Van Kampen Modes for Bunch Longitudinal Motion

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Content

- Main equations
- Steady state solutions
- Integral equation for modes
- Van Kampen modes: continuous and discrete spectrum
- Classical plasma result
- Single-harmonic RF results
- Two-harmonics RF: BS mode
- Two-harmonics RF: BL mode

Main equations

$$H(z,p)=\frac{p^2}{2}+U(z)+V(z)$$ Hamiltonian

$$U(z)=U_{\mathrm{rf}}(z)-\int\lambda(z')W(z-z')dz'$$ Steady state potential

$$V(z)=-\int\rho(z')W(z-z')dz'$$ Perturbation of the potential

$$\lambda(z)=\int F(I)dp$$ Steady state linear density

$$\rho(z)=\int f(I,\phi)dp$$ Linear density perturbation

$$\frac{\partial f}{\partial t}+\Omega(I)\frac{\partial f}{\partial\phi}-\frac{\partial V}{\partial\phi}F'(I)=0$$ Vlasov equation

Steady state distribution

$$U(z) = U_{\mathrm{rf}}(z) - \int \lambda(z')W(z-z')dz' \equiv U_{RHS}[\lambda]$$

$$\lambda(z) = 2\int_{U(z)}^{H_{\max}} \frac{F(I(H))}{\sqrt{2(H-U(z))}}dH$$

$$I(H) = \frac{1}{\pi}\int\sqrt{2(H-U(z))}dz$$

Numerical solution: use of an artificial time:

$$U(z) = U_{RHS}[\lambda] \quad \rightarrow \quad \frac{\partial U}{\partial t} = -\frac{\varepsilon}{\Delta t}\left(U - U_{RHS}[\lambda]\right)$$

$$\varepsilon \cong 0.05 - 0.25$$

$U(z, t=0) = U_{\mathrm{rf}}$; $\lambda(z, t=0) = \lambda_{\mathrm{rf}}(z)$ ⟵ as if wake=0 at t=0

Takes 2-10 min with my Mathematica code

Existence and Uniqueness of Solution

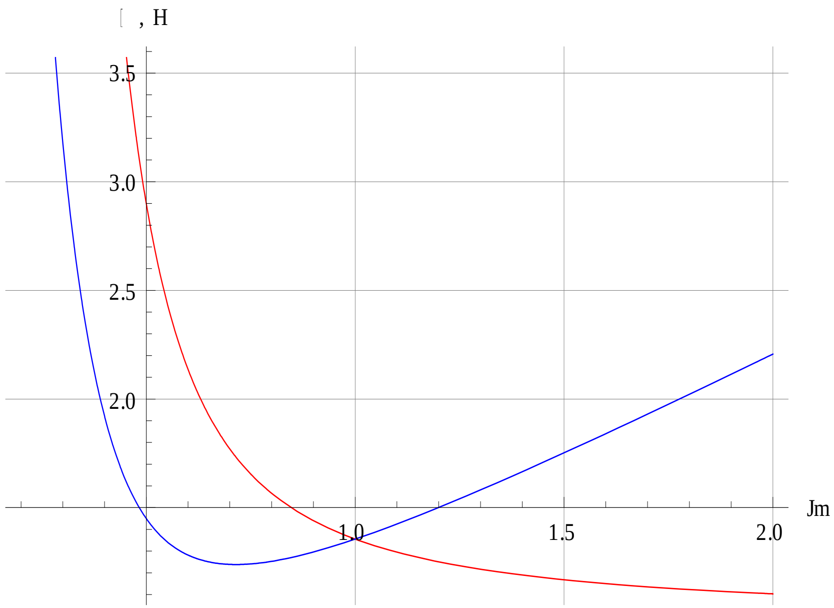
- For thermal equilibrium, F~exp(-H/T), the resulted Haissinski equation may have no solution for negative mass and space charge impedance.
- However, we assume F(I), not F(H). It does make a difference. Formally, $F(I)$ is normalized a priori, while $F(H)$ normalization

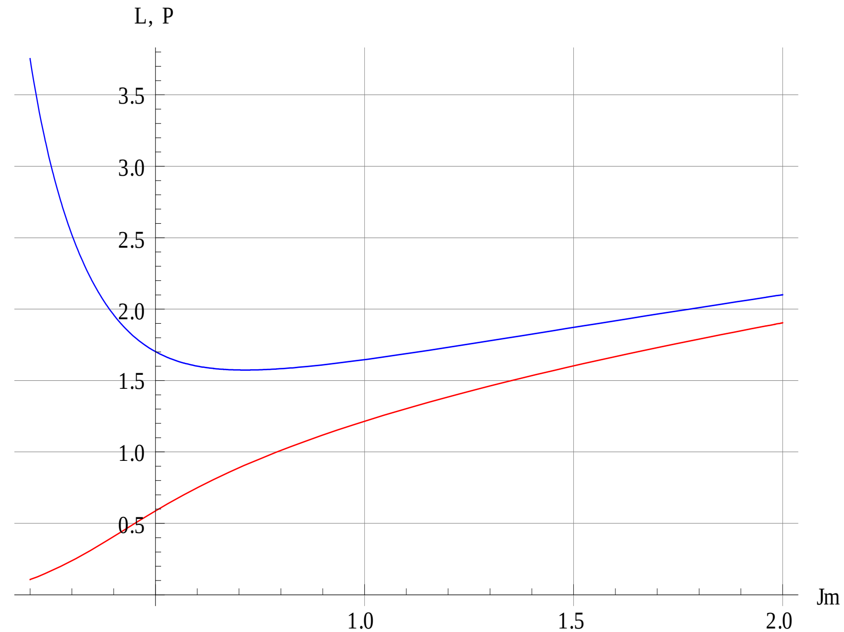
$$2\pi \int F(H)dI = 1$$

cannot be given from the beginning, but solved jointly with the entire set of integral equations.

- In other words, for given number of particles, rms momentum spread $<p>$ may have a minimum as a function of emittance I: too small temperatures may be impossible.

Example: H-P $F(I) \propto \sqrt{I_m - I}$ for space charge





Incoherent frequency and maximal Hamiltonian (left),
Bunch length and maximal momentum offset (right) as functions of emittance I_m.

Note minima for the max momentum and max Hamiltonian. $H_m = H(I_m) = \Omega(I_m) I_m$

That is why a solution always exists for action-distributions, as $F(I) \propto \sqrt{I_m - I}$,

and not always for Hamiltonian-distributions, as $F(H) \propto \sqrt{H_m - H}$.

Existence and Uniqueness of I-Solution

- Theorem I (action domain):

  If the impedance grows not too fast,

$$\lim_{q\to\infty} \mathrm{Im}(Z_{\|}(q)/q^2) = 0$$

  there is at least one solution for the I-type steady state. This asymptotic is satisfied for all known impedances.

In general, the solutions relate to the steady-state equation

$$J^2 q^4 = 1 + Dq^2\,\mathrm{Im}(Z_{\|}(q)) = 0;$$

$$p = qJ\,; \quad \Omega = q^2 J\,.$$

where $D = \dfrac{N\eta r_0 c\omega_{\mathrm{rf}}^2}{\Omega_0^2 \gamma C}$ is intensity parameter, J is the emittance.

