NIU
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A New Paradigm for Modeling, Simulation, and Analysis of Intense Beams

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Overview

- Purpose is to create a transfer map which includes the effects of space charge.
- It is done self consistently.
- It can be used for fitting and optimization.
- Includes a fast multipole method solver for high accuracy without averaging.

Outline

- Overview of differential algebras and numerical derivatives
- Overview of normal form methods
- Calculation of space charge maps and their inclusion into the element.
- Example systems
- Further expansions
- Conclusions

Differential Algebras and Numerical Derivatives

First we introduce the ${}_1D_1$ Differential Algebra, which is a first order one variable vector.

$$\begin{aligned}
(q_0, q_1) + (r_0, r_1) &= (q_0 + r_0, q_1 + r_1) \\
n(q_0, q_1) &= (nq_0, nq_1) \\
(q_0, q_1)(r_0, r_1) &= (q_0 r_0, q_0 r_1 + r_0 q_1)
\end{aligned}$$

It follows that,

$$\begin{aligned}
(f(x), f'(x)) &= f(x + d) \\
d &= (0, 1)
\end{aligned}$$

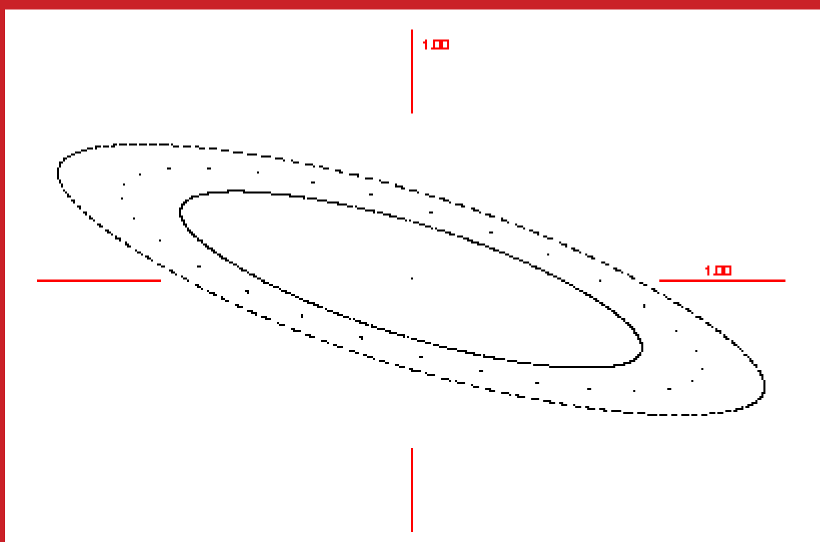
As an example,

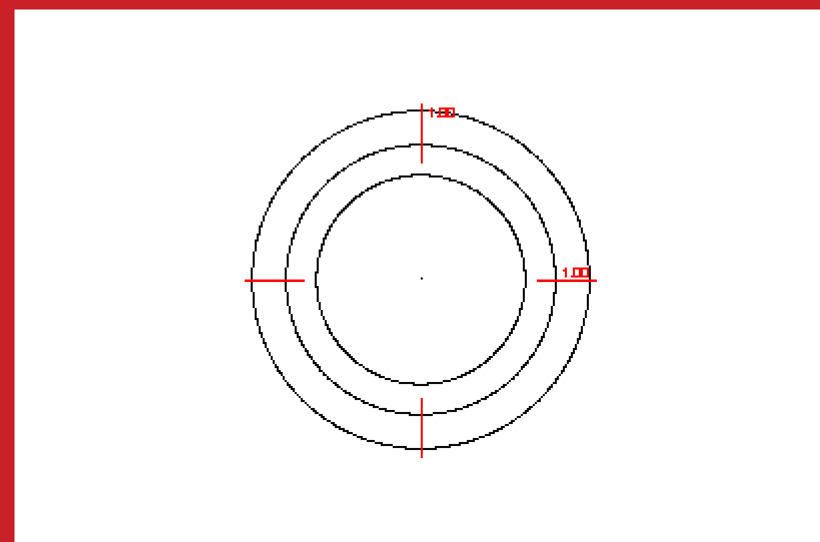
$$\begin{aligned}
f(x) &= 2x^2 - x + 3 \\
f'(x) &= 4x - 1
\end{aligned}$$

$$\begin{aligned}
f(2) &= 9 \\
f'(2) &= 7
\end{aligned}$$

$$\begin{aligned}
f(2 + d) &= f(2, 1) \\
&= 2(2, 1)(2, 1) - (2, 1) + (3, 0) \\
&= 2(4, 4) - (2, 1) + (3, 0) \\
&= (8, 8) - (2, 1) + (3, 0) \\
&= (9, 7)
\end{aligned}$$

Normal Form Methods





$$\mathcal{N} = \mathcal{A} \circ \mathcal{M} \circ \mathcal{A}^{-1}$$

Adding in Space Charge

- We need to solve Poisson's Equation

- Fast
- Accurate
- Valid throughout the distribution.

The Distribution Function

$$\rho(x,y)=\sum_i \delta(x-x_i)\delta(y-y_i) \rightarrow \rho(x,y)=\sum_j \sum_k C_{jk} x^j y^k$$

- Mathematically if two distributions have the same moments then they are mathematically indistinguishable.

Distribution: (Cont'd)

$$M_{ij} = \int\int x^i y^j \sum_l \sum_m C_{lm} x^l y^m dx dy$$

$$M = TC$$

$$C = T^{-1} M$$

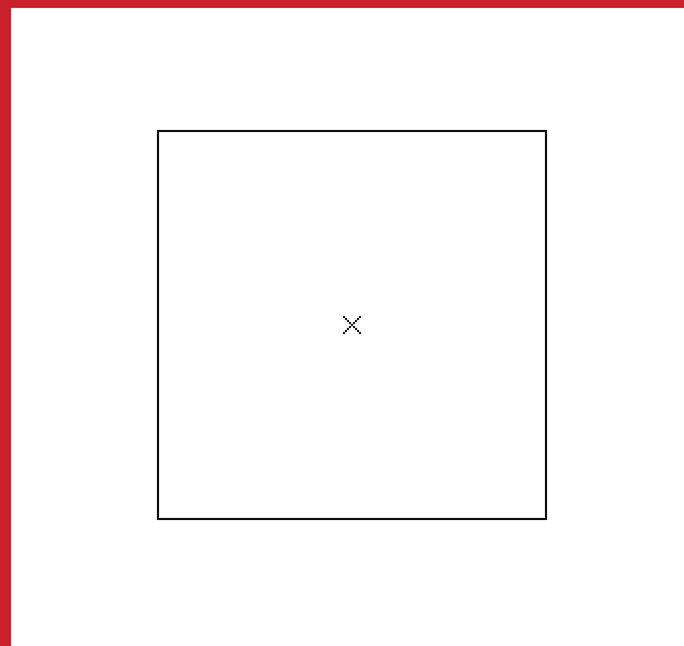
Potential Calculation

- Since we now have a Taylor series for the distribution can't we just integrate and find the potential.

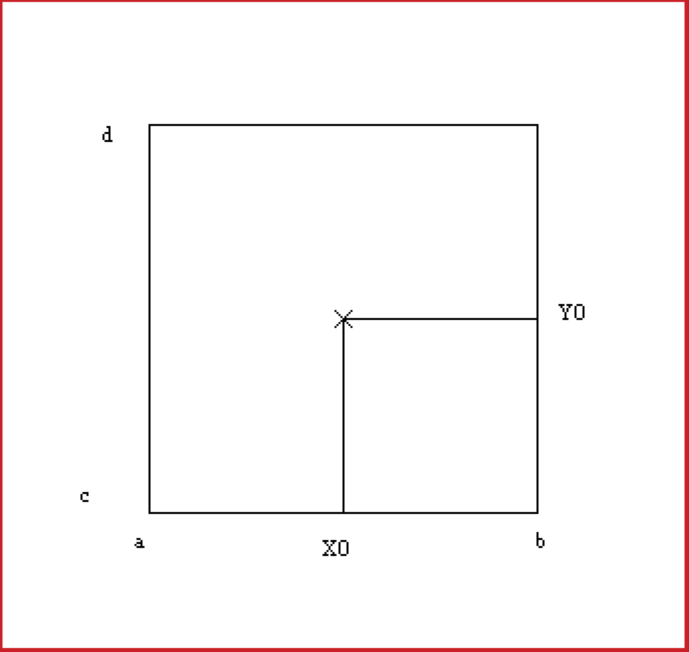
- No, Since the expansion is occurring inside the distribution Singularities become an issue.

$$G(r,r') = \ln(|\ r - r'|); \lim_{r \to r'} (G(r,r')) = \infty$$

Duffy Transformation

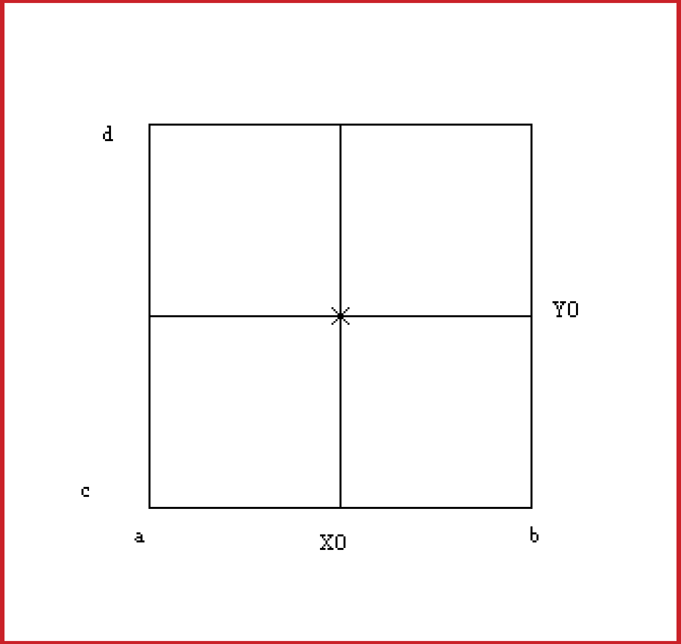


Duffy Transformation



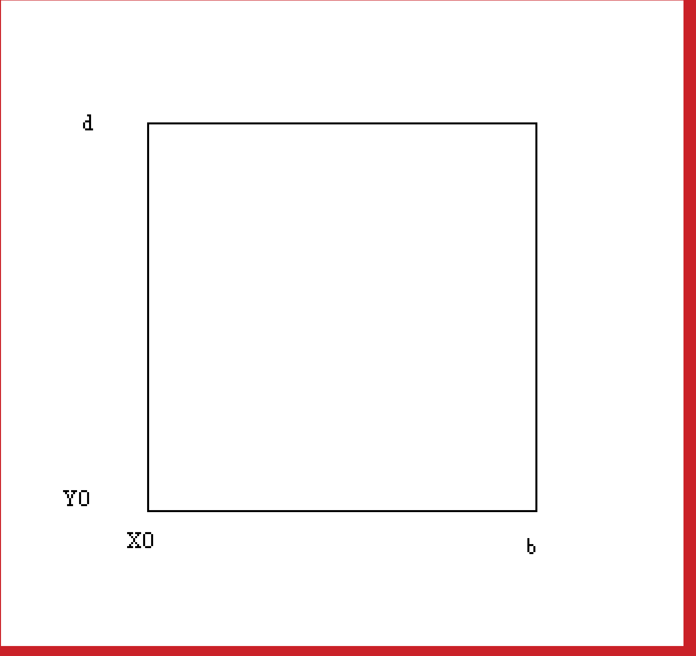
$$\int_c^d \int_a^b \ln\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right) dx\, dy$$

