Effect of Space Charge on Transverse Instabilities

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Introduction

- Transverse instabilities of bunched beams are discussed
  at dominant Space Charge Impedance,
  which is very characteristic case for high-brightness proton synchrotrons.

- In such conditions, SCI almost completely controls intra-bunch oscillations (head-tail modes): frequency, shape, and threshold of possible instability.

- However, SCI cannot cause the beam instability itself.
  Retarding (wake) field is directly responsible for this.
  Though it is relatively small, it controls the beam collective modes
  including the instability growth rate.

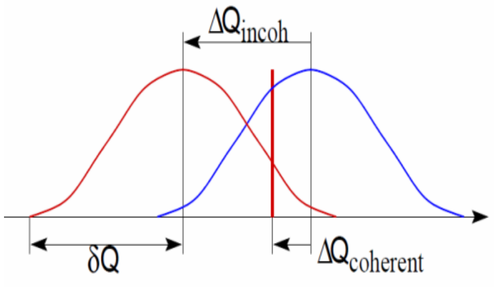
- Landau Damping is the most important Space Charge exhibition
  in the overall picture of the instability, and the main object of the talk.

Preliminary notices: coasting beam

- Stability condition of a coasting beam is well known:
  space charge tune shift should be about less than the tune spread.

- This rule has a simple origin: self-sustaining coherent oscillations are impossible if the coherent frequency falls within the incoherent range.

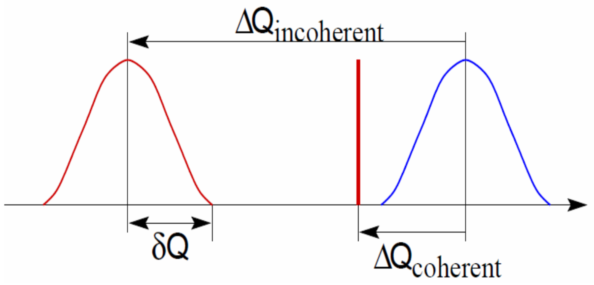
- Indeed, electric field attendant such coherent oscillations would excite contra-phase oscillations of different parts of the beam, resulting in energy transfer from coherent motion to incoherent one, that is beam heating and the coherence decay.



The effect is known as Landau Damping (LD)

- At higher intensity or lower tune spread, coherent tune leaves the incoherent range.

- Then all the particles are exited in-phase, supporting the coherence. Landau damping does not spring up.



- So, the instability condition is:

$\boxed{\Delta Q_{incoherent} - \Delta Q_{coherent} > \delta Q}$ that is $\boxed{\Delta Q_{incoherent} >\sim \delta Q}$

Bunched beams – qualitative glance

- In coasting beams, the tune spread is presented mostly as a product of chromaticity on momentum spread.

- However, chromaticity does not affect instability threshold of bunched beams (C. Pellegrini, M. Sands, 1969).

- Main source of its tune spread is space charge tune shift itself because it depends on the particle position in the bunch (V. Balbekov, 1976).

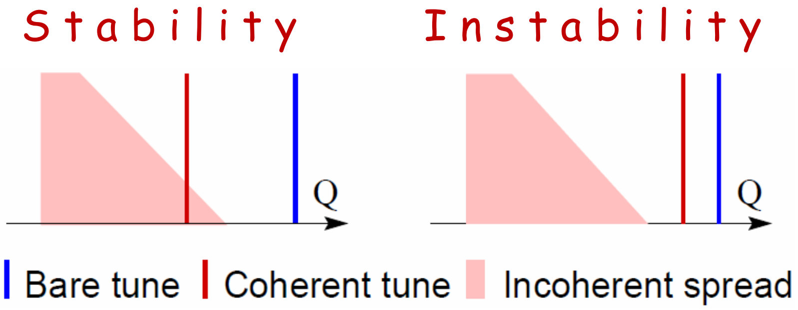
- Small addition can be produced by nonlinearity of external (focusing) field.

- However, space charge nonlinearity does not affect the coherent motion and Landau damping (V. Balbekov, 2006) .

So stability condition of a bunched beam is:

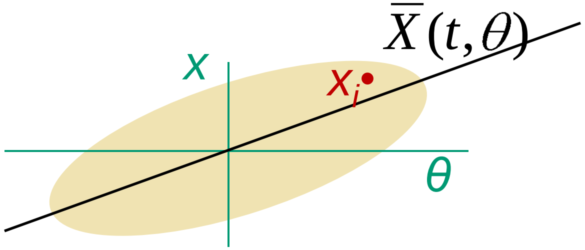
$$\boxed{(Q_{incoh})_{min} < Q_{coh} < (Q_{incoh})_{max}}$$

with the tune shifts averaged over time.