Except for a narrow-band impedance, the solution is unique.

Existence and Uniqueness of H-Solution

- Theorem H (Hamiltonian domain):

  If the impedance grows so slow that

$$\lim_{q\to\infty} \operatorname{Im} Z_{\parallel}(q) = 0$$

  there is at least one solution for the H-type steady state. This asymptotic is not satisfied for space charge (SC) and resistive wall (RW).

In general, the solutions relate to the steady-state equation

$$Hq^2 = 1 + Dq^2 \operatorname{Im} Z_{\parallel}(q) = 0;$$

$$p = \sqrt{H}; \quad \Omega = q\sqrt{H}.$$

where $\kappa \propto N$ is intensity parameter, H is the average Hamiltonian.

For SC and RW, there are either no or 2 solutions.

Vlasov Equation

$$\frac{\partial f}{\partial t} + \Omega(I)\frac{\partial f}{\partial \phi} - \frac{\partial V}{\partial \phi}F'(I) = 0$$

$$V(z) = -\int \rho(z')W(z-z')dz'$$

Following Oide & Yokoya (1990) :

$$f(I,\varphi,t) = e^{-i\omega t}\sum_m \; f_m(I)\cos m\varphi + g_m(I)\sin m\varphi$$

$$\left[\omega^2 - m^2\Omega^2(I)\right] f_m(I) = -2m^2\Omega(I)F'(I)\sum_n V_{mn}$$

$$V_{mn} = -\frac{2}{\pi}\int dI' \int_0^{\pi} d\varphi \int_0^{\pi} d\varphi' \cos(m\varphi)\cos(n\varphi')W(z-z')\, f_n(I')$$

matrix elements

$$z(I,-\varphi) = z(I,\varphi)\,; \quad p(I,-\varphi) = -p(I,\varphi)$$

$$z(I,\varphi=0) = z_{\min}(I) \quad z(I,\varphi=\pi) = z_{\max}(I)$$

phase definition

Matrix Elements

$$V_{mn}(I) = -2\,\mathrm{Im}\int dI' f_n(I') \int_0^\infty dq \frac{Z_{\|}(q)}{q} G_m(q,I) G_n^*(q,I')$$

$$G_m(q,I) = \int_0^\pi \frac{d\phi}{\pi} \cos(m\phi) e^{iqz(I,\phi)}$$

Note: No bunch-to-bunch interaction here yet

These equations solve the problem.

CPU time ~ (number of azimuthal modes)^2.

If the wake is not strong compared with the RF, azimuthal mode coupling can be neglected, at least as a first step analysis.

Particle loss, emittance growth, or instability may happen because of:

- Finite bucket capacity. For a full RF bucket, and effectively repulsive wake, the threshold is 0.

- Azimuthal or radial mode coupling.
  - Azimuthal mode coupling requires rather high intensity
  - Radial mode coupling may happen either for non-monotonic distributions $F(I)$, or for significantly asymmetric distorted potential well.

- Loss of Landau damping (LLD).

- Strong bunch-to-bunch interaction.

- Improper tuned damper or feedback.

Integral Equation

Neglecting azimuthal mode coupling:

$$\left[\omega^2 - m^2\Omega^2(I)\right] f(I) = -m^2\Omega(I)F'(I)\int K_m(I,I')f(I')dI'$$

$$K_m(I,I') = -\frac{2}{\pi}\int_0^{\pi} d\phi \int_0^{\pi} d\phi' \cos(m\phi)\cos(m\phi')W(z-z') =$$

$$= -2\,\mathrm{Im}\int_0^{\infty} dq\,\frac{Z(q)}{q} G_m(q,I)G_m^*(q,I')$$

Van Kampen modes for plasma

- What are the eigenfunctions and eigenvalues for the Vlasov equation?

$$\frac{\partial \tilde{f}}{\partial t} + p\frac{\partial \tilde{f}}{\partial z} - \alpha\frac{\partial \tilde{\lambda}}{\partial z}F' = 0 \ ; \quad \tilde{\lambda} \equiv \int \tilde{f} dp \ ; \quad \alpha = u_p^2 = \omega_p^2 / k^2$$

- With $\tilde{f} = \exp(ikz - i\omega t)\tilde{f}_\omega(p)$, it leads to

$$(v - p)\tilde{f}_v(p) = -\alpha F'(p)\int \tilde{f}_v(p')dp' \ ; \quad v = \omega / k$$

- Eigenfunctions $\tilde{f}_\omega(p)$ can be normalized as $\int \tilde{f}_\omega(p)dp = 1$. With that, an infinite set of solutions follows:

$$\tilde{f}_v(p) = -\text{P.V.}\frac{\alpha F'(p)}{v - p} + A(v)\delta(v - p) \ ; \quad A(v) = 1 + \alpha\,\text{P.V.}\int\frac{F'(p')dp'}{v - p'}$$

Van Kampen modes for plasma (2)

$$\tilde{f}_v(p) = -\mathrm{P.V.}\frac{\alpha F'(p)}{v-p} + A(v)\delta(v-p)\,; \quad A(v) = 1 + \alpha\,\mathrm{P.V.}\int\frac{F'(p')dp'}{v-p'}$$

Here v is an arbitrary real number. The eigenfrequency ◆$=kv$.
This infinite set of eigenmodes was found for plasma oscillations in 1955 by a Dutch physicist N. G. van Kampen.

If the beam (or plasma) is stable, this continuous set of eigenmodes is complete. Any smooth initial condition can be expanded over that singular basis. The Landau damping results as phase mixing of the van Kampen modes.

In case of unstable beam, the continuous spectrum is not complete. On top of that, there is a finite number of mode pairs (just one pair in our space charge example):

$$\mathrm{Im}(\omega_\mu) = \pm\Gamma_\mu$$

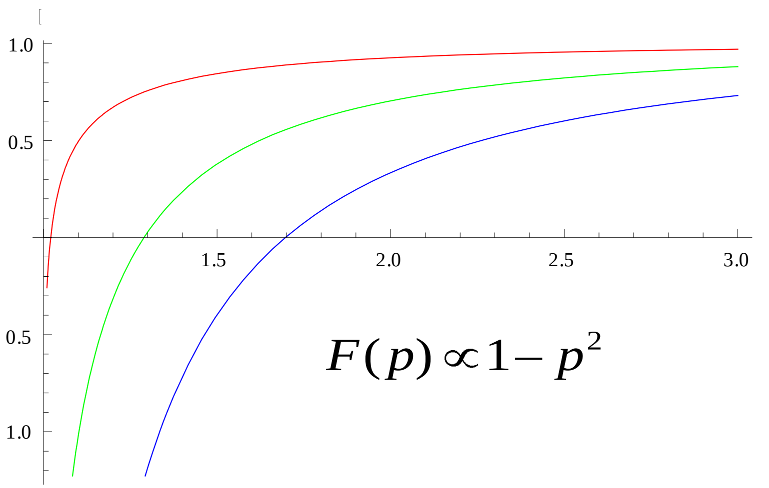
Contrary to the continuous spectrum, these discrete modes are smooth functions.