Duffy Transformation



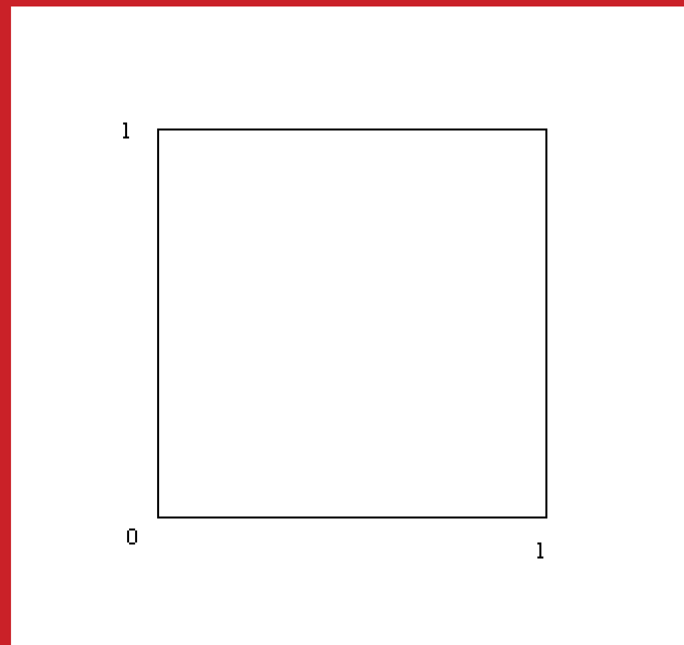
$$\int_c^{y_0}\int_a^{x_0}\ln(\sqrt{(x-x_0)^2+(y-y_0)^2})dxdy+\int_c^{y_0}\int_{x_0}^{b}\ln(\sqrt{(x-x_0)^2+(y-y_0)^2})dxdy+\int_{y_0}^{d}\int_a^{x_0}\ln(\sqrt{(x-x_0)^2+(y-y_0)^2})dxdy+\int_{y_0}^{d}\int_{x_0}^{b}\ln(\sqrt{(x-x_0)^2+(y-y_0)^2})dxdy$$

Duffy Transformation



$$\int_{y_0}^{d}\int_{x_0}^{b}\ln\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right)dx\,dy$$

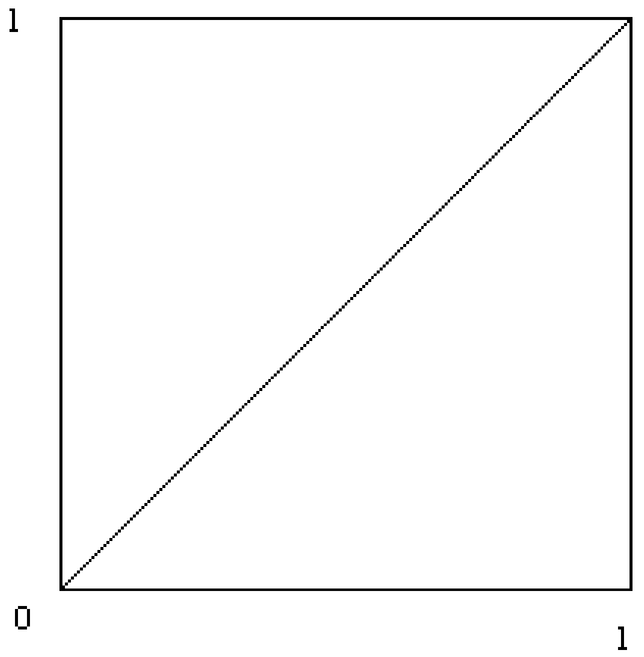
Duffy Transformation



$$u_1 = \frac{x - x_0}{b - x_0}; u_2 = \frac{y - y_0}{d - y_0}; \lambda_1 = (b - x_0); \lambda_2 = (d - y_0);$$

$$\lambda_1 \lambda_2 \int_0^1 \int_0^1 \ln\left(\sqrt{(\lambda_1 u_1)^2 + (\lambda_2 u_2)^2} \right) dx dy$$

Duffy Transformation



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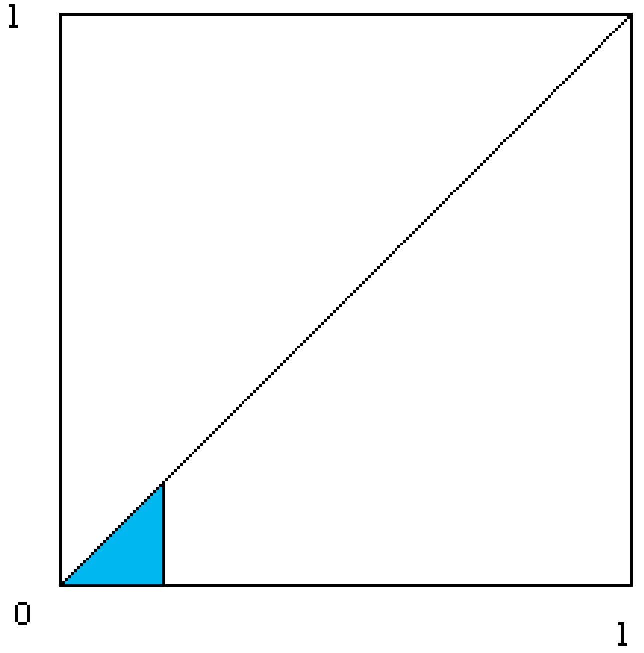

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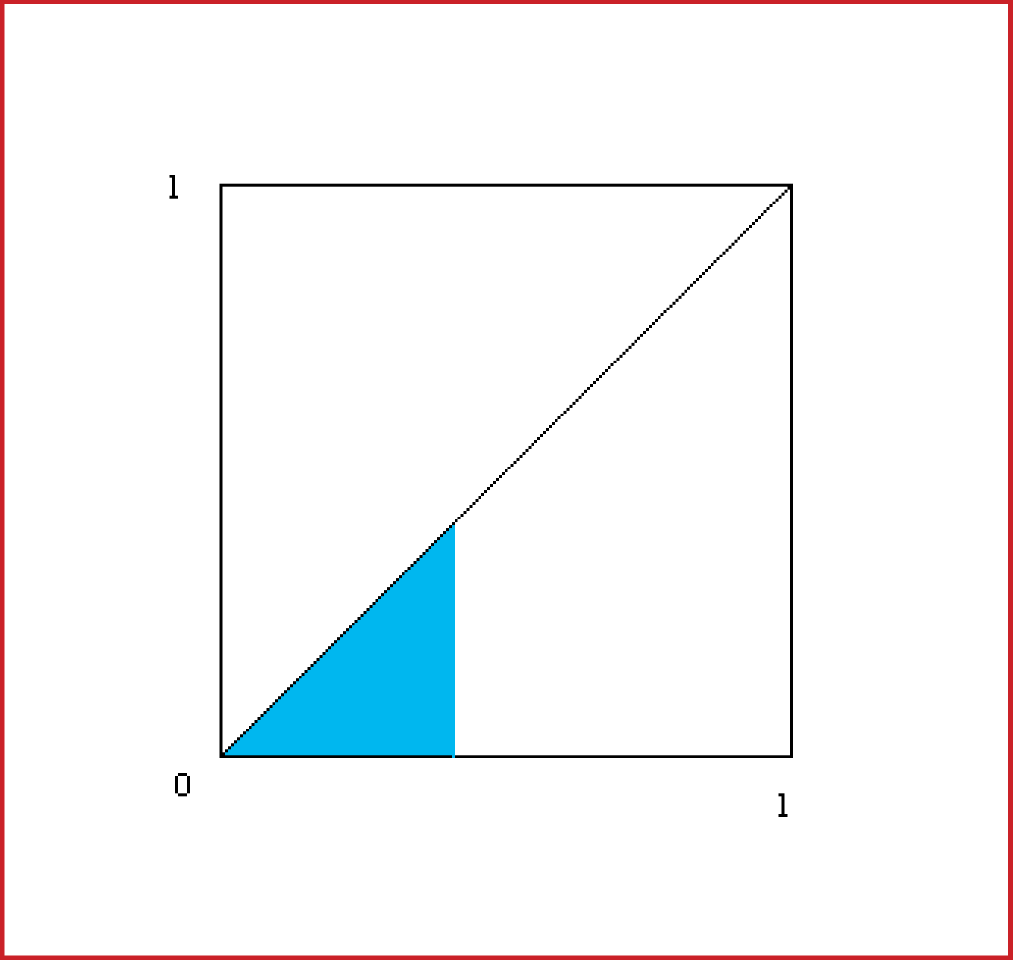