Bunched beam: equation of coherent betatron oscillation

- Equation of betatron oscillations with space charge is



$$\boxed{\frac{d^2 x_i}{dt^2} + \Omega^2 Q^2 x_i = \frac{eE(\theta, x_i - \bar{X}(t,\theta), y_i)}{m\gamma^3} + \cancel{(wake)}}$$

where $E(\theta,x,y)$ is steady-state beam field, $\bar{X}(t,\vartheta)$ – coherent displacement.

- Generally, *E* is *nonlinear function*, and space charge tune shift depends on amplitude.

- However, averaging in transverse phase space results in *linear* equation for function $X(t,\theta,p)$ describing coherent displacement in point (θ,p) of longitudinal phase space:

$$\boxed{\frac{d^2 X}{dt^2} + \Omega^2 Q^2 X = 2\Omega^2 Q \Delta Q(\theta)(X - \bar{X})}$$

Nonlinearity of space charge field and related tune spread do not affect coherent motion and the instability threshold.

- At constant density, ΔQ coincides with usual incoherent tune shift. For Gaussian beam, it is exaclty a half of small oscillations shift.

- The operator $\dfrac{d}{dt}$ is total time derivative: $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + \Omega_{syn}\dfrac{\partial}{\partial \varphi_{syn}} = -i\omega + \Omega_{syn}\dfrac{\partial}{\partial \varphi_{syn}}$

Equation of coherent betatron oscillation (cont'd)

- Operator $\dfrac{d}{dt}$ is the total time derivative:

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\Omega_{syn}\frac{\partial}{\partial\varphi_{syn}}=-i\omega+\Omega_{syn}\frac{\partial}{\partial\varphi_{syn}}$$

- Betatron frequency depends on the particle momentum (chromaticity). However, the dependence can be excluded by transformation:

$$X(t,\theta,p)=Y(\theta,p)\exp(-i\tfrac{d(\Omega Q)}{d\Omega}\theta-i\omega t)$$

- New variable $Y(\theta,p)$ satisfies the equation which does not include chromaticity:

$$(\omega-\Omega_0 Q_0)Y+i\Omega_s\frac{\partial Y}{\partial\varphi}=-\Omega_0\Delta Q(\theta)(Y-\bar{Y})$$
$$\rho(\theta)\bar{Y}(\theta)=\int F(\theta,p)Y(\theta,p)dp$$

$F(\theta,p)$ is longitudinal distribution function

$\rho(\theta)$ – corresponding linear density.

Instability threshold of a bunched beam does not depend on chromaticity

- All eigenfrequencies of these equations are real numbers. The statement does not contradict the possibility of Landau damping, but it means that LD is a non-exponential process.

Rigid mode

- At any distribution function, the equation has a special solution: *rigid mode* which does not depend on space charge and synchrotron frequency at all:

$$Y(\theta, p) = \bar{Y}(\theta) = 1, \qquad \omega = \Omega_0 Q_0$$

At zero chromaticity, the bunch oscillates as a solid without twist and rotation. With the chromaticity, traveling wave is superimposed on the bunch oscillations.

- This mode is especially important because it is universal and not prone to Landau damping, i.e. it is potentially unstable at very low intensity (threshold depends on nonlinearities which are neglected in this approximation).

Further results are presented graphically in terms of normalized parameters:

$$\mu = \frac{\Omega_s}{\Omega_0 \Delta Q_{\max}}, \qquad \nu = \frac{\omega - \Omega_0 Q_0}{\Omega_0 \Delta Q_{\max}}$$