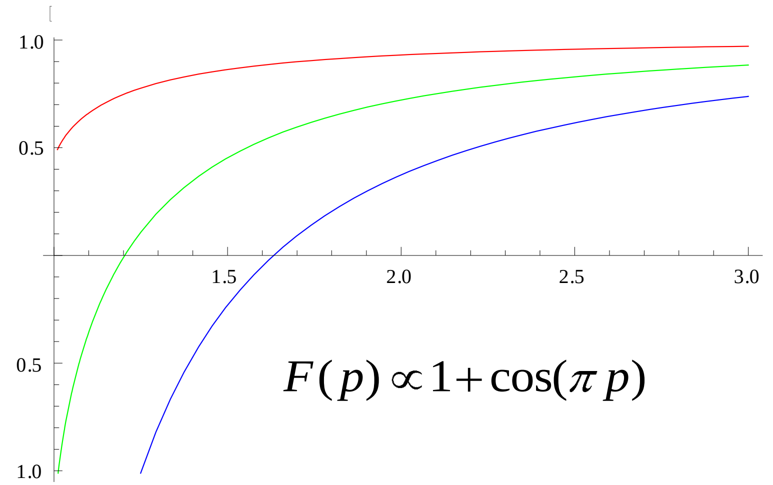
Van Kampen modes normally appear as numerical solutions of Vlasov equation.

Loss of Landau damping for van Kampen modes

For finite width of the distribution, $F(|p| > p_{\max}) = 0$, a discrete undamped mode can appear outside the continuous spectrum. The condition for that is

$$\varepsilon(v) = 1 + u_p^2 \int \frac{F'(p)dp}{v - p} = 0\,, \quad u_p = \omega_p / k$$





$u_p =$ 0.5, 1.0, 1.5

For the hard-edge distribution (left), the discrete mode appears at any interaction parameter.

For the soft-edge case (right), there is a threshold for LLD.

Van Kampen modes for a bunch

$$\left[\omega^2 - m^2\Omega^2(I)\right] f(I) = -m^2\Omega(I)F'(I)\int K_m(I,I')f(I')dI'$$

This is a linear equation for a spectrum of eigen-system.

Similar problem for Langmuir oscillations in collisionless plasma was solved in 1955 by van Kampen, who found that in general the spectrum consists of a continuous and discrete parts.

The continuous modes are singular, their spectrum coincides with the particle spectrum in the distorted potential well, and their decoherence describes Landau damping:

$$f_\omega(I) = A(I)\,\mathrm{P.V.}\frac{1}{m\Omega(I)-\omega} + B(I)\delta(m\Omega(I)-\omega) + C(I)$$

Instead, the discrete modes are smooth regular functions. They are either unstable, or without Landau damping. The discrete spectrum not necessarily exists.

At zero current limit, there are either no discrete modes, or they go infinitesimally close to the continuous spectrum.

Parabolic RF: LLD w/o threshold

- For parabolic RF potential, there is always at least a single mode of the discrete spectrum: bunch motion as a whole. There are no resonant particles there, since all the incoherent frequencies are either suppressed or elevated by the potential well distortion.

- Although $\mathrm{Im}(\omega_{11}) = 0$, a slightest bunch-to-bunch talk would make this mode unstable. This is an example of loss of Landau damping (LLD).

- Check for this mode is a good tool to verify the code.

- In general, there is no that mode for other RF shape.

Units

$$U_{\rm rf}(z) = (1-\cos z) + \frac{\alpha_2}{4}(1-\cos 2z)$$

$$\alpha_2 = \begin{cases} 0 & \Rightarrow \mathrm{SH} \\ 1 & \Rightarrow \mathrm{BS} \\ -1 & \Rightarrow \mathrm{BL} \end{cases}$$

- SH: single RF harmonic
- BS: bunch shortening 2nd RF harmonic
- BL: bunch lengthening 2nd RF harmonic

Conventional eVs for the action are obtained from its dimensionless value by $\frac{E_0}{\omega_{\rm rf}} \frac{\nu_{s0}}{\eta h_{\rm rf}}$

For the energy offset this factor is given by $\delta E / E_0 = -p \frac{\nu_{s0}}{h_{\rm rf}\eta}$

With ν_{s0} as the zero-amplitude bare synchrotron tune in SH mode.

RW Interaction constant

Below, the results are shown for a pure resistive wake, and $m = 1$

$$W(s) = -k_W / \sqrt{-s}$$

$$Z(q) = k_W (1 - i \operatorname{sgn} q)\sqrt{\pi q / 2}$$

For these dimensionless units, the interaction constant k_W is:

$$k_W = \frac{N r_0 \eta h_{RF}^2}{\gamma b C \nu_{s0}^2} \sqrt{\frac{2 C h_{RF} c}{\pi \sigma}}$$

Note that k_W does not depend on the beam energy.

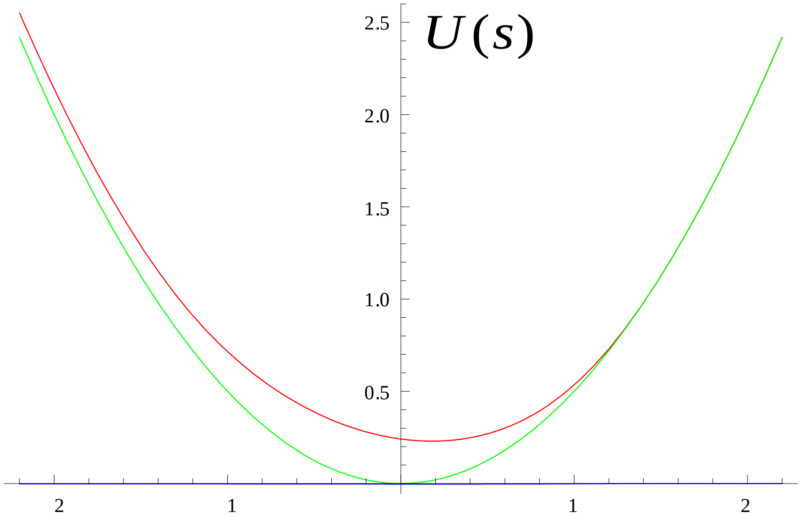
Why Landau damping may be lost

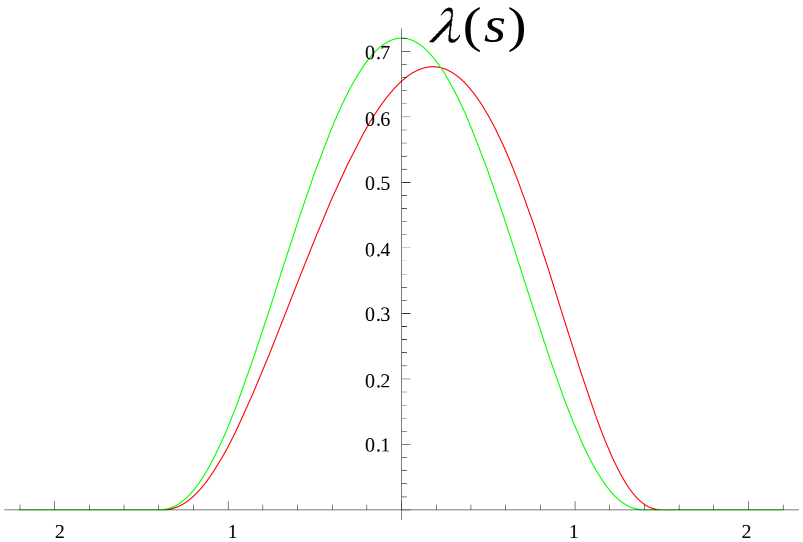
- Particle interaction always acts stronger on the incoherent frequencies (potential well distortion), than on the mode frequencies.