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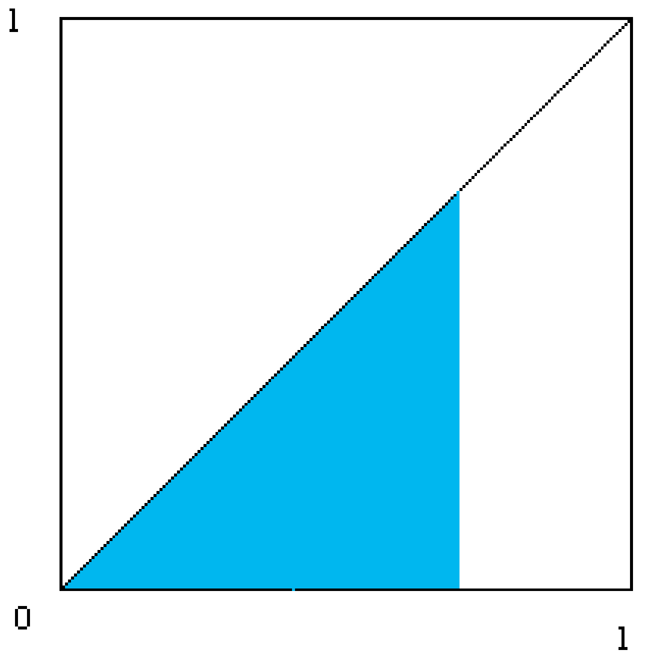
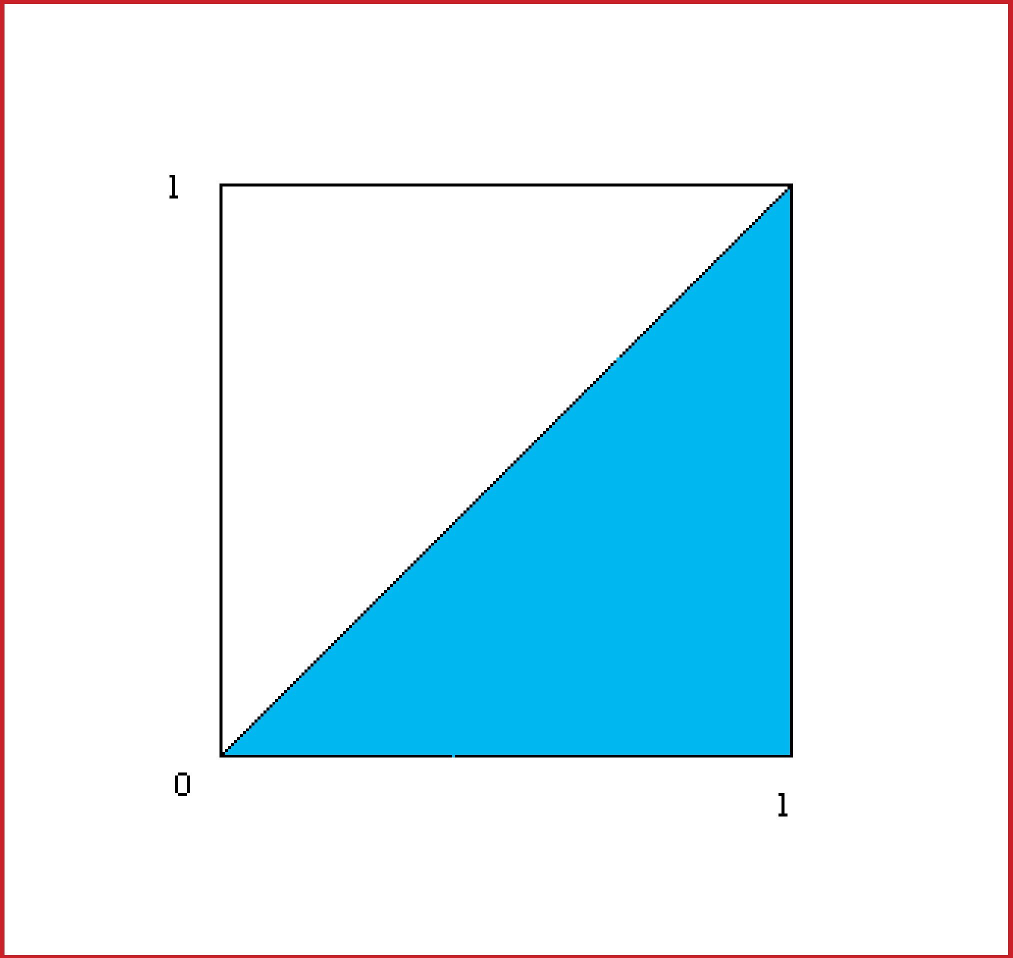


Duffy Transformation



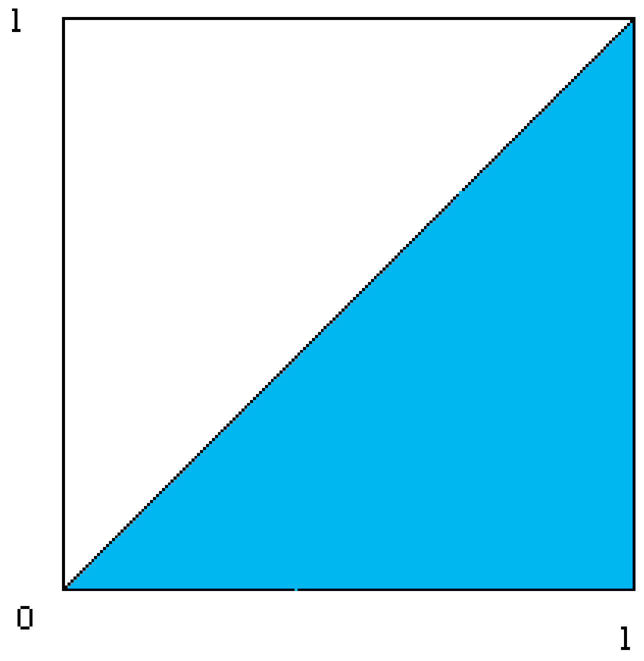
Duffy Transformation

- This is done using the following set of coordinate transformations:

$$w_1 = u_1; w_2 = \frac{u_2}{u_1}$$

$$\lambda_1 \lambda_2 \int_0^1 \int_0^1 w_1 \ln(\sqrt{\lambda_1^2 w_1^2 + \lambda_2^2 w_1^2 w_2^2})\, dw_1 dw_2$$

Duffy Transformation



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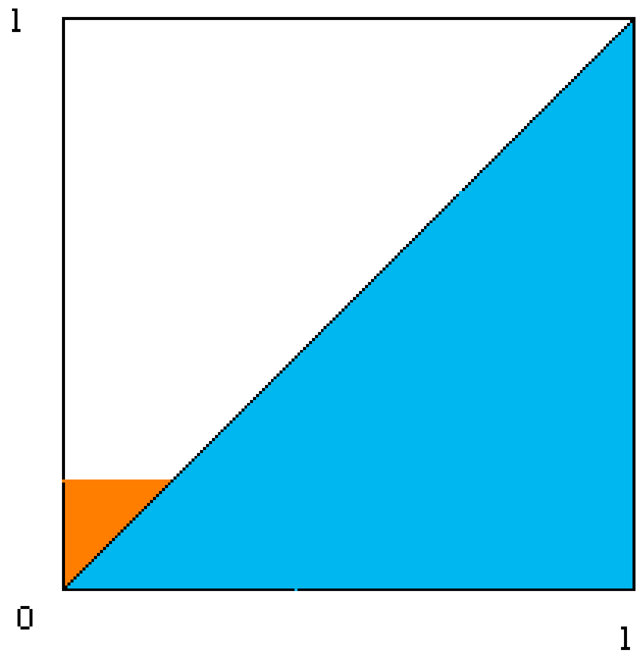

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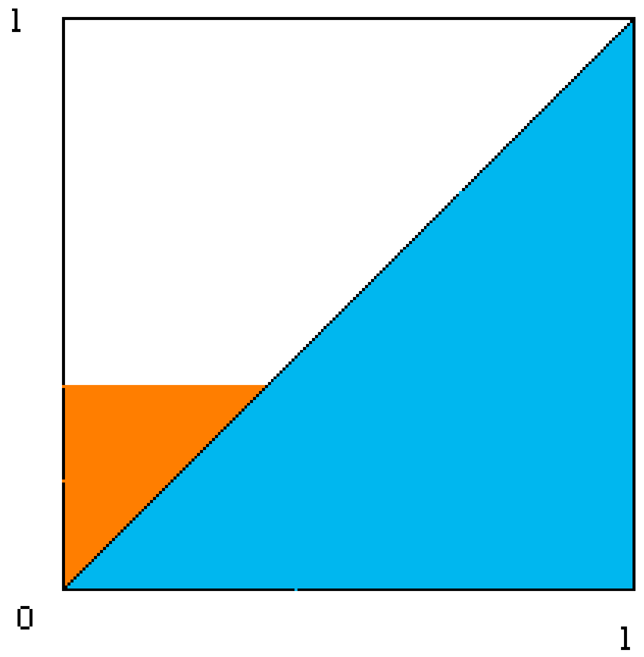
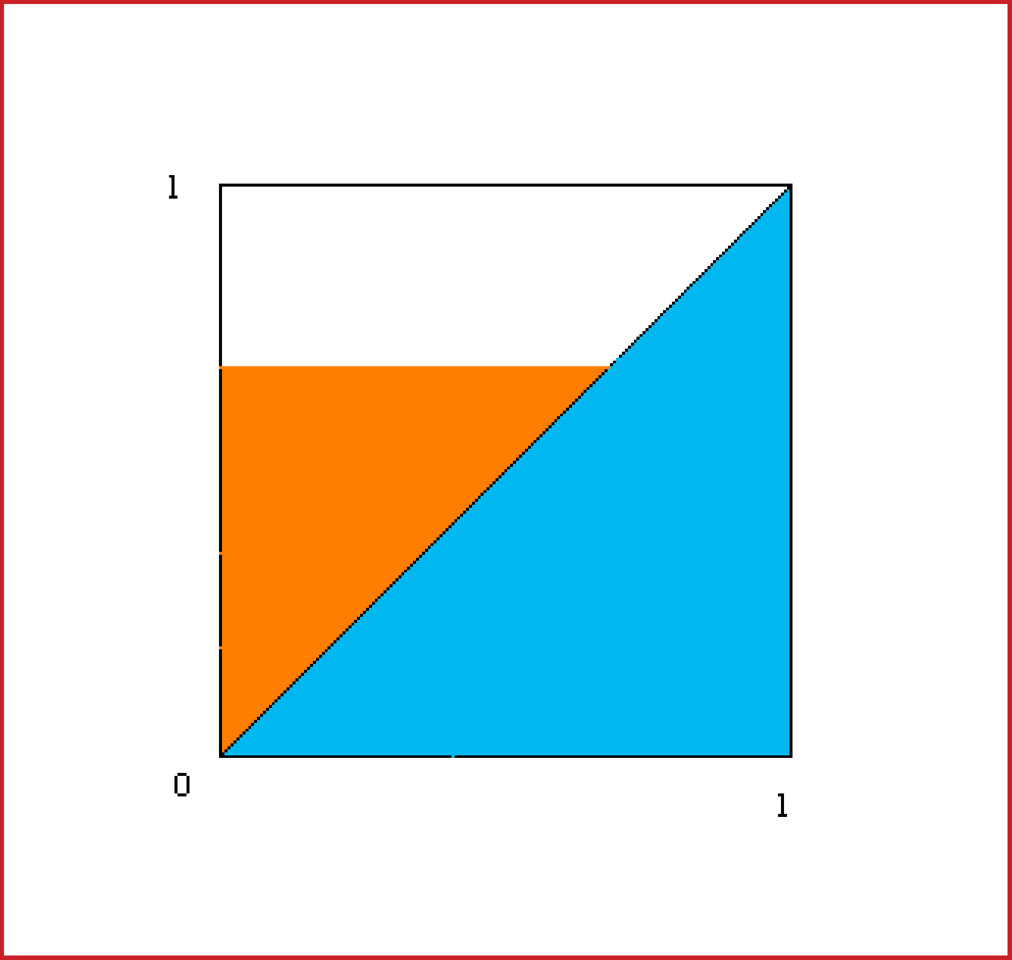
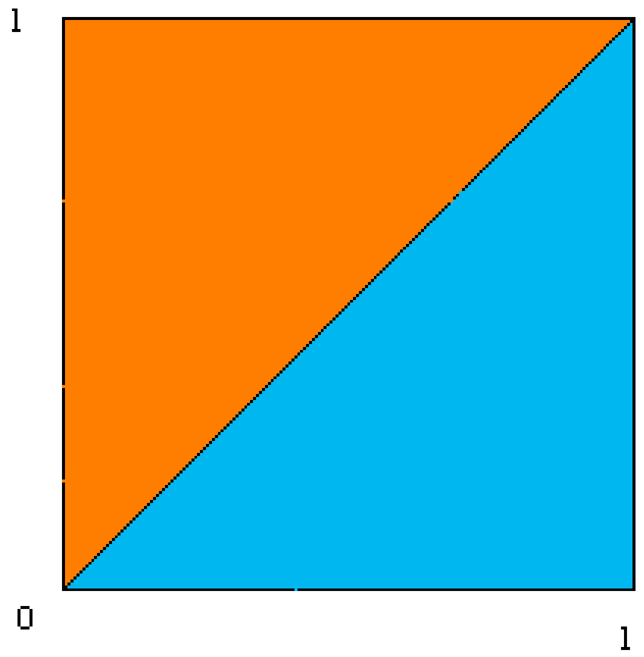


Duffy Transformation



Duffy Transformation



Duffy Transformation

- This is done using the similar set of coordinate transformations:

$$w_2 = u_2 ; w_1 = \frac{u_1}{u_2}$$

$$\lambda_1 \lambda_2 \int_0^1 \int_0^1 w_2 \ln\left(\sqrt{\lambda_2^2 w_2^2 + \lambda_1^2 w_2^2 w_1^2}\right) dw_1 dw_2$$

Duffy Transformation

$$\lambda_1\lambda_2 \int_0^1\int_0^1 w_1 \ln(w_1) + \ln(\sqrt{\lambda_1^2 + \lambda_2^2 w_2^2})\, dw_1 dw_2$$

$$+ \lambda_1\lambda_2 \int_0^1\int_0^1 w_2 \ln(w_2) + \ln(\sqrt{\lambda_1^2 w_1^2 + \lambda_2^2})\, dw_1 dw_2$$

$$\underset{x\to 0}{Lim}(x \ln(x)) = 0$$

External Fields

- These are modeled using Strang Splitting.