Boxcar model – bunch of constant density

- At $\rho = const$, there are solutions in form of Legendre polynomials (F. Sacherer, 1972):

$$\overline{Y}_n(\theta) = P_n(\theta)$$

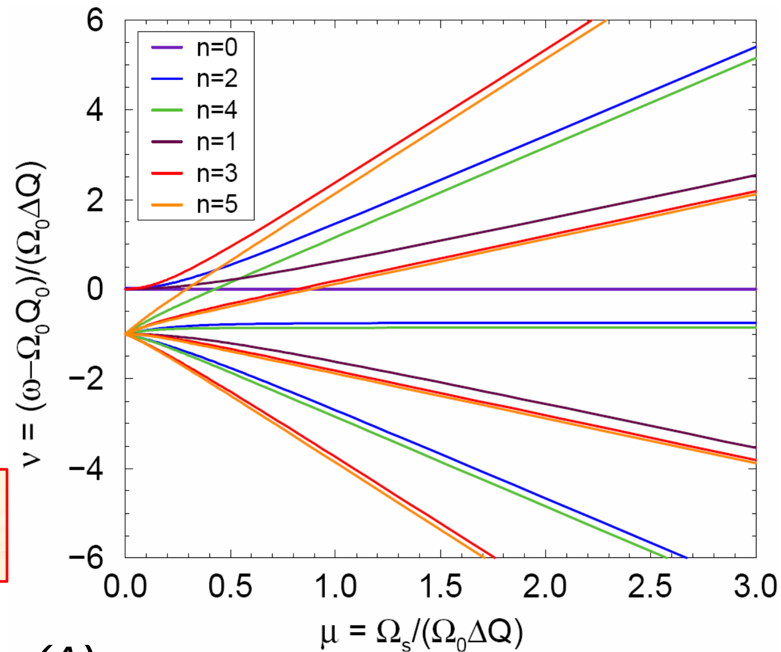
- At given n, there are $n+1$ different eigenfunctions $Y_{mn}(\theta,p)$ with eigentunes ν_{mn}.

- At $\mu = 0$, the eigentunes start either from point $\nu = 0$ (i), or from point $\nu = -1$ (ii) (bare and actual bertatron frequencies).

- At $\mu \ll 1$, eigenfuctions have almost linear polarization: either along θ (i), or along p (ii)

- At $\mu \gg 1$, all of them converge in groups with almost circular polarization (multipoles):

$$Y_{mn}(A,\varphi) \approx R_{mn}(A)\exp(im\varphi), \qquad \nu \approx m\mu$$



- At any m, there are a lot of *radial modes* $R_{mn}(A)$ that is eigenfunctions with different dependence on synchrotron amplitude, associated with different Legendre polynomials.

Revised Landau damping condition

- In normalized variables, the condition of Landau damping transforms:

$$\boxed{\Delta Q_{incoh,\,min} < \Delta Q_{coh} < \Delta Q_{incoh,max}} \Longrightarrow \boxed{-\bar{\rho}(0) < \nu < -\bar{\rho}(1)}$$

where $\bar{\rho}(A)$ is linear bunch density averaged over syncrotron phase ($\bar{\rho}(0) = 1$).

- However, at coherent frequency ω, the bunch spectrum includes frequencies $\omega + m\Omega_{syn}$ that is $\nu + m\mu$ in the normalized parameters.
Landau damping springs up if any of them falls within incoherent tunes range.

- Therefore the stability condition has to be modified:

$$\boxed{-1 < \nu - m\mu < -\bar{\rho}(1)}$$

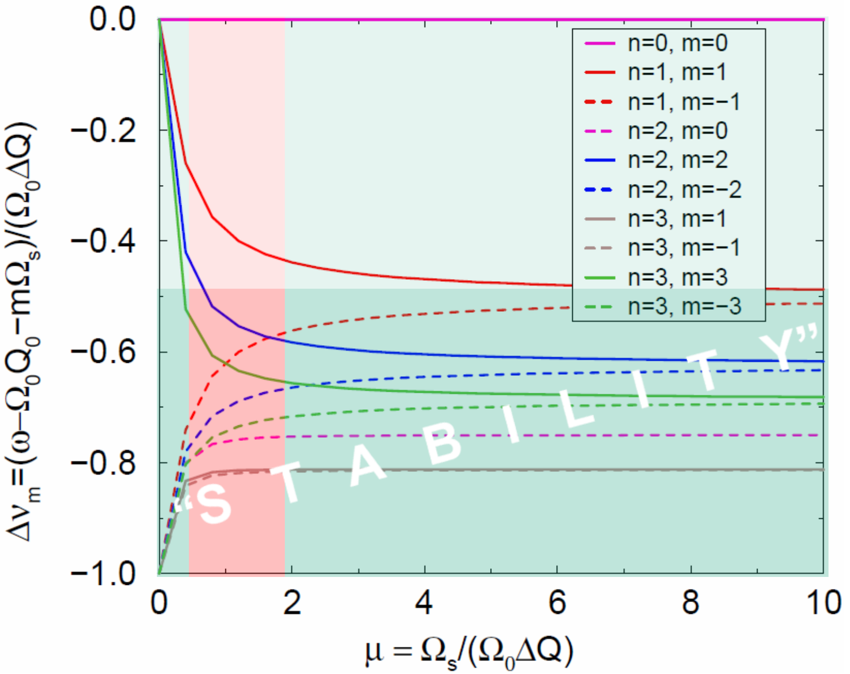
All the multipoles presented in the eigenfunction $Y_{mn}(A,\varphi)$ should be examined.
The multipole remaining at $\mu \to \infty$ is the most "suspicious"
(for example, $m = 0$ is the only harmonic inherent in the rigid mode).