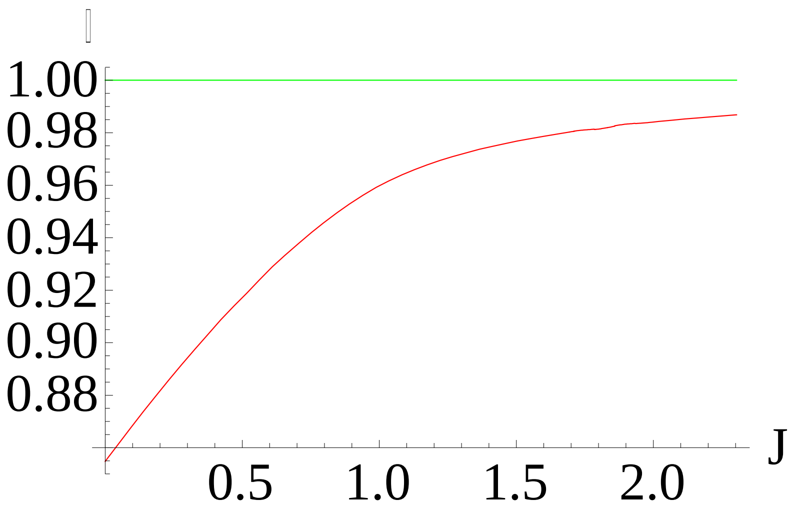
- An example: parabolic potential, where the first mode ω_{11} does not depend on the impedance at all.

- Thus, when all the particle frequencies are wake-suppressed, the highest-frequency mode jumps out of the continuous spectrum, and becomes discrete. For the SH RF, it means that lowest-amplitude particles are mostly excited by this mode.