Diffeq1(L) → Solution1(L)

Diffeq2(L) → Solution2(L)

Diffeq1(L)+Diffeq2(L) → ?

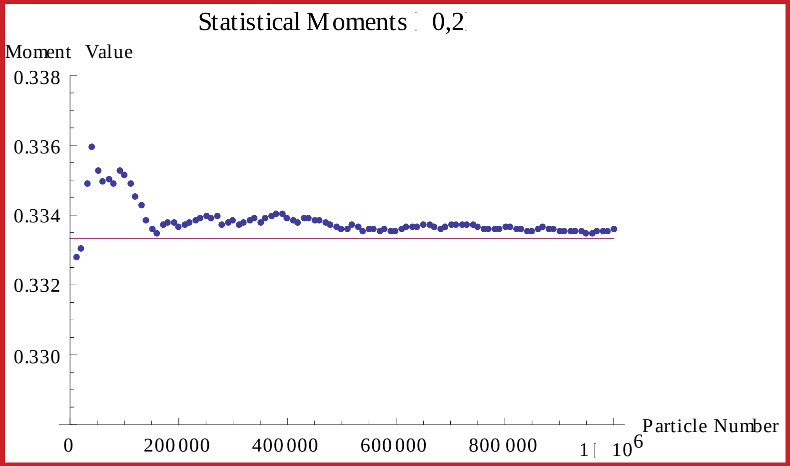
External Fields (cont’d)

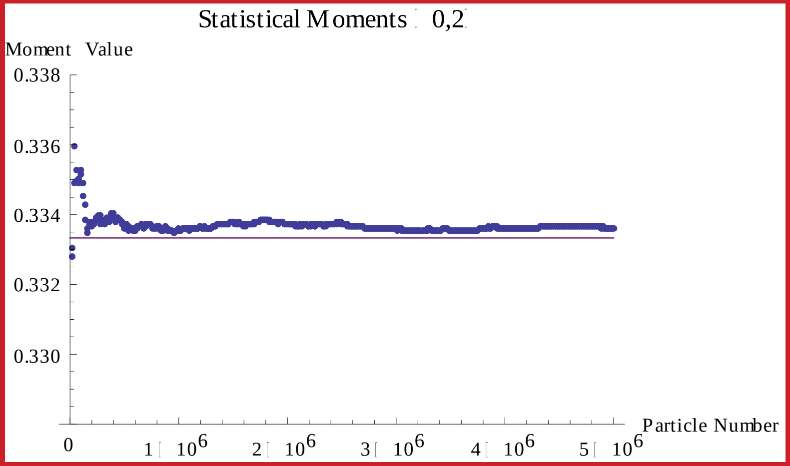
Solution1(L) $\longrightarrow$ $\mathcal{M}(L)$

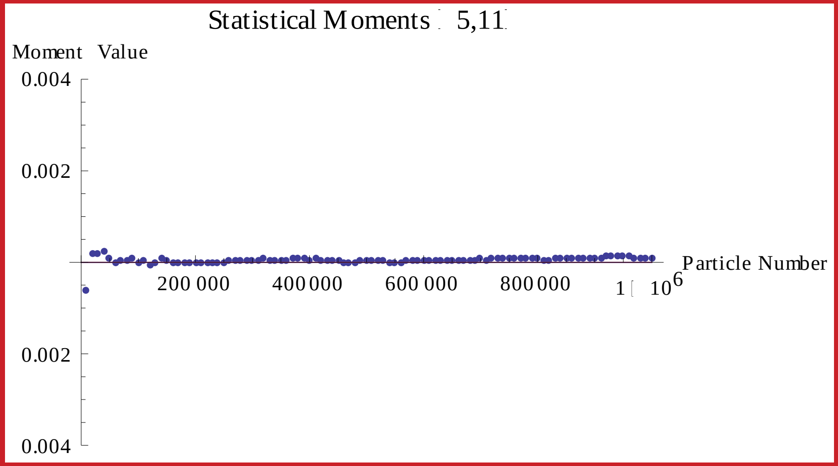
Solution2(L) $\longrightarrow$ Kick(L)

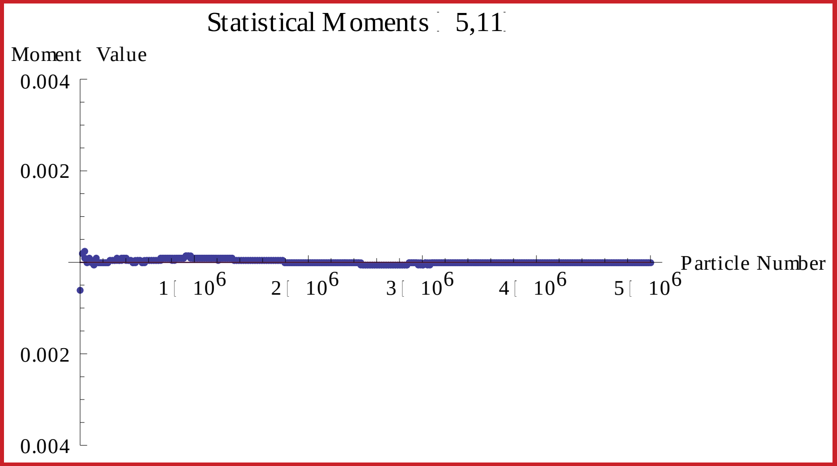
$$\mathcal{M}(L/2)\ \mathrm{Kick}(L)\ \mathcal{M}(L/2)\ + O(L^3)$$