Transformed eigentunes of the boxcar model are plotted

- Landau damping is impossible in this case because boxcar bunch has no tune spread.

- However, *let us assume for a moment* that some real bunch has the same coherent tunes and spread corresponding $\overline{\rho}(1) = 0.5$

- Then region of stability appears, which is marked in the picture as darker zones.



It offers some preliminary conclusions:

- The modes starting from point *Δv = μ = 0* are potentially unstable at low *μ* but can become stable at higher *μ.*

- The modes starting from point *μ* = 0, *Δv = -1* demonstrate opposite behavior but probably cannot reach the unstable zone.

- Higher modes are more stable (prone to Landau damping) in all the cases.

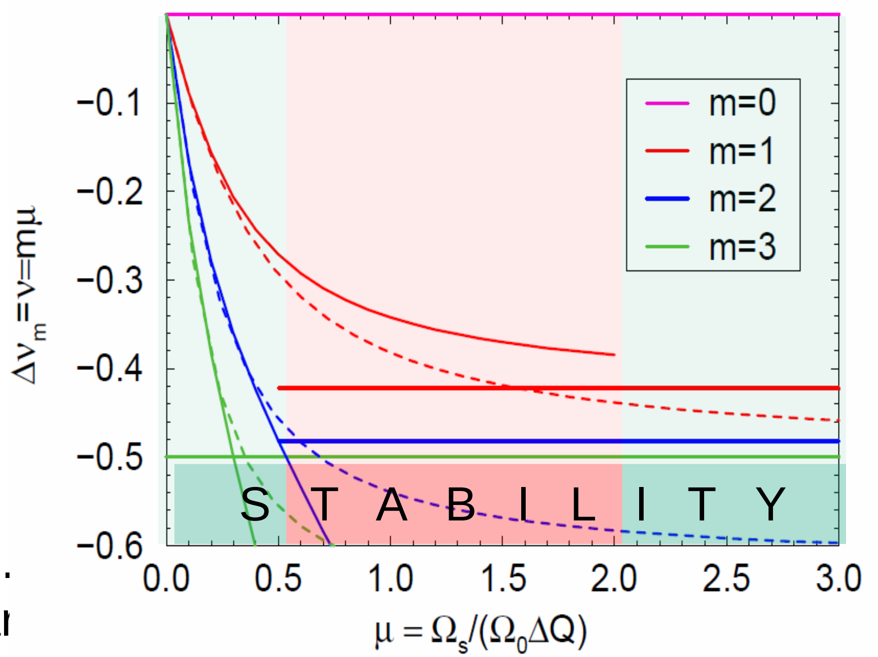
Actually, numerical solutions are achievable only at *μ* «1 and *μ* »1 (green zones). Extrapolations to the "alien" red zone *μ*~1 are needed to complete the picture.

Parabolic bunch

$$\rho(\vartheta) = 1 - \vartheta^2, \qquad \rho(A) = 1 - \frac{A^2}{2}, \qquad |\vartheta|, A \le 1$$

- Solid lines -- numerical solutions with approximations μ «1 and μ »1 extrapolated to the “doubtful” red zone.

  Dashed lines -- boxcar (for comparison)



It is seen

- A lot of potentially unstable modes at μ<~0.5. Their eigentunes are very close to the boxcar ones (confirmation of the assumption).

- However, almost all of them fall to region of stability at more μ. Only three unstable modes remain: m = 0, 1, 2 (lower radial mode each time).

- They have a continuation in μ » 1 zone, so the extrapolation looks very reasonably.