Parabolic Potential Well Distortion





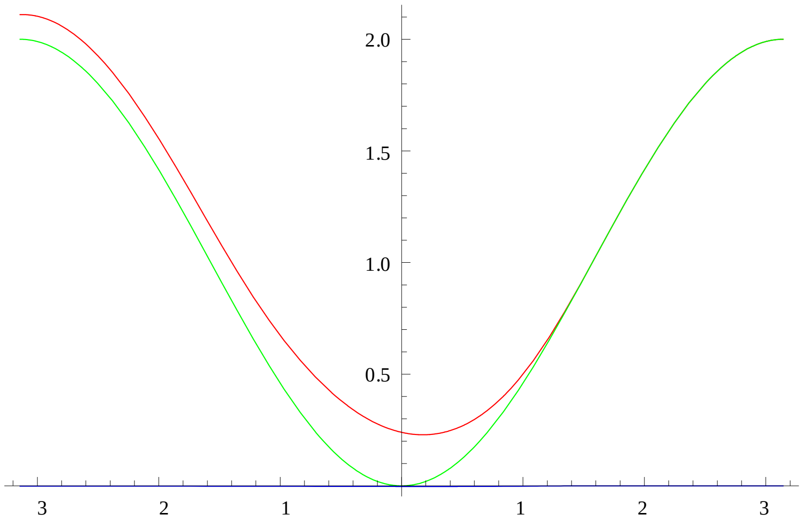


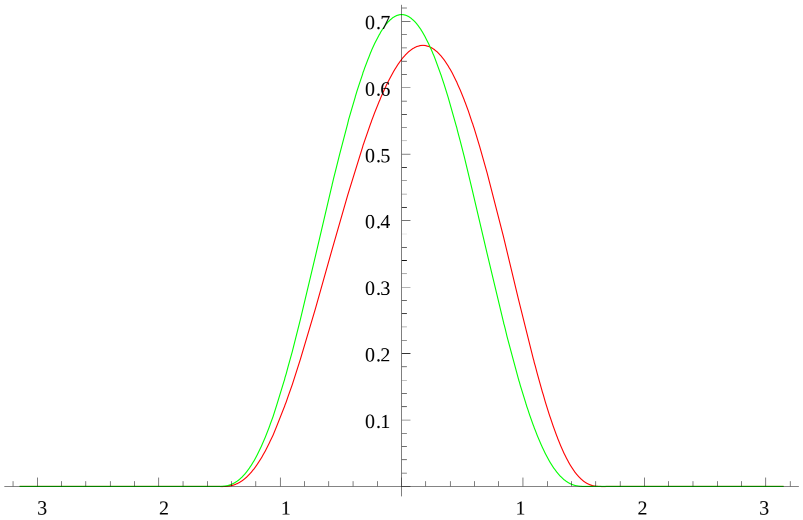
$$k_W = 0.2$$

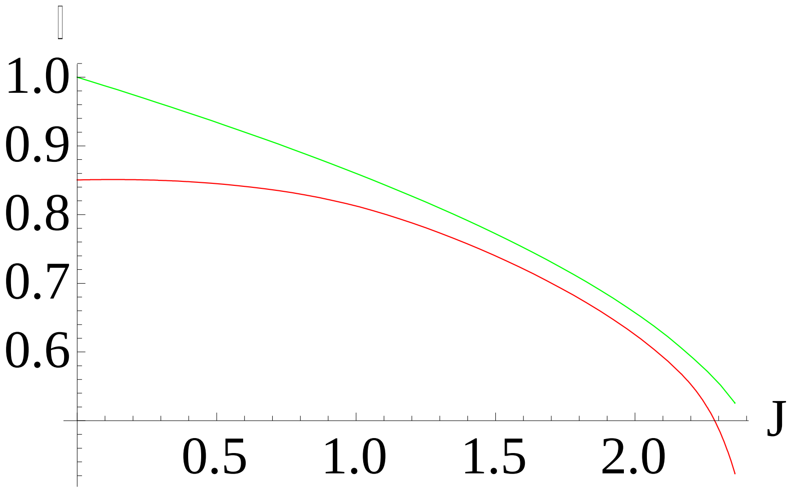
$$F(J) \propto (1 - J / J_{\lim})^2$$

$$J_{\lim} = 1.0$$

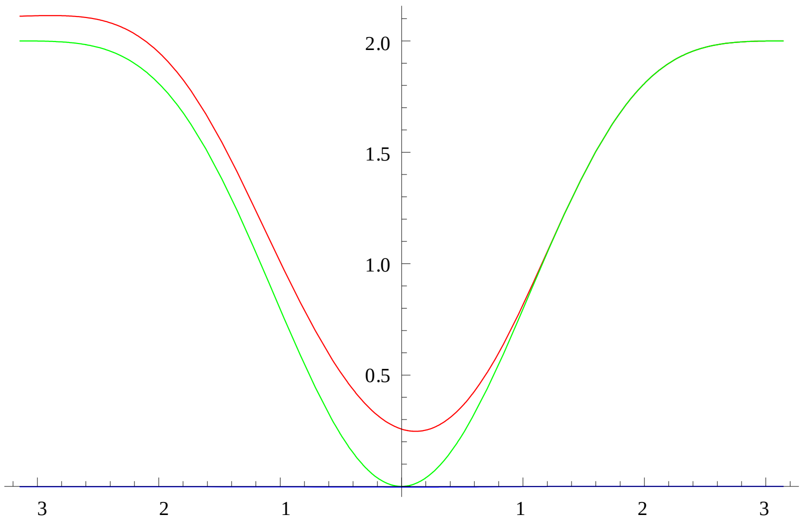
Same for SH

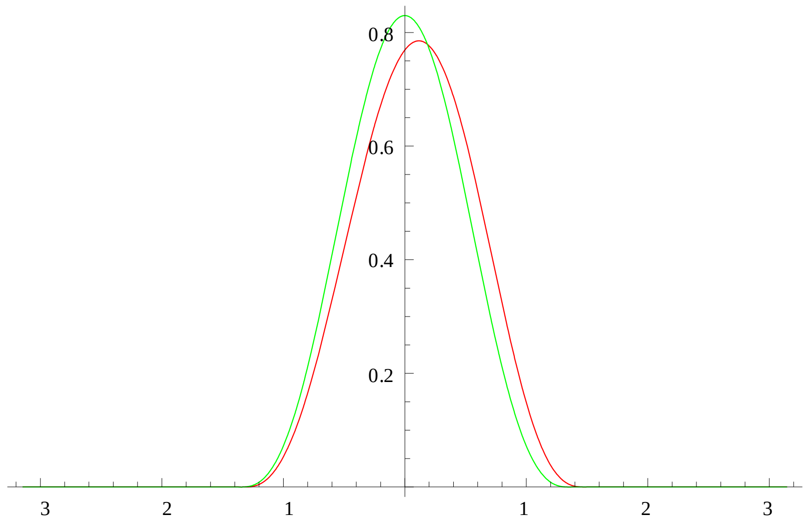

2.0
1.5
1.0
0.5
3
2
1
1
2
3

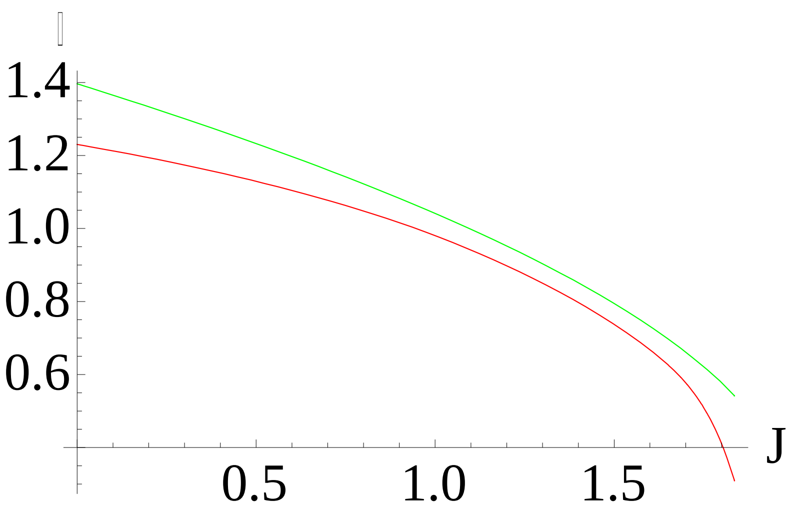

0.7
0.6
0.5
0.4
0.3
0.2
0.1
3
2
1
1
2
3


1.0
0.9
0.8
0.7
0.6
0.5
1.0
1.5
