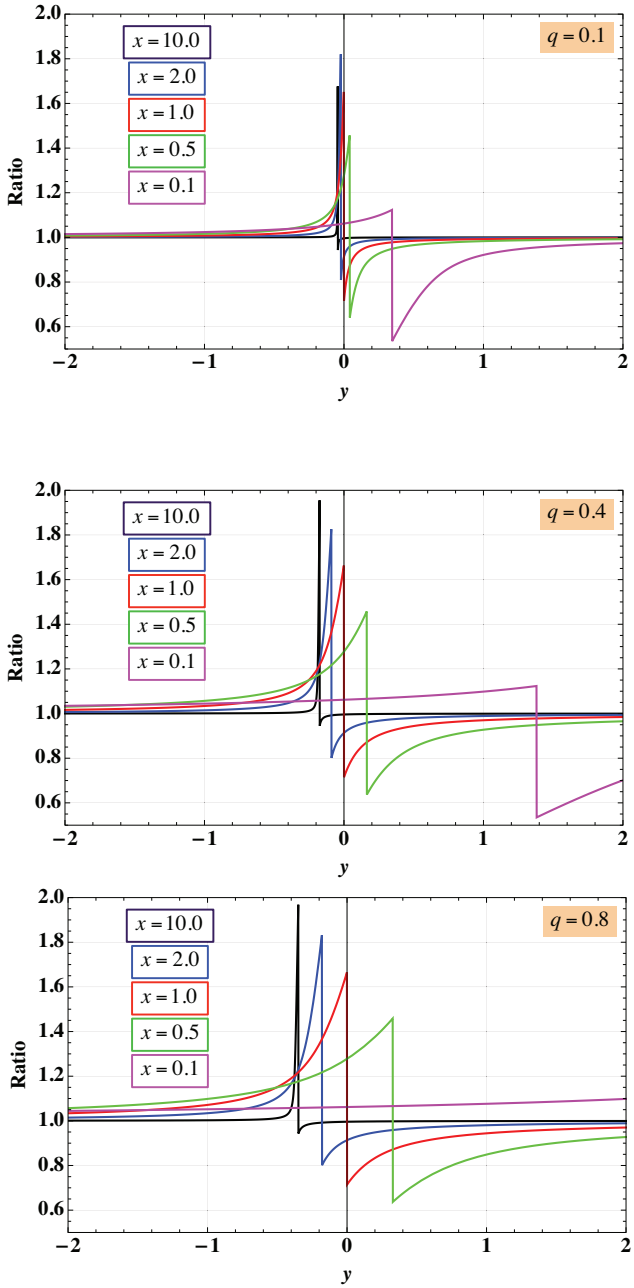

Figure 4: Ratio between the new formula from Eq. (28) and the usual formula from Eq. (1) as a function of y for different values of x: (a) for q = 0.1, (b) for q = 0.4 and (c) for q = 0.8.

CONCLUSION

A new general formula, giving the link between the incoherent direct space charge tune spread and the measured intensity-dependent shift of the quadrupolar mode, has been derived taking into the account the coupling between the transverse planes introduced by space charge and which is neglected in the usual formula (see Eq. (1)). This more involved formula is given in Eq. (28) and depends also on the tune distance between the low-intensity transverse tunes. The ratio between the new and the usual formula is plotted in Fig. 3 as a function of the ratio between the vertical and horizontal equilibrium beam sizes and the distance between the transverse tunes, for the example case of a measured quadrupolar tune shift of 0.4. Some example cases are also shown in Fig. 4 on 2D plots, where it can be seen that differences from the usual formula can be as large as a factor ~ 1.5-2. These results should be checked by simulations and beam-based measurements in the running machines.

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