Particle Number

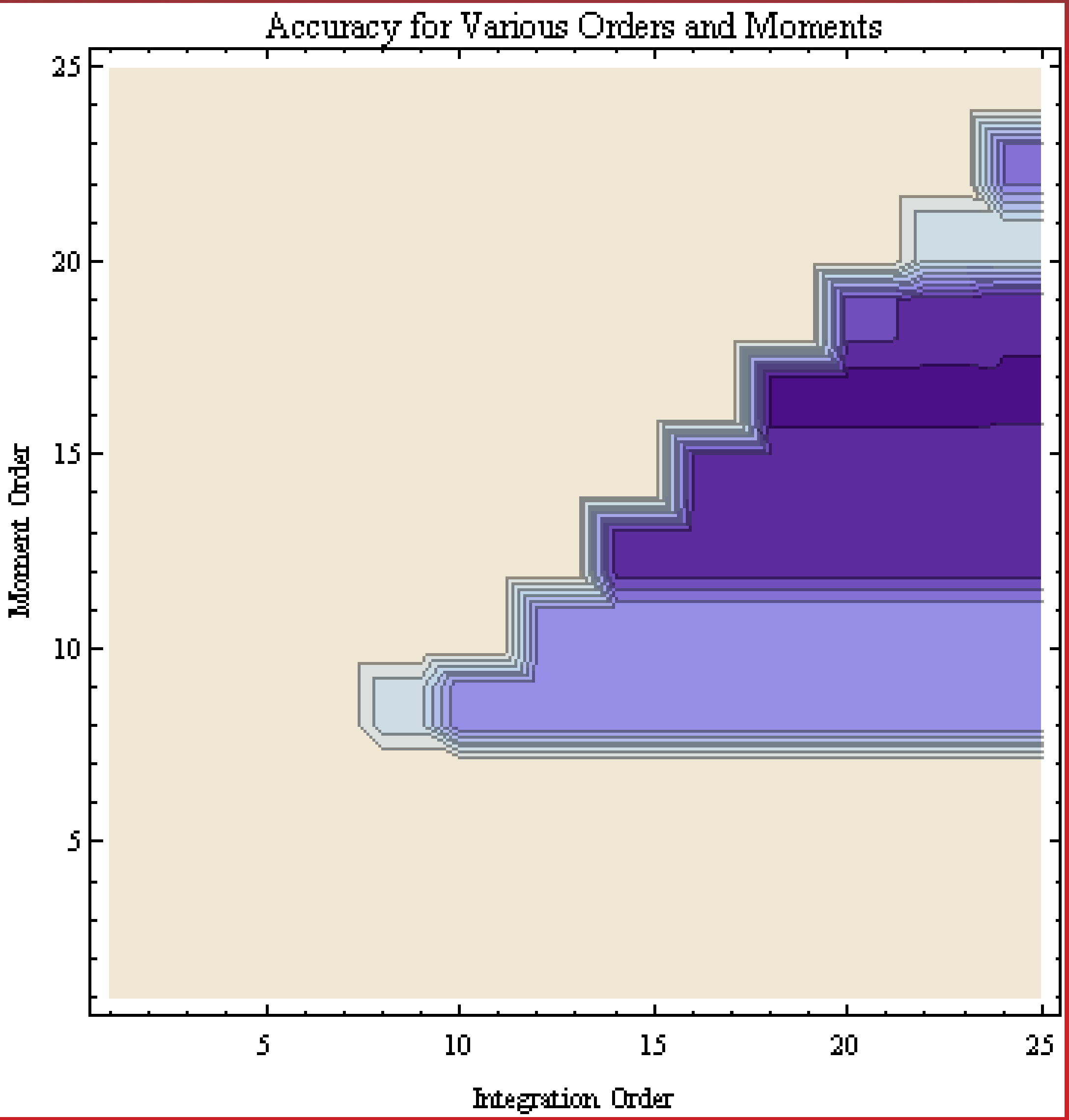








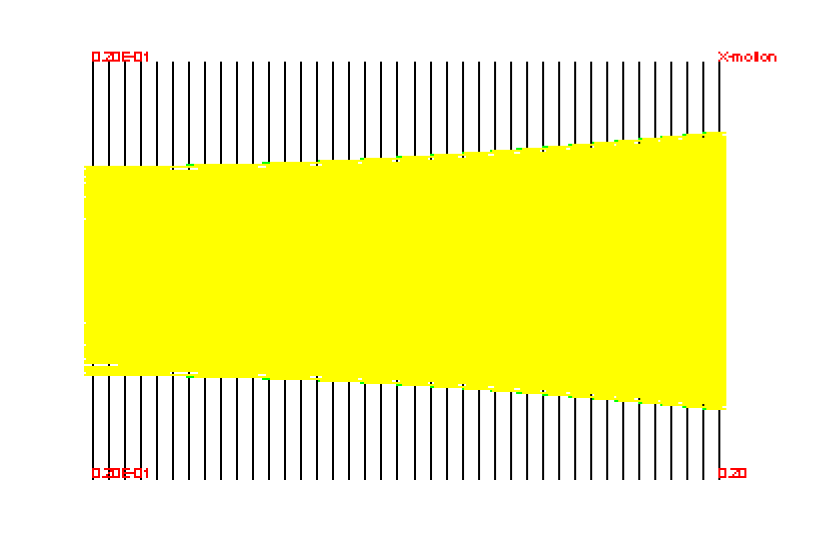
Accuracy

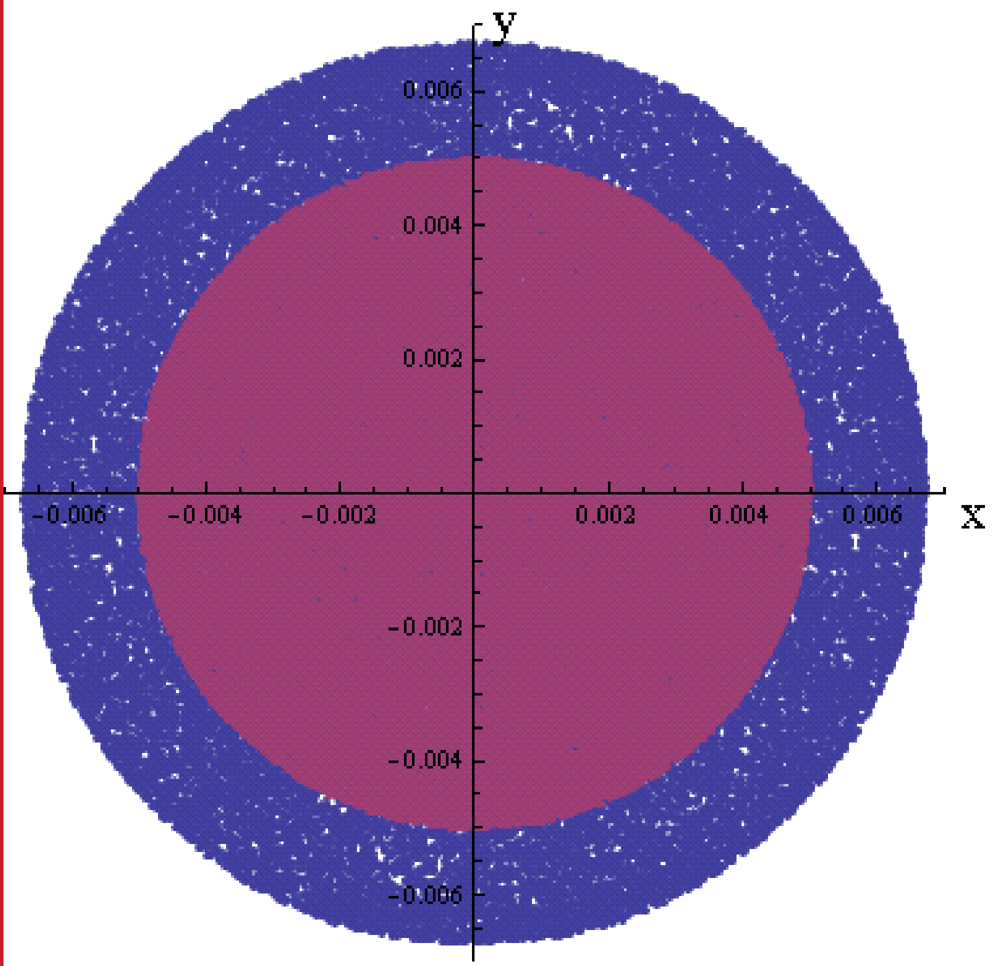

Accuracy for Various Orders and Moments
Moment Order
Integration Order
5
10
15
20
25
5
10
15
20
25

Example

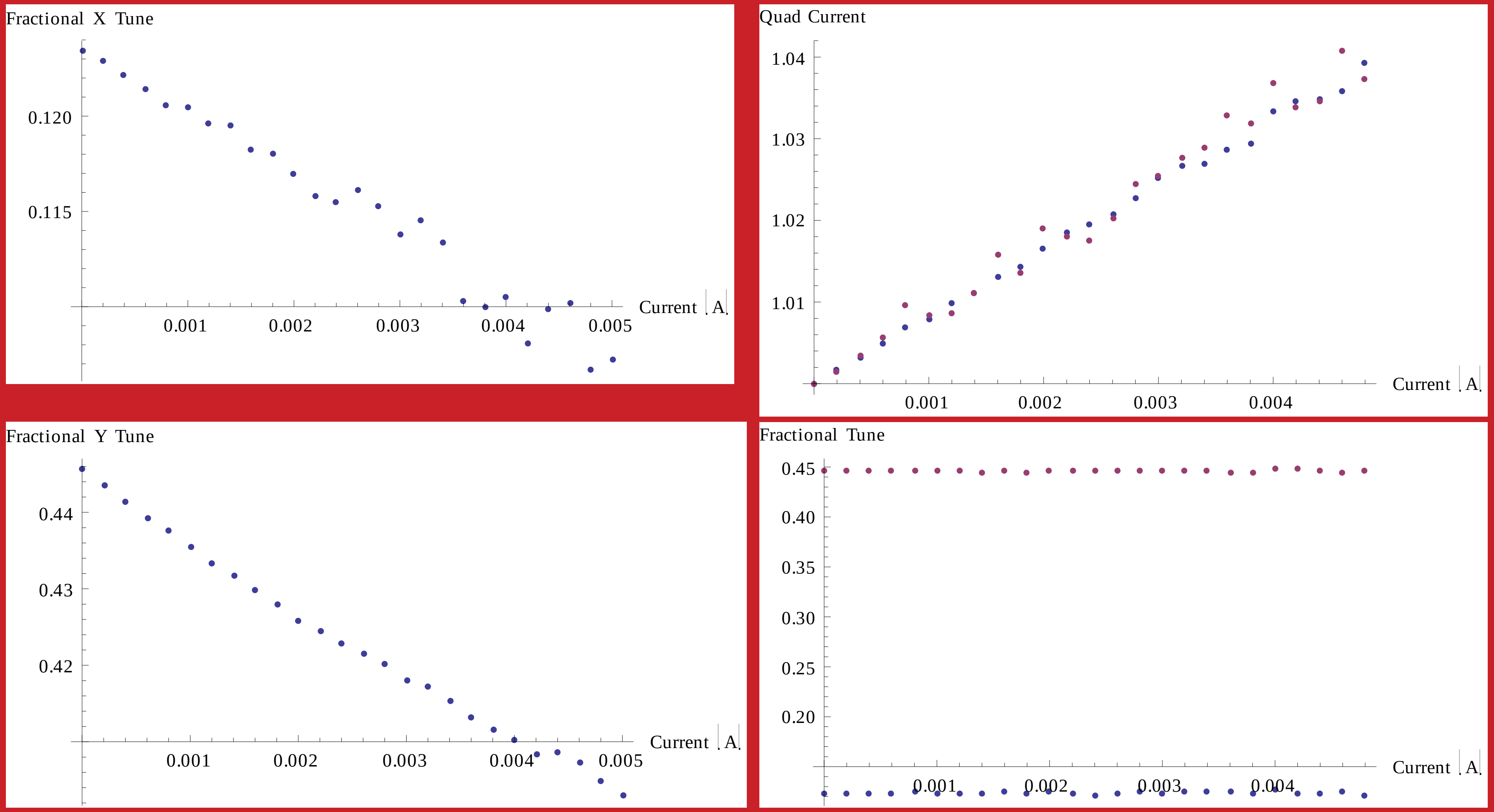
$$\frac{r_m}{r_0} = 1 + 5.87 \times 10^{-5} \frac{I}{(\gamma^2 - 1)^{\frac{3}{2}}} (\frac{z}{r_0})^2$$

Method	Growth
Edge Point x	35.27 %
Edge Point y	35.30 %
Map Element x	31.21 %
Map Element y	31.34 %