2.0
J

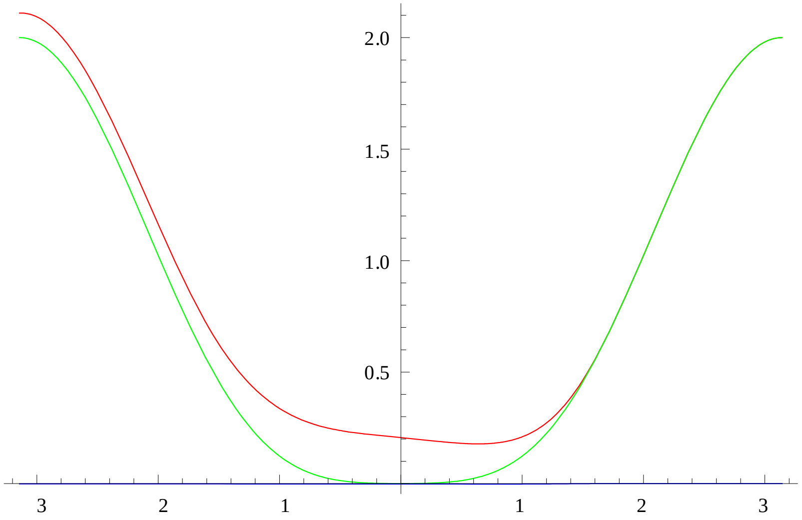
Same for BS

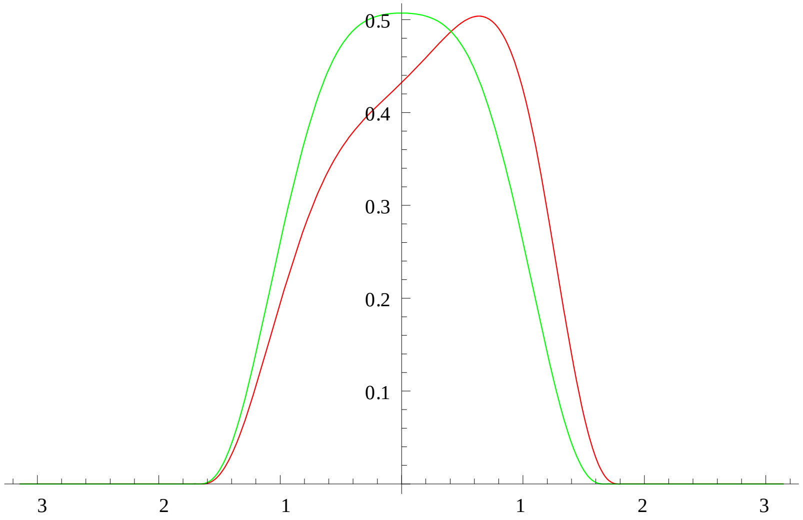


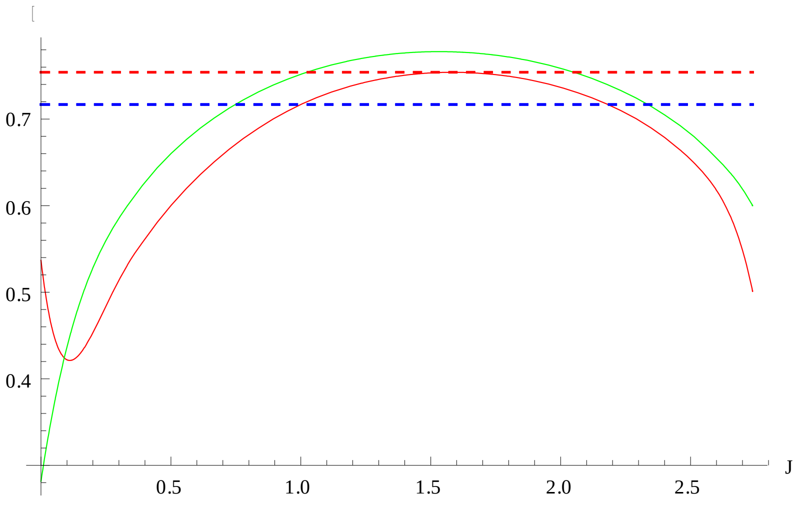




Same for BL

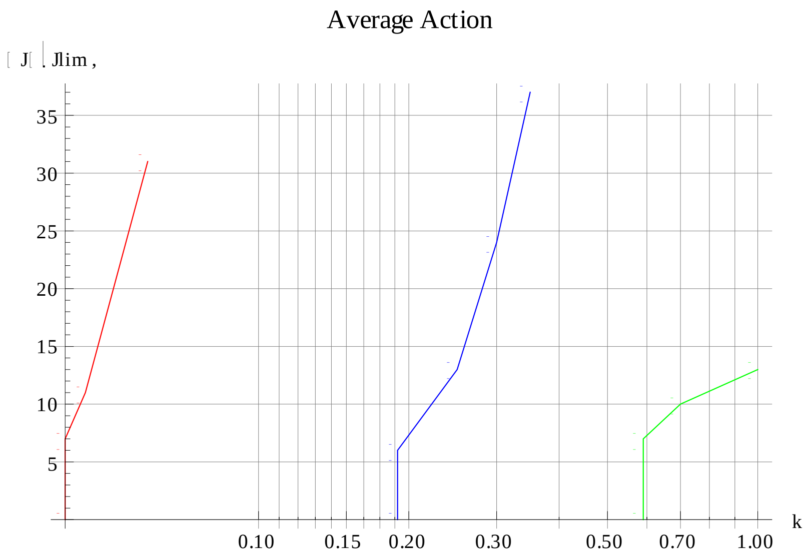


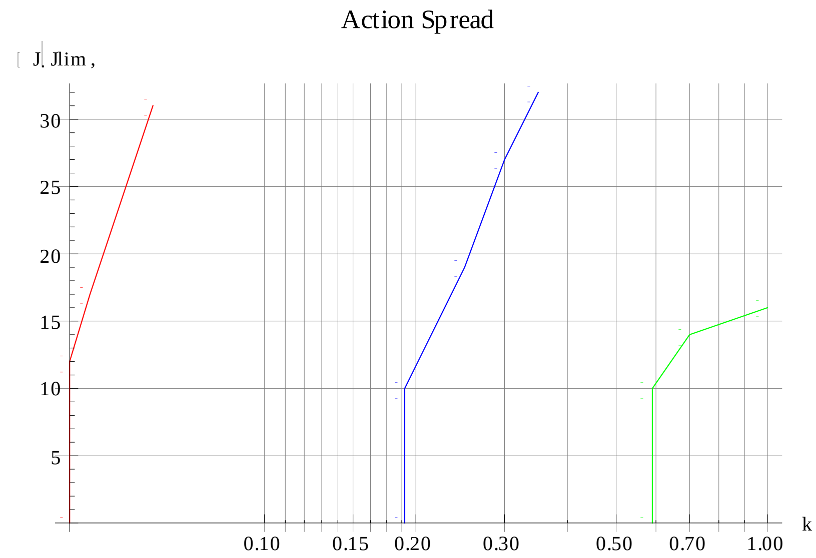




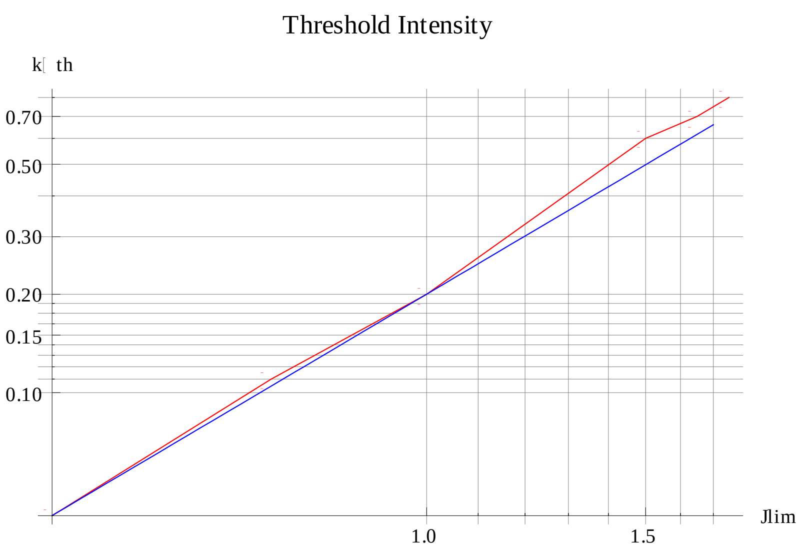
← radical LLD

SH case, for the phase space density $F(J) \propto \sqrt{1 - J / J_{\mathrm{lim}}}$





Relative average action and action spread for $J_{\mathrm{lim}} =$ 0.5 , 1.0 , 1.5