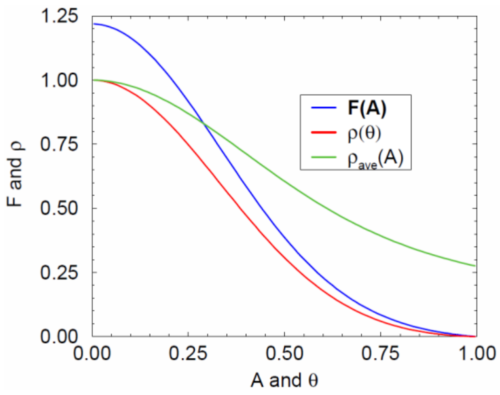
- The unstable “parabolic” tunes are slightly more than the comparable boxcar tunes. It looks like incoherent tunes push out the coherent ones from their room.

- All other modes are singular at μ »1, and form continuous spectrum at $\Delta\nu_m$< -0.5.

Gaussian bunch eigentunes

- Gaussian distribution truncated on *3σ* level is plotted.

$$F \propto \exp\left(\frac{1-A^2}{2\sigma^2}\right), \qquad A \le 1$$

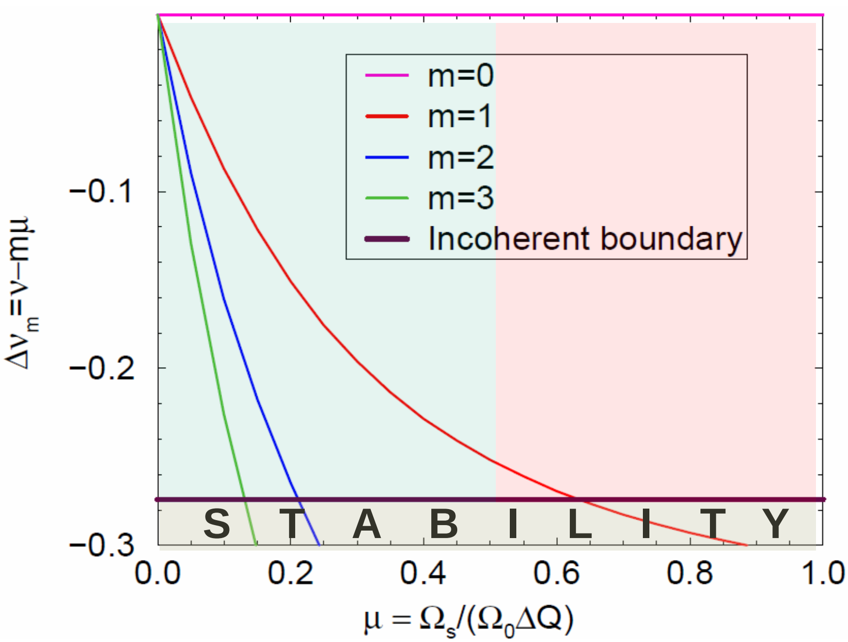


- Low *μ* approximation is shown in the plot.

- All the modes are unstable in the beginning, but become stable as *μ* increases (except rigid mode).

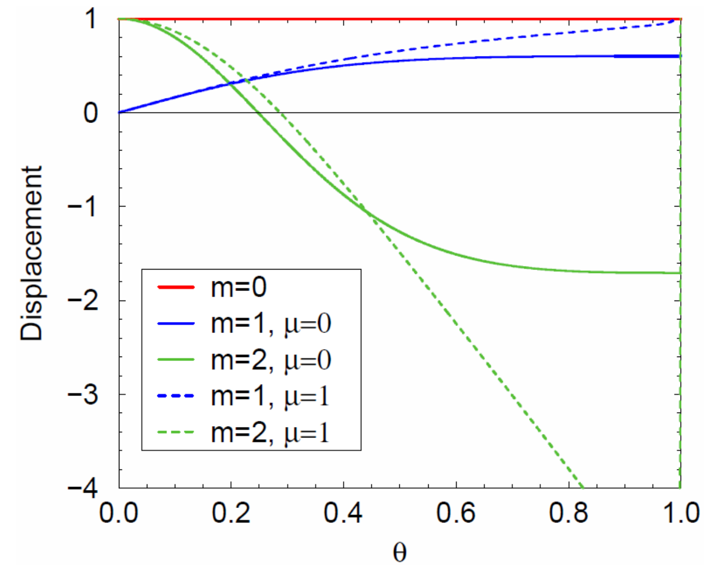
- It is confirmed by high *μ* approximation: all solutions (except rigid) are singular, prone to Landau damping, and form continuous spectrum at $\Delta\nu < -\rho(1) = -0.274$.