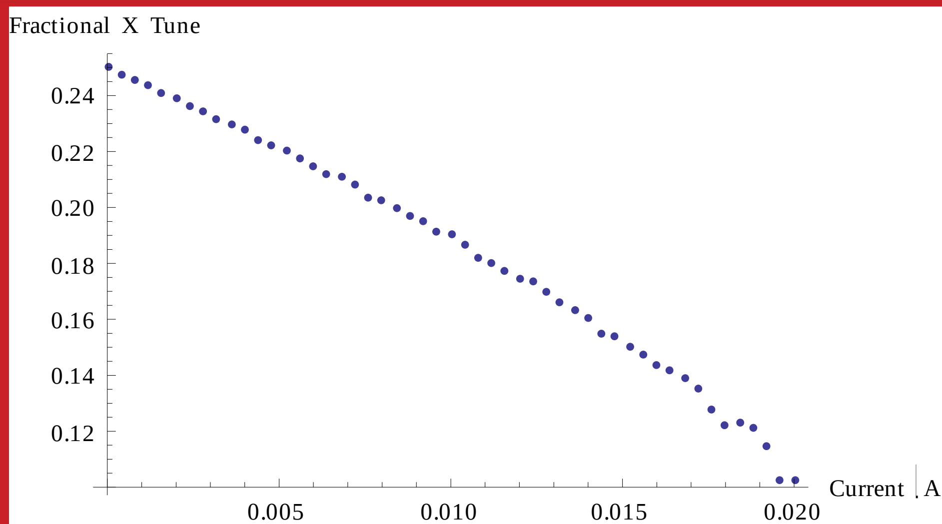


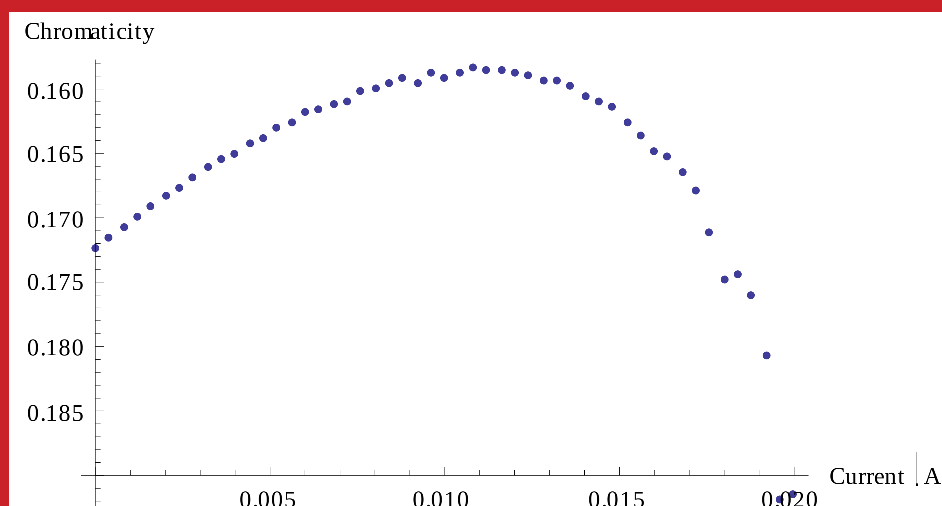


Tune Measurement

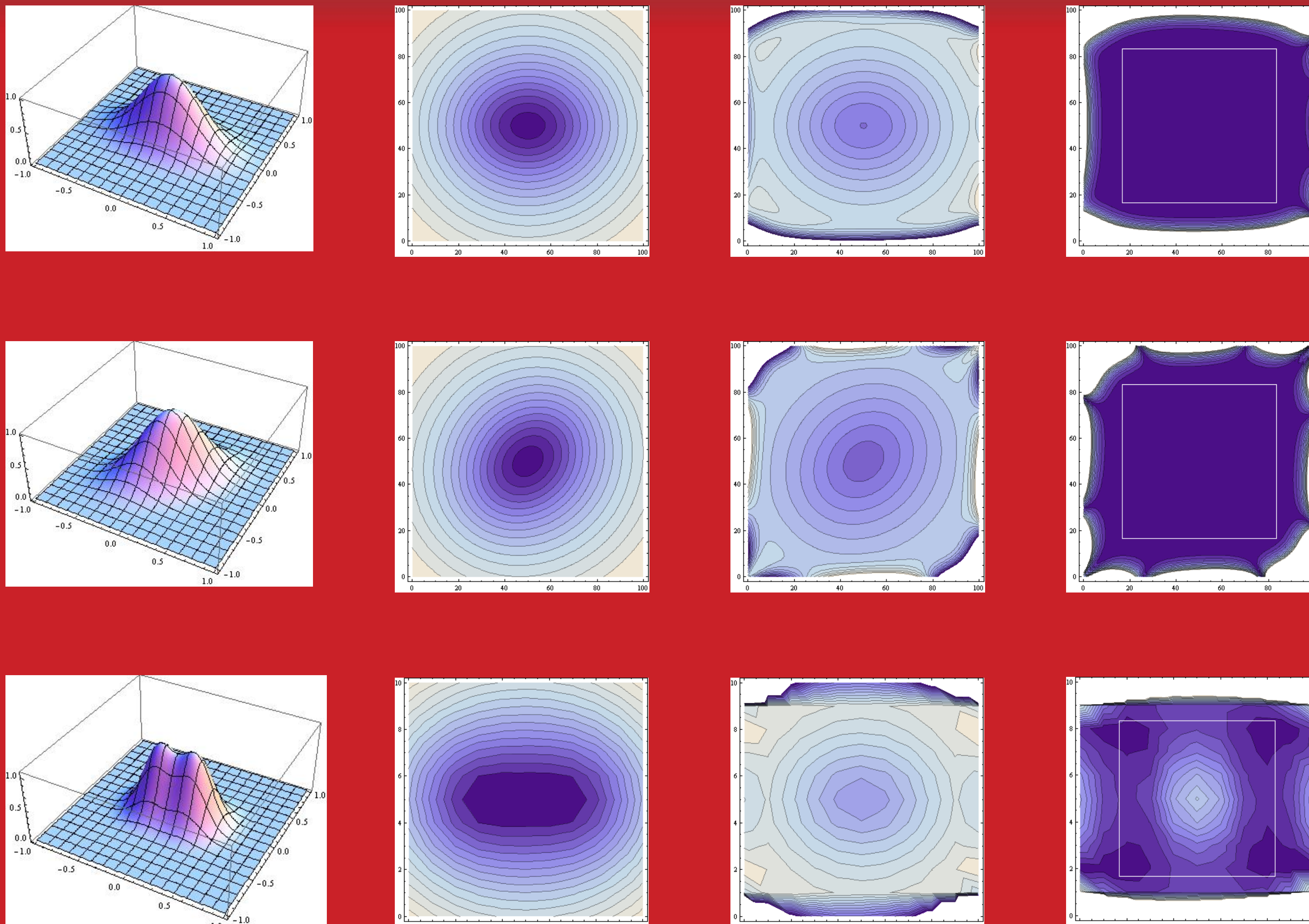


Chromaticity





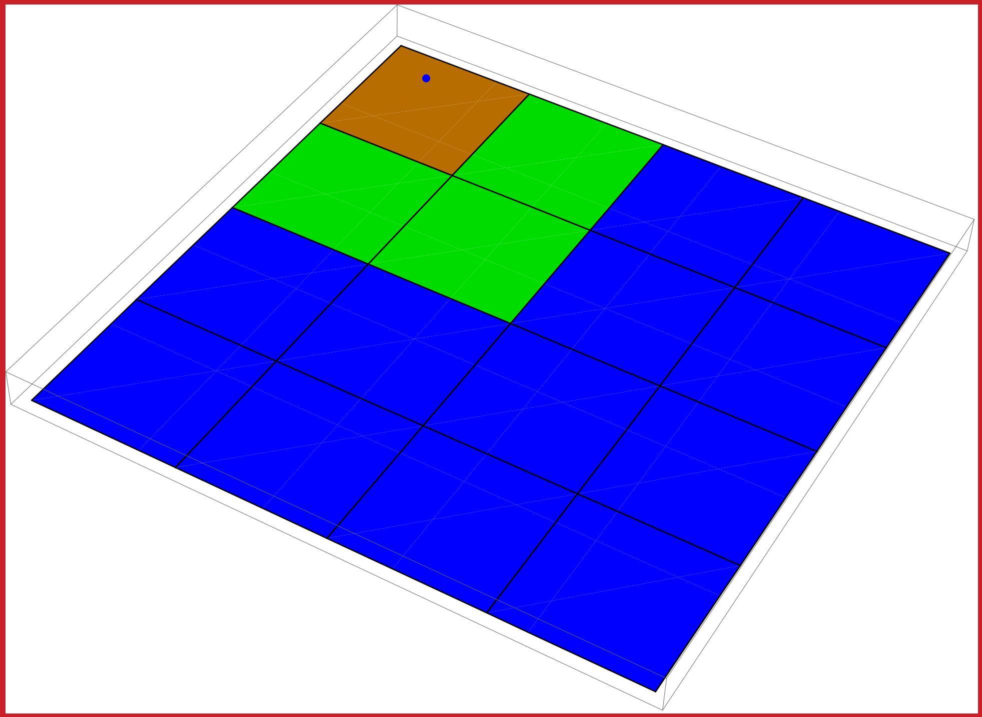
Limitations



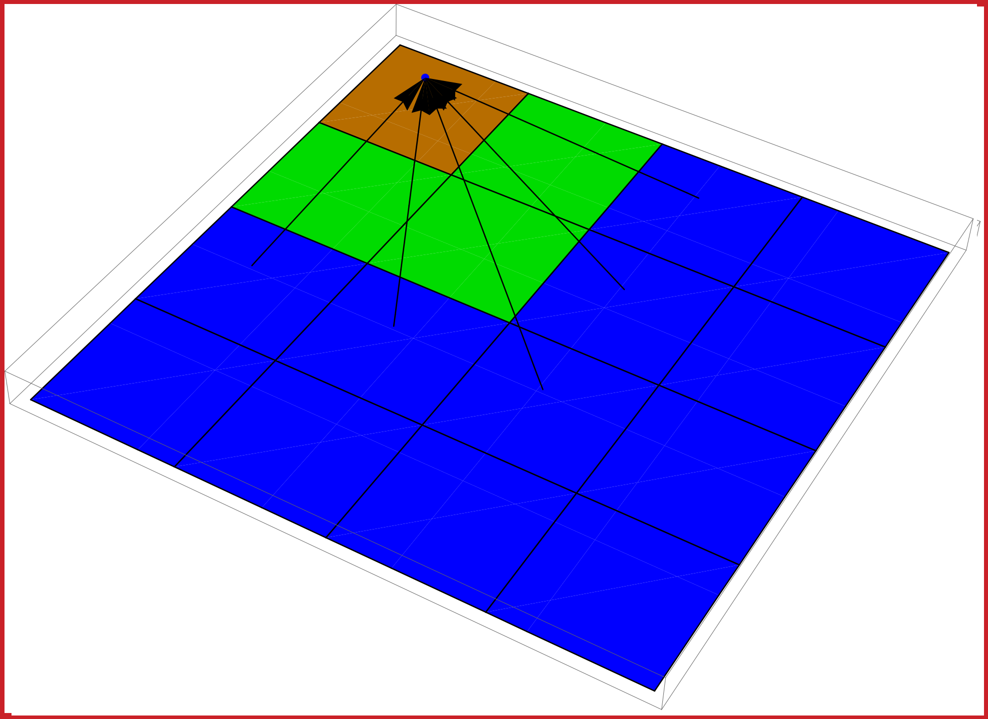
Fast Multipole Method

- Uses multipole expansions for distant particles
- Uses direct coulomb interactions for close particles
- Currently used in solid state physics, fluids, and chemistry