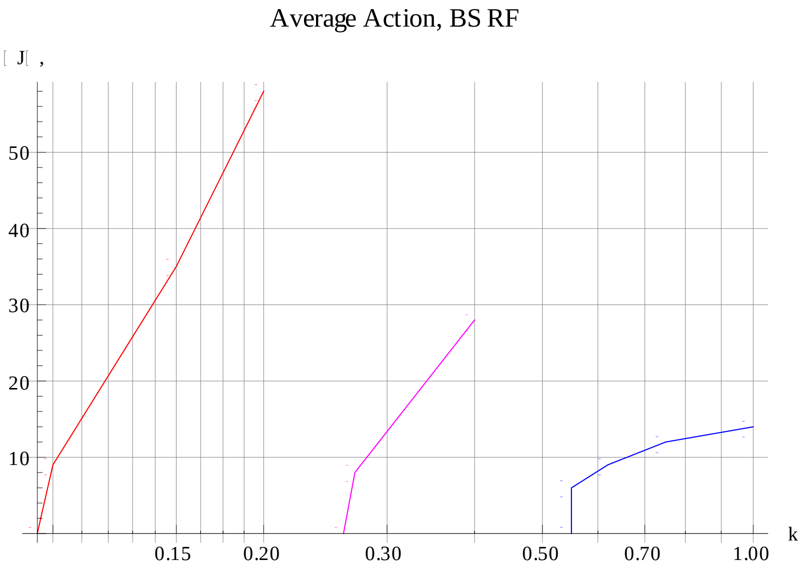


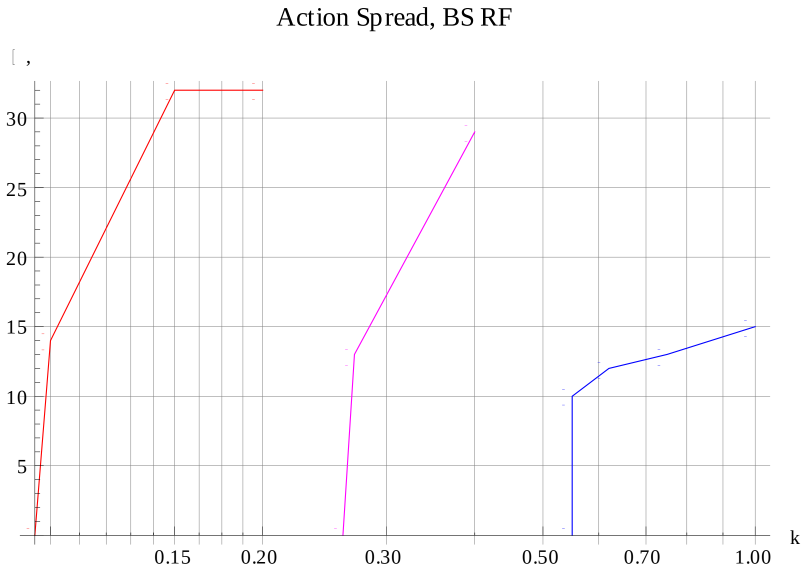
Threshold intensity versus emittance.

Blue line – fit $k_{\mathrm{th}} = 0.2 J_{\mathrm{lim}}^{9/4}$

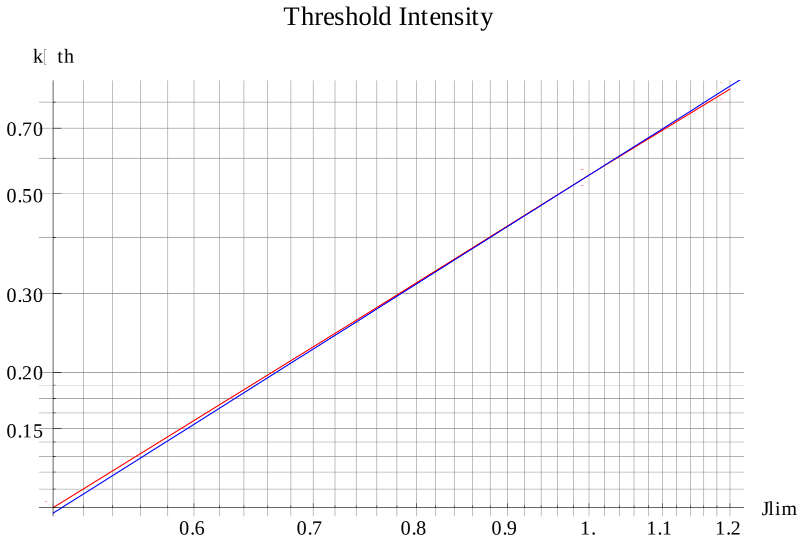
Power 9/4 agrees with naïve rigid-bunch model.

BS RF mode





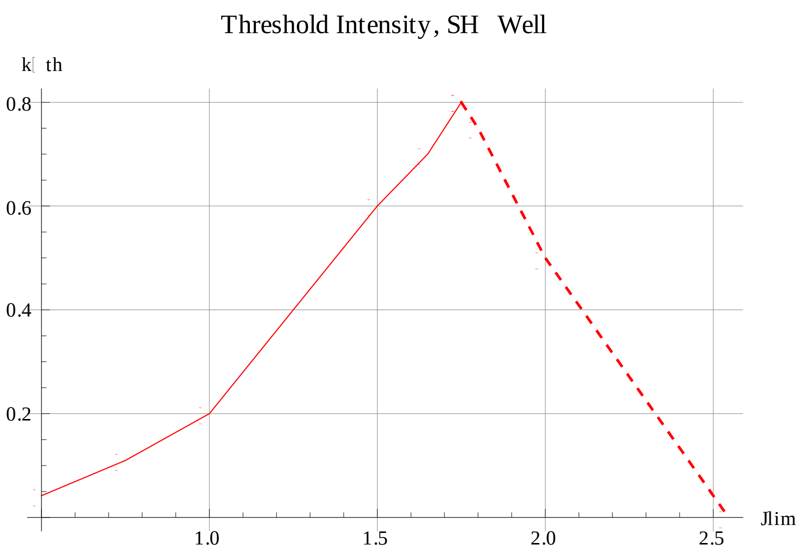
Relative average action and action spread for $J_{\text{lim}} =$ 0.5 , 0.75 , 1.0

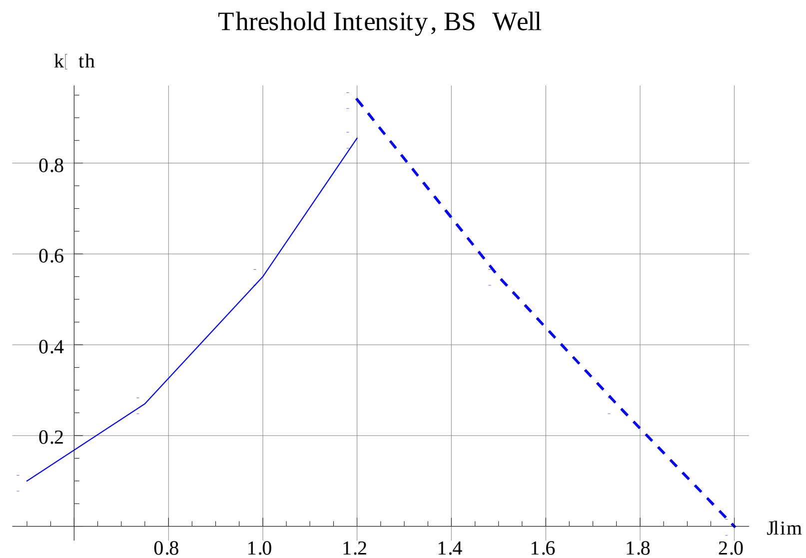


Threshold intensity versus emittance.

