# Nonlinear Optics as a Path to High-Intensity Circular Machines

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COLLIDING BEAMS: PRESENT STATUS; AND THE SLAC PROJECT*

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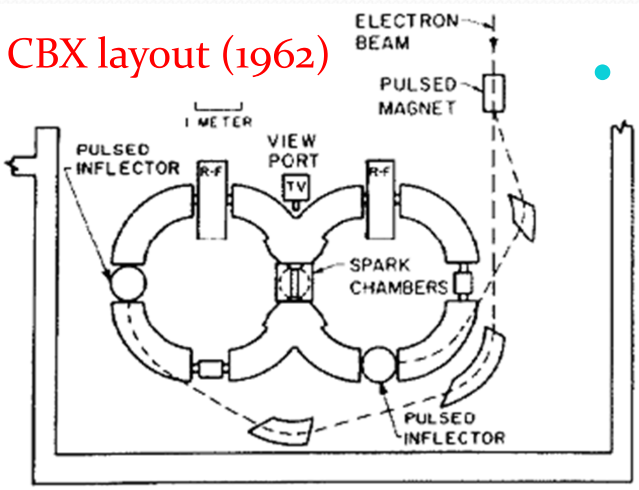
**Report at HEAC 1971**

The discovery in the early '60's at the Princeton-Stanford ring of what was thought to be the resistive wall instability brought the realization that circular accelerators are fundamentally unstable devices because of the interaction of the beam with its environment. Stability is achieved only through Landau damping and/or some external damping system.

CBX layout (1962)




- 1965, Priceton-Stanford CBX: First mention of an 8-pole magnet
  - Observed vertical resistive wall instability
  - With octupoles, increased beam current from ~5 to 500 mA
- CERN PS: In 1959 had 10 octupoles; not used until 1968
  - At $10^{12}$ protons/pulse observed ($1^{st}$ time) head-tail instability. Octupoles helped.
  - Once understood, chromaticity jump at transition was developed using sextupoles.
  - More instabilities were discovered; helped by octupoles and by feedback.

# How to make a high-intensity machine?
# (OR, how to make a high-intensity beam stable?)

1. Landau damping – the beam's "immune system". It is related to the spread of betatron oscillation frequencies. The larger the spread, the more stable the beam is against collective instabilities.

2. External damping (feed-back) system – presently the most commonly used mechanism to keep the beam stable.
   - Can not be used for some instabilities (head-tail)
   - Noise
   - Difficult in linacs

# Most accelerators rely on both

- LHC
  - Has a transverse feedback system
  - Has 336 Landau Damping Octupoles
  - Provide tune spread of 0.001 at 1-sigma at injection
- In all machines there is a trade-off between Landau damping and dynamic aperture.
  - ...But it does not have to be.

# Today's talk will be about…

- … How to improve beam's immune system (Landau damping through betatron frequency spread)
  - Tune spread not ~0.001 but 10-50%
- What can be wrong with the immune system?
  - The main feature of all present accelerators – particles have nearly identical betatron frequencies (tunes) by design. This results in two problems:
    I. Single particle motion can be unstable due to resonant perturbations (magnet imperfections and non-linear elements);
    II. Landau damping of instabilities is suppressed because the frequency spread is small.

# Courant-Snyder Invariant

Equation of motion for betatron oscillations

$$z'' + K(s)z = 0,$$
$$z = x \text{ or } y$$

- Courant and Snyder found a conserved quantity:

$$J = \frac{1}{2\beta(s)}\left(z^2 + \left(\frac{\beta'(s)}{2}z - \beta(s)z'\right)^2\right)$$

where $\left(\sqrt{\beta}\right)'' + K(s)\sqrt{\beta} = \frac{1}{\sqrt{\beta^3}}$ -- auxiliary equation

$$H(J_x, J_y) = \omega_x J_x + \omega_y J_y$$

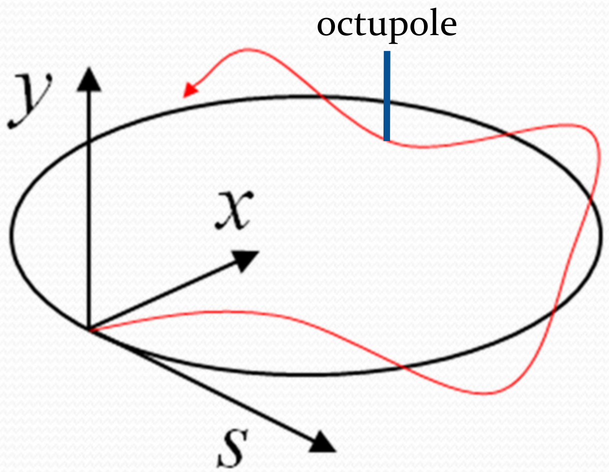
$J_x, J_y$ -- are Courant-Snyder integrals of motion

$\omega_x = \frac{\partial H}{\partial J_x}$ $\omega_y = \frac{\partial H}{\partial J_y}$ -- betatron frequencies