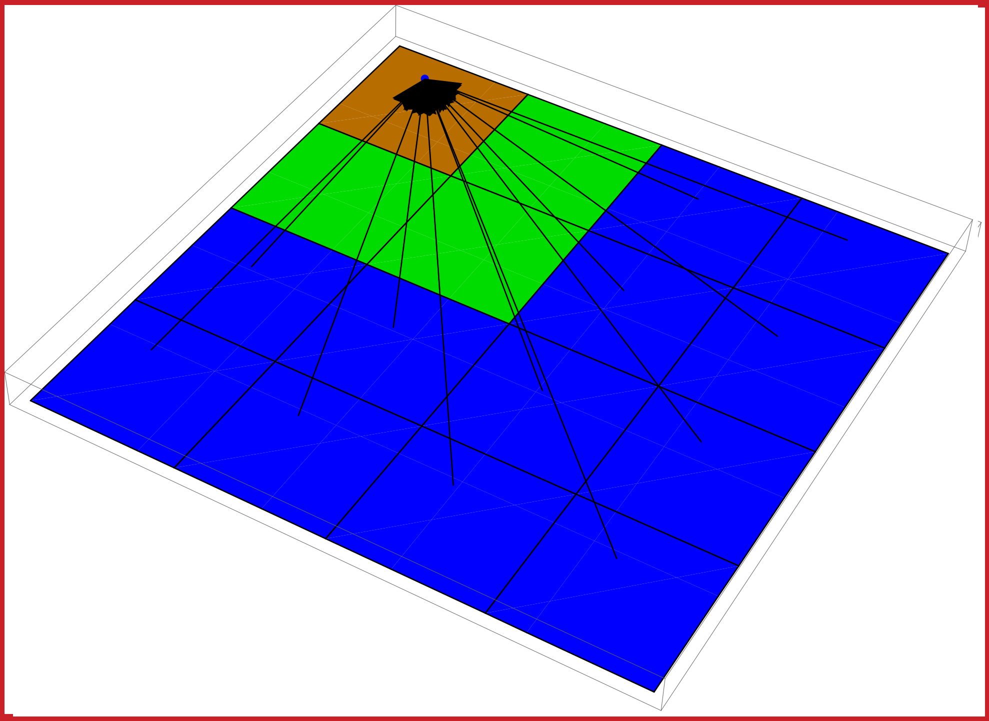
FMM: Examples



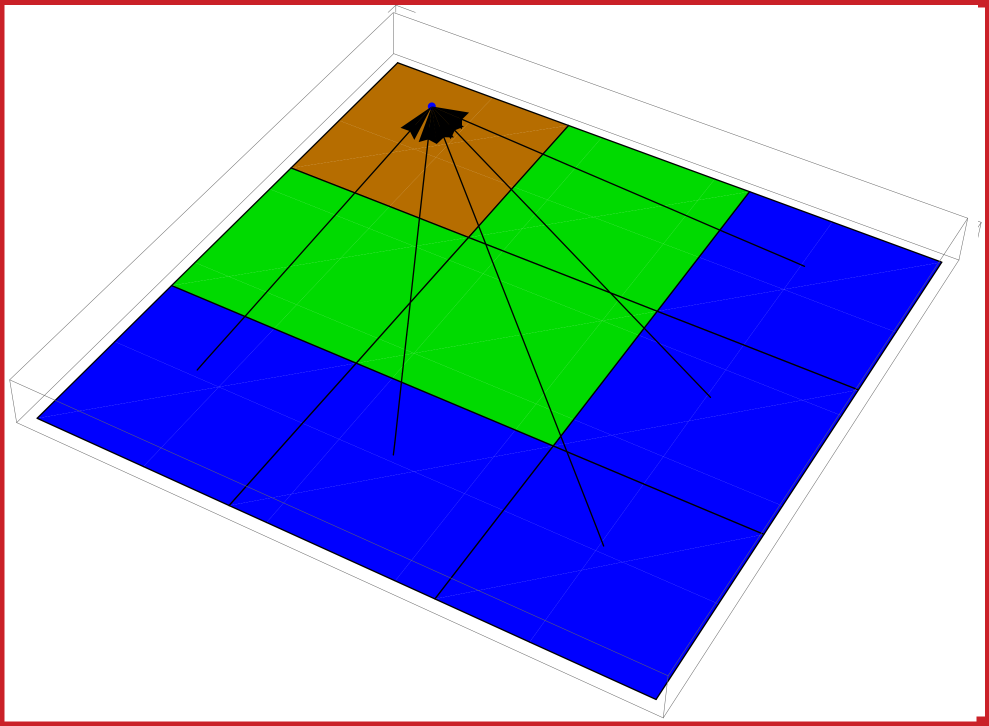
FMM: Examples



FMM: Examples

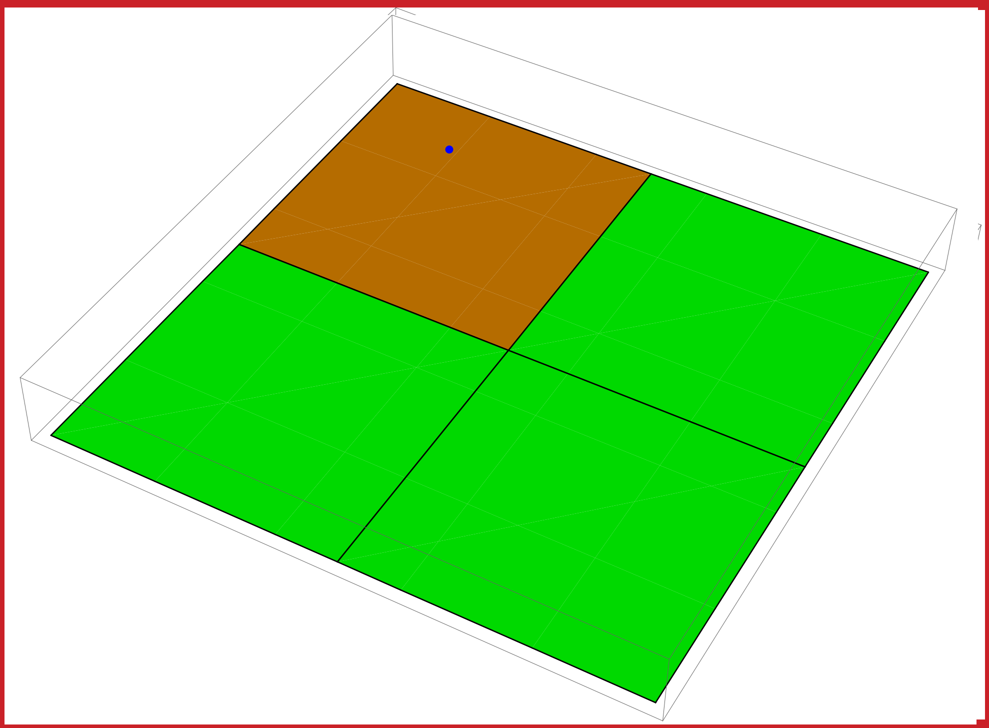


FMM: Examples



$$\phi(z) = \sum_i \left(Q \log(z_i) + \sum_k \frac{a_k}{z_i^k} \right)$$

FMM: Examples



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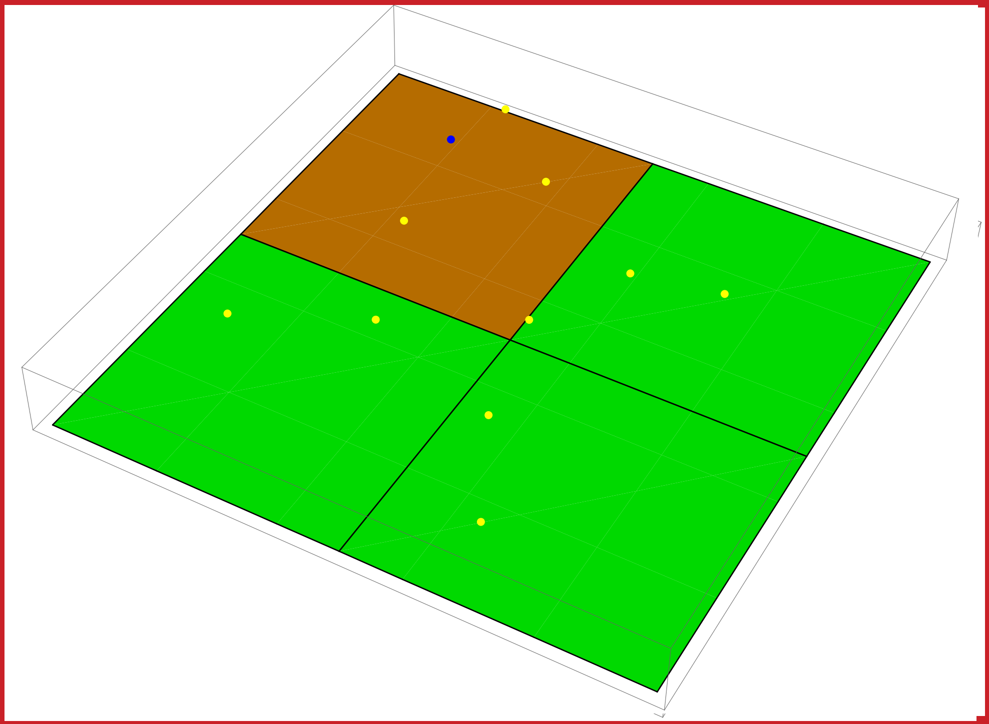
FMM: Examples



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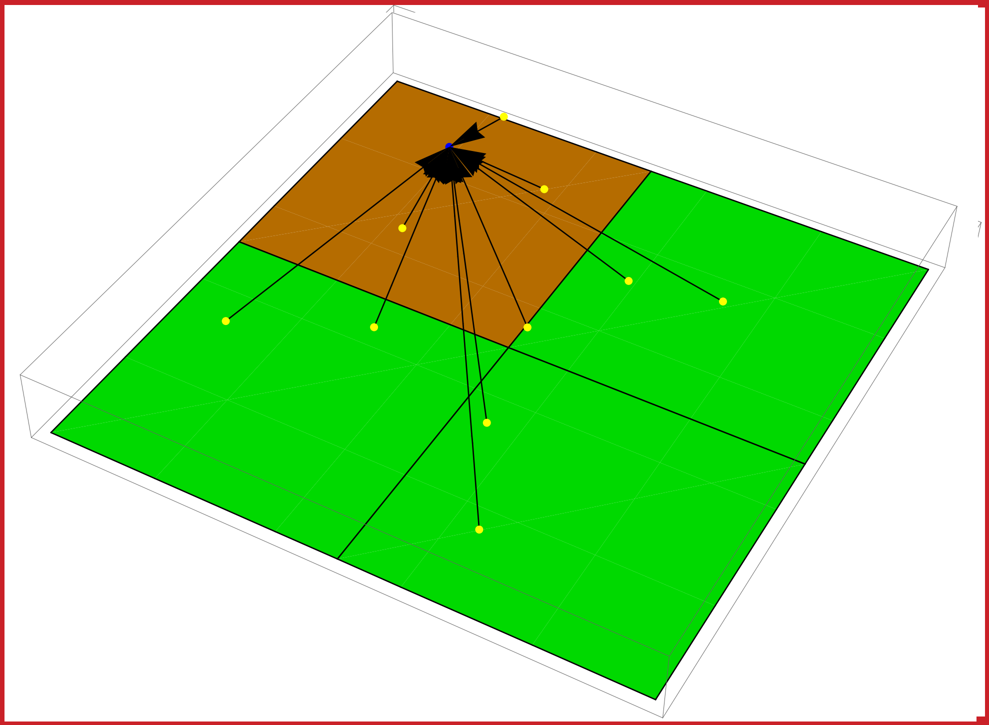
FMM: Examples



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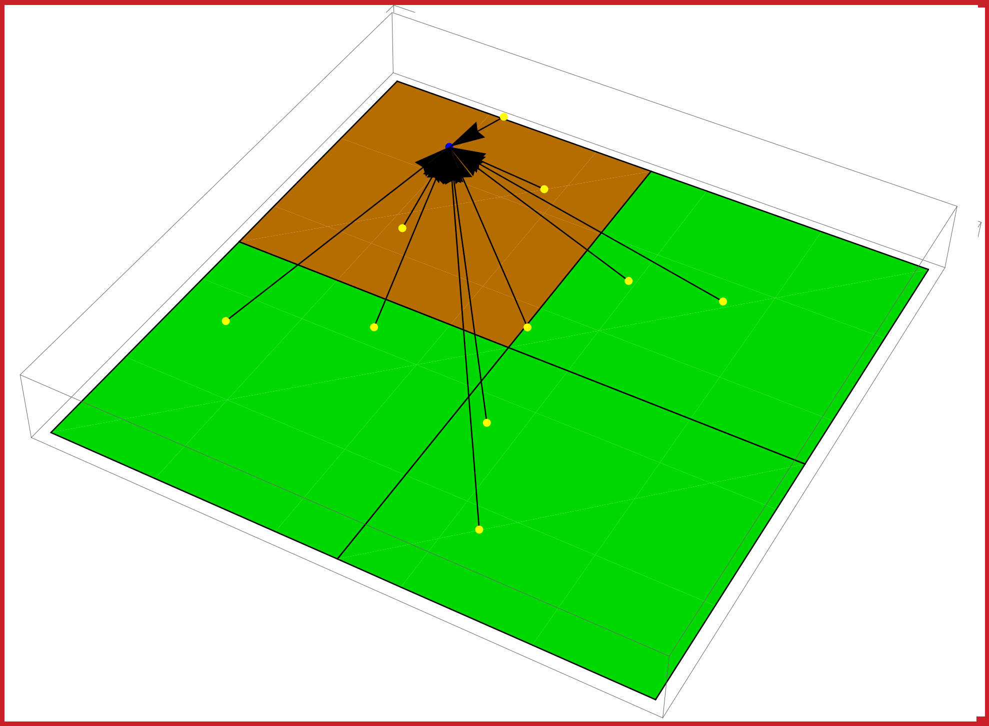
FMM: Examples



$$\phi(z) = \sum_i \left(Q \log(z_i) + \sum_k \frac{a_k}{z_i^k} \right) + \sum_j q_j \log(z_j)$$

Conclusions

- Adding the effects of space charge to the transfer map of a system is both possible and feasible
- New insights can be found using this method in conjunction with normal form analysis
- Can even analyze complex distributions using the fast multipole method

Questions?