Blue line – fit $k_{\text{th}} = 0.55 J_{\text{lim}}^{5/2}$

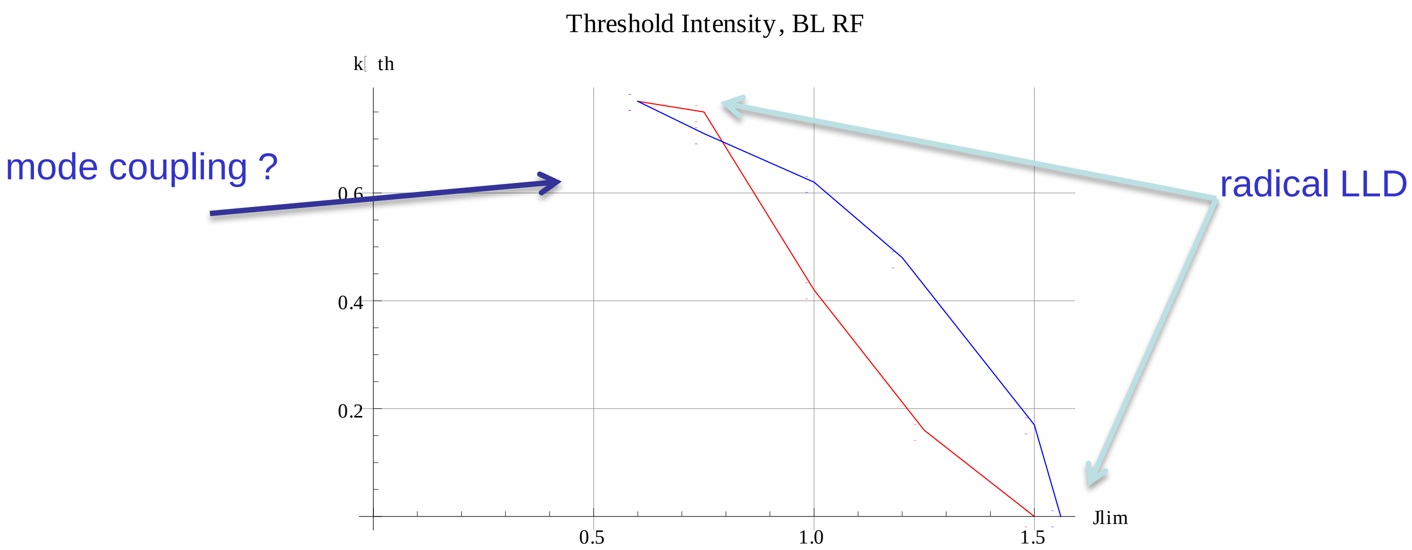
SH and BS, bucket capacity is taken into account





For the Tevatron, with $\mathrm{Re}(Z/n) = 0.8$ at 53 MHz (Ng, Run II HB), the resistive impedance gives $k_w = 0.08$. For $J_{\mathrm{lim}} \approx 0.9$ at the top energy, this corresponds to ~ 15 times below the red-line threshold. The inductive impedance is calculated ~ 2-3 times higher, so the gap may be expected to reduce. Also, the real distribution drops faster than HP. All that together may explain the observed dancing bunches. Computations for the specific Tevatron impedance are needed.

BL RF mode

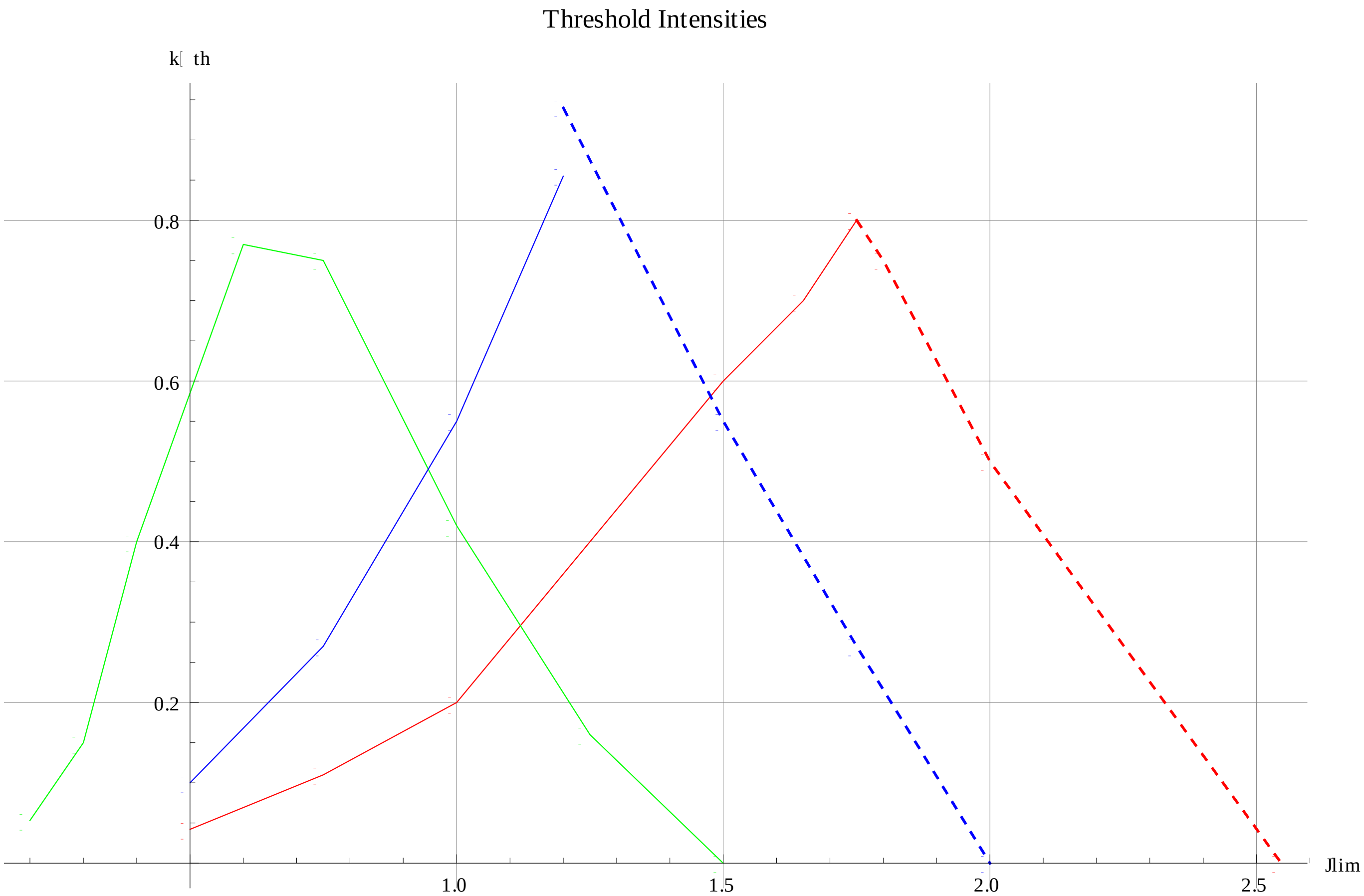


blue - $F(J) \propto (1 - J/J_{\lim})^2$ red - $F(J) \propto \sqrt{1 - J/J_{\lim}}$

For BL RF, the discrete mode excites mostly the tail particles.

NB: Low-current formal bucket capacity $J_{max} = 3.0$.
LLD reduces is twice.

Stability areas for all the 3 RF modes



Next steps

- Azimuthal mode coupling.
- Space charge / inductive impedance. Comparison with Tevatron.
- Multiple bunches and over-revolution wakes.
- Dampers and feedbacks.

Summary

- Existence theorems for steady state are formulated for distributions in action and Hamiltonian domains.

- Radical LLD concept is introduced and discussed.

- Intensity-Emittance areas of availability are found for RW impedance and single harmonic, bunch shortening and bunch lengthening RF modes.