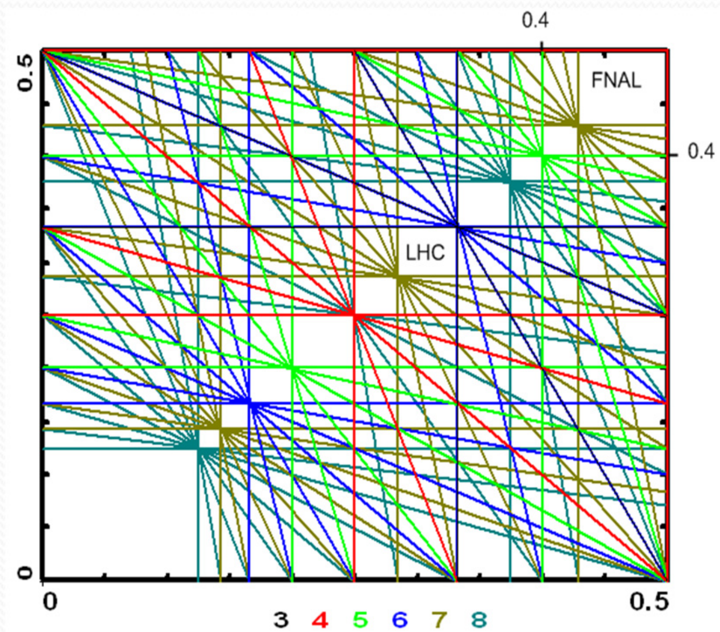
# Linear function of actions: good or bad?

$$H(J_x, J_y) = \omega_x J_x + \omega_y J_y$$

- It is convenient (to have linear optics), easy to model, …but it is NOT good for stability.
- We did not know (until now) how to make it any other way!
- To create the tune spread, we add non-linear elements (octupoles) as best we can
  - Destroys integrability!




Tune spread depends on a linear tune location
1-D system:
Theoretical max. spread is 0.125
2-D system:
Max. spread < 0.05



# Long-term stability

- The first paper on the subject was written by Nikolay Nekhoroshev in 1971

Russian Math. Surveys 32:6 (1977), 1–65
*From* Uspekhi Mat. Nauk 32:6 (1977), 5–66

AN EXPONENTIAL ESTIMATE OF THE TIME OF STABILITY OF NEARLY-INTEGRABLE HAMILTONIAN SYSTEMS

N. N. Nekhoroshev

**1.1 Nearly-integrable Hamiltonian systems. Perpetual stability and stability during finite intervals of time.** In this article we investigate the behaviour of the variables $I$ in the Hamiltonian system of canonical equations

$$\dot{I} = -\frac{\partial H}{\partial \varphi}, \quad \dot{\varphi} = \frac{\partial H}{\partial I}$$

with the Hamiltonian

$$H = H_0(I) + \varepsilon H_1(I, \varphi), \tag{1.1}$$

where $\varepsilon \ll 1$ is a small parameter, the perturbation $\varepsilon H_1(I, \varphi)$ is $2\pi$-periodic in $\varphi = \varphi_1, \ldots, \varphi_s$, and $I$ is an $s$-dimensional vector, $I = I_1, \ldots, I_s$.

- He proved that for sufficiently small ε $\quad \|I(t) - I(0)\| \le R_* \epsilon^b \quad \text{for} \quad |t| \le T_* \exp(\epsilon^{-a})$
  provided that $H_o(I)$ meets certain conditions know as steepness
  - Convex and quasi-convex functions $H_o(I)$ are the steepest
- An example of a NON-STEEP function is a linear function

$$H_0(I_1, I_2) = \nu_1 I_1 + \nu_2 I_2$$

- Another example of a NON-STEEP function is

$$H_0(I_1, I_2) = I_1^2 - I_2^2$$

# Non-linear Hamiltonians

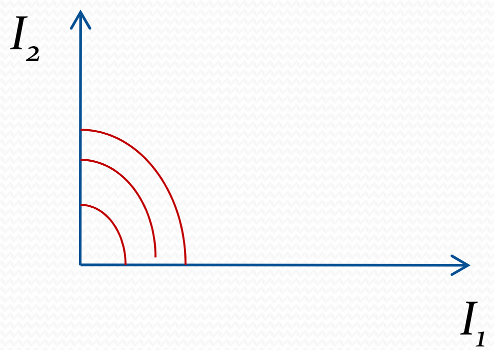
- We were looking for (and found) non-linear 2-D steep Hamiltonians that can be implemented in an accelerator
- Other authors worked on this subject: Yu. Orlov (1962-65), E. McMillan (1967-71), and recently, J. Cary et al., S. Danilov, E. Perevedentsev
  - The problem in 2-D is that the fields of non-linear elements are coupled by the Laplace equation.
  - An example of a steep (convex) Hamiltonian is