Map Overview

- A map is a method of advancing particles which takes the form

$$z_f = Mz_i$$

$$\begin{pmatrix} x_f \\ p_{xf} \end{pmatrix} = \begin{pmatrix} (x_f \mid x_i) & (x_f \mid p_{xi}) \\ (p_{xf} \mid x_i) & (p_{xf} \mid p_{xi}) \end{pmatrix} \cdot \begin{pmatrix} x_i \\ p_{xi} \end{pmatrix}$$

Map Overview

- A map is a method of advancing particles which takes the form

$$z_f = M z_i$$

$$
\begin{aligned}
x_f &= (x_f \mid x_i) x_i + (x_f \mid p_{xi}) p_{xi} \\
p_{xf} &= (p_{xf} \mid x_i) x_i + (p_{xf} \mid p_{xi}) p_{xi}
\end{aligned}
$$

Map Overview

- A map is a method of advancing particles which takes the form

$$z_f = M z_i$$

$$x_f = (x_f \mid x_i)x_i + (x_f \mid p_{xi})p_{xi} + (x_f \mid x_i^2)x_i^2 + (x_f \mid x_i p_{xi})x_i p_{xi} + (x_f \mid p_{xi}^2)p_{xi}^2$$

$$p_{xf} = (p_{xf} \mid x_i)x_i + (p_{xf} \mid p_{xi})p_{xi} + (p_{xf} \mid x_i^2)x_i^2 + (p_{xf} \mid x_i p_{xi})x_i p_{xi} + (p_{xf} \mid p_{xi}^2)p_{xi}^2$$

Legendre Polynomials

- Legendre Polynomials have the following orthogonality property.

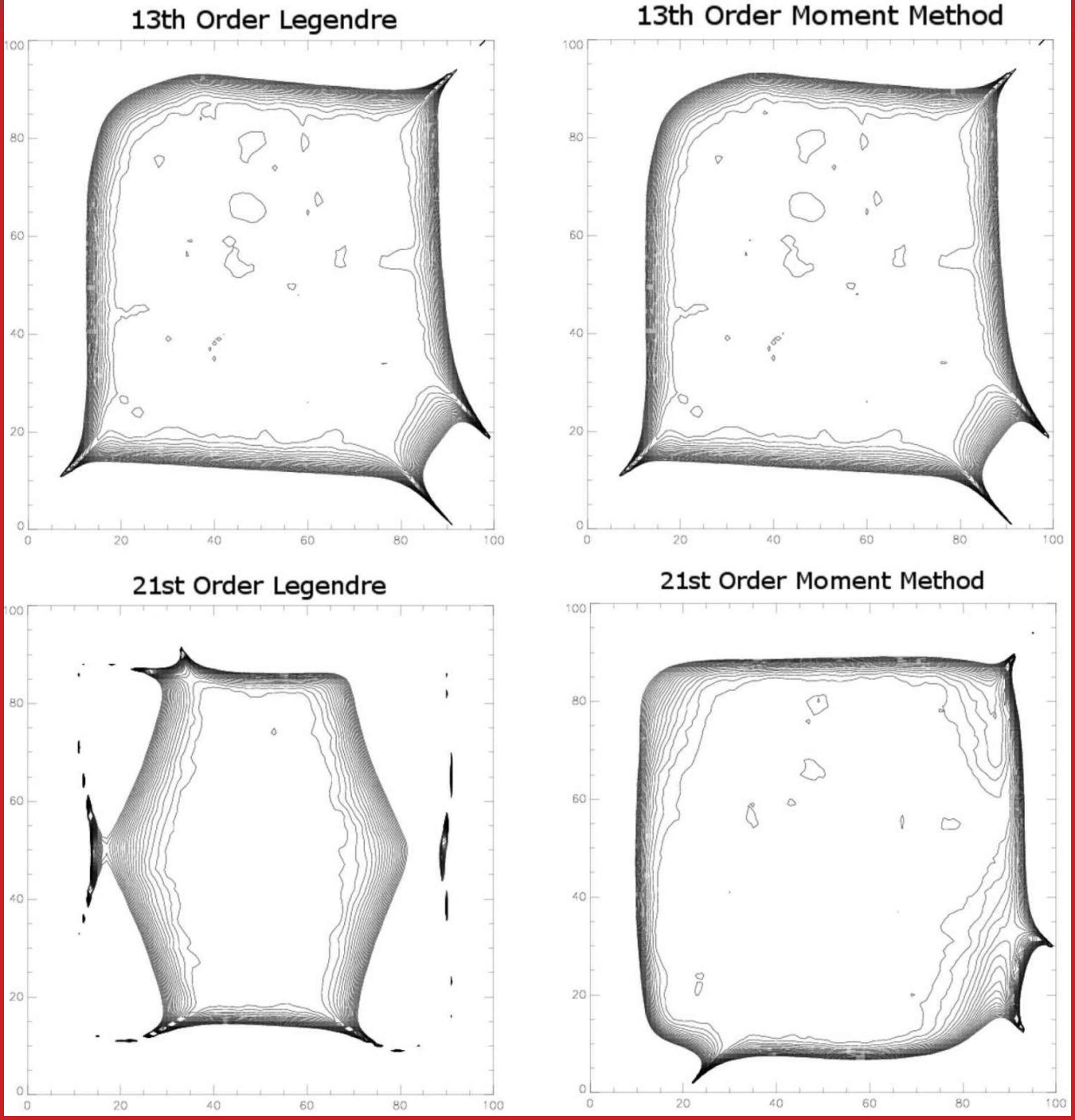
$$\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn}$$

- If we assume that the distribution can be modeled as a sum of legendre polynomials, we can easily find the coefficients.

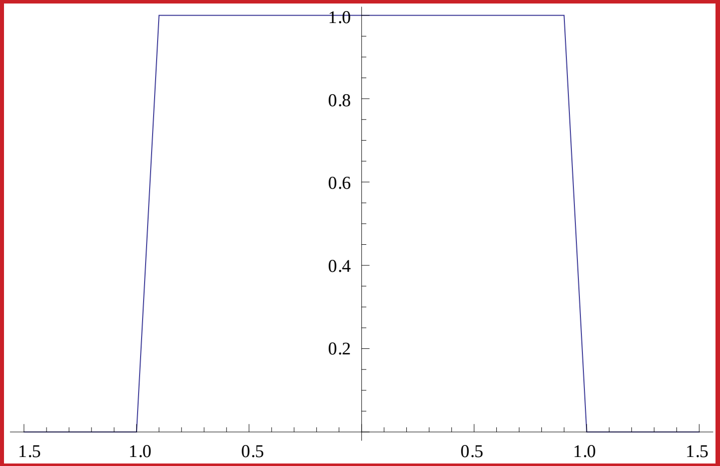
$$\rho(x) = \sum_n C_n P_n(x)$$

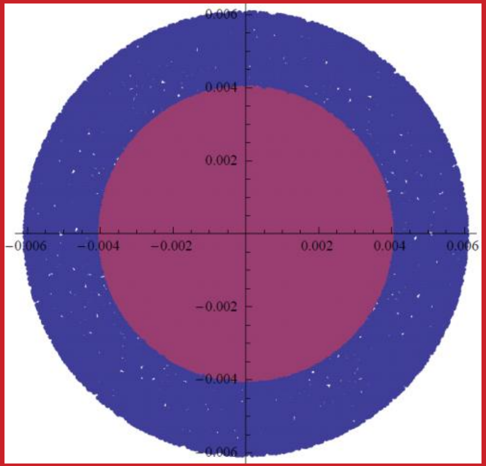
$$\int_{-1}^{1} P_m(x)\rho(x)dx = C_m \frac{2}{2m+1}$$

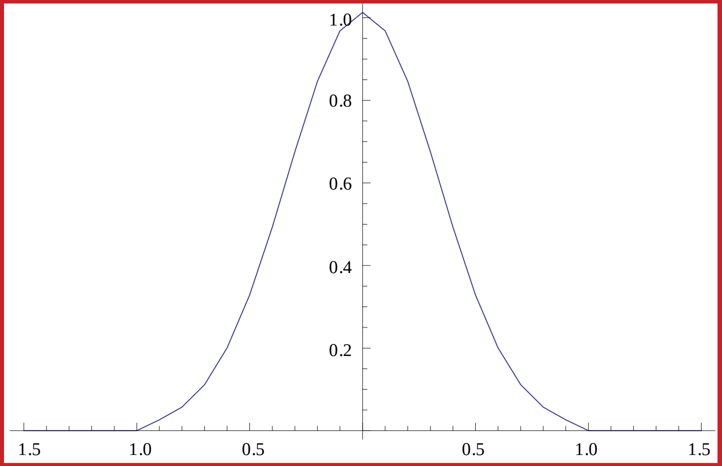
Method Comparison

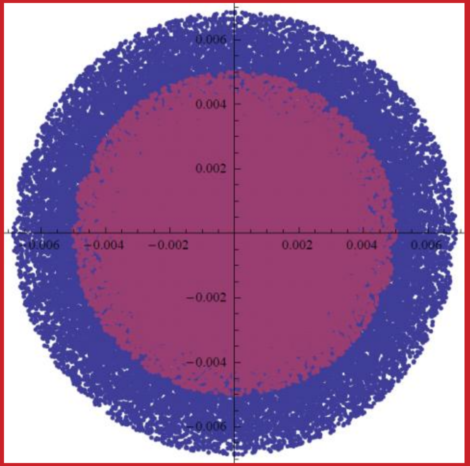


Particle Advancement









Particle Advancement Cont'd