- Instability thresholds of lowest modes are:
  1st : $\mu <\sim 0.6$, that is $\Delta Q_{max} >\sim 1.7 Q_{syn}$
  2nd : $\mu <\sim 0.2$, that is $\Delta Q_{max} >\sim 5 Q_{syn}$,... etc.

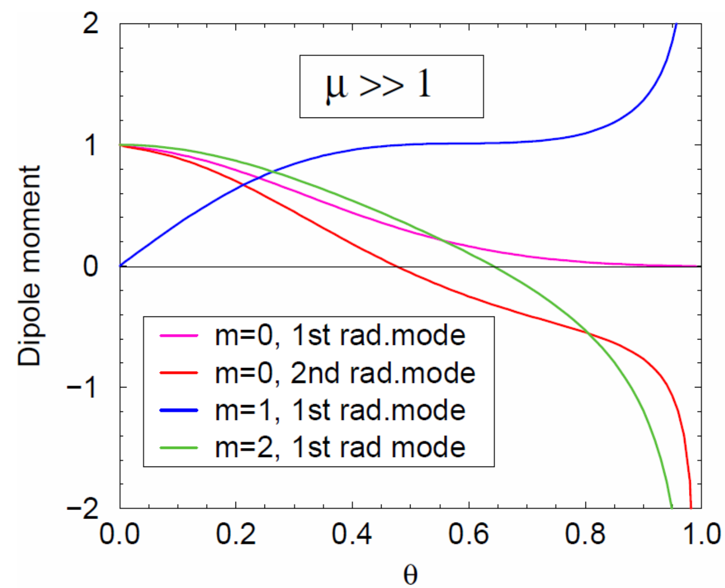


Gaussian bunch eigenfunctions

- *Low μ approximation*: lower eigenfunctions are plotted actually at $\mu=e\text{-}4$ (solid) and $\mu=1$ (dashed)
- Amplitude of the bunch tail increases at higher μ, because coherent frequency comes closer to the incoherent boundary.
- However, the displacement remains finite that is LD does not spring up in this approach.



- *High μ approximation*: The same eigenfunctions are presented at $\mu \gg 1$ (dipole moment against longitudinal coordinate).
- Rigid mode coincides with the bunch linear density in such a form (magenta curve).
- Other modes demonstrate a steep growth at the bunch tails what is a sign of Landau damping.



Bunched beam collective modes

- Self-maintaining oscillation which are not subjected to Landau damping can become really unstable under influence of a wake field.

- At given intra-beam mode $\bar{Y}(\vartheta)$ (=*1* for the most important rigid mode), linear density of the beam dipole moment in laboratory frame is:

$$\boxed{D(t,\vartheta) = \sum_j A_j N_j \rho(\vartheta - \vartheta_j - \Omega_0 t)\bar{Y}(\vartheta - \vartheta_j - \Omega_0 t)\exp[-i\chi(\vartheta - \vartheta_j - \Omega_0 t) - i\omega t]}$$

where A_j, N_j, θ_j are amplitude, intensity, position of j-th bunch

Wake field should be found at this dipole moment.

- Substituting it to equation of transverse oscillations as a small perturbation, one can find characteristics of the collective modes: relative amplitudes A_j, and frequency ω, including growth/decay rate.

- Additional investigation is needed at very low μ because of degeneration of the basic system (all eigentunes are coalesced into $\nu = 0$, any $Y(\theta)$ is a solution).

Summary

- Dominating SCI determines parameters of intra-bunch coherent oscillations including their frequency, shape, and threshold of possible instability.
- It create incoherent tune spread and specifies position of coherent tune, and, by doing so, governs Landau Damping.
- Generally, increasing SCI seeks to push out the coherent tune from the incoherent range. However, the effort is not sufficient if maximal incoherent shift is about less than synchrotron tune. Then the coherent tune remains in the incoherent range resulting in Landau damping and beam immunity to the instability.
- At higher SCI, coherent tune leaves the incohertent range, and Landau damping disappears making possible instability (lower mode first).
- However, at any distribution, there is a special *rigid mode* which is not sensitive to space charge and synchrotron oscillations, and therefore is not vulnerable to LD.
- Even at dominant SCI, wake field should be invoked to determine characteristics of collective motion including the instability growth rate.