$$H_0(I_1, I_2) = \alpha_1 I_1^2 + \alpha_2 I_2^2,\ \alpha > 0$$

    but we DO NOT know how to implement it with magnetic fields…

# What are we looking for?

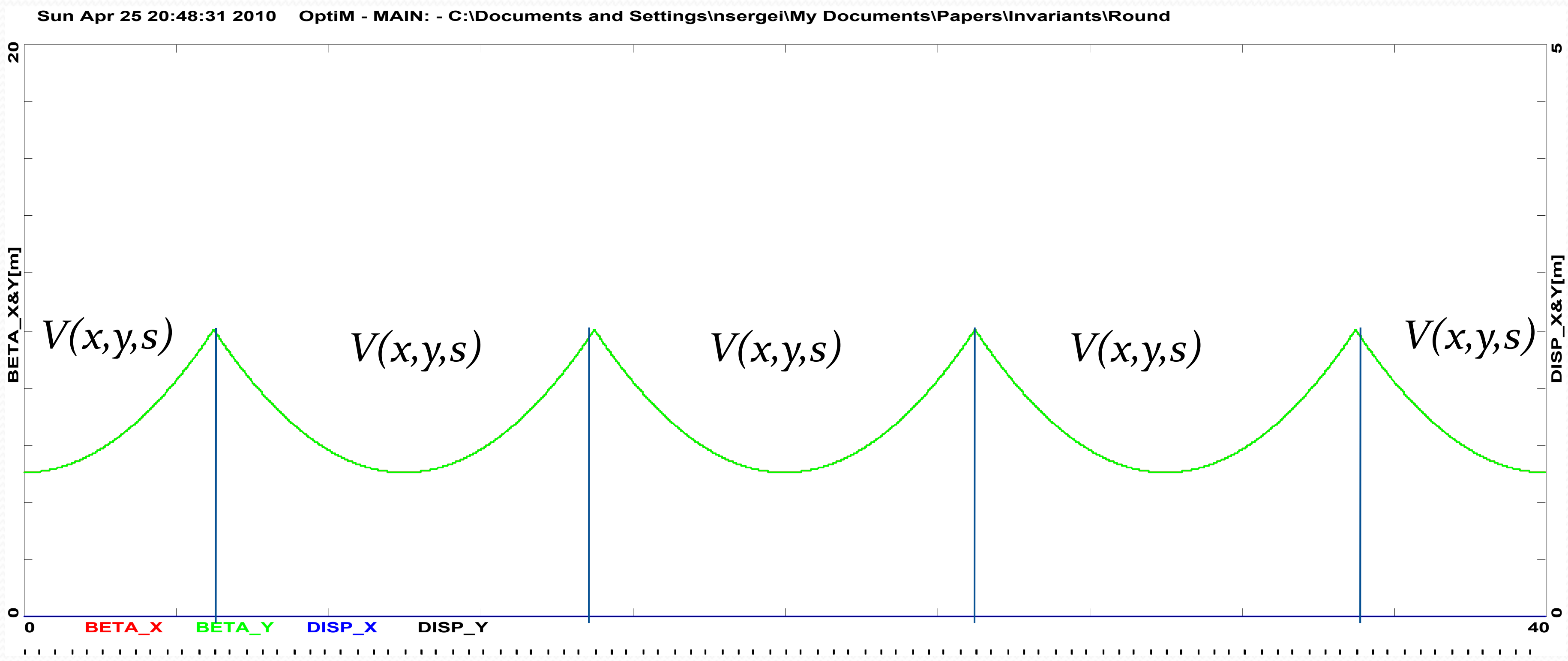
- We are looking for a 2-D integrable convex non-linear Hamiltonian, $H_0(I_1, I_2) = h(I_1, I_2)$
  - $h(I_1, I_2) = const$ -- convex curves



# Our approach

- See: Phys. Rev. ST Accel. Beams 13, 084002
- Start with a round axially-symmetric LINEAR focusing lattice (FOFO)
  - Add special non-linear potential *V(x,y,s)* such that

$$\Delta V(x,y,s) \approx \Delta V(x,y) = 0$$



# Special time-dependent potential

- Let's consider a Hamiltonian

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + K(s)\left(\frac{x^2}{2} + \frac{y^2}{2}\right) + V(x,y,s)$$

where *V(x,y,s)* satisfies the Laplace equation in 2d:

$$\Delta V(x,y,s) \approx \Delta V(x,y) = 0$$

- In normalized variables we will have:

$$z_N = \frac{z}{\sqrt{\beta(s)}},$$

$$p_N = p\sqrt{\beta(s)} - \frac{\beta'(s)z}{2\sqrt{\beta(s)}},$$

$$H_N = \frac{p_{xN}^2 + p_{yN}^2}{2} + \frac{x_N^2 + y_N^2}{2} + \beta(\psi)V\left(x_N\sqrt{\beta(\psi)}, y_N\sqrt{\beta(\psi)}, s(\psi)\right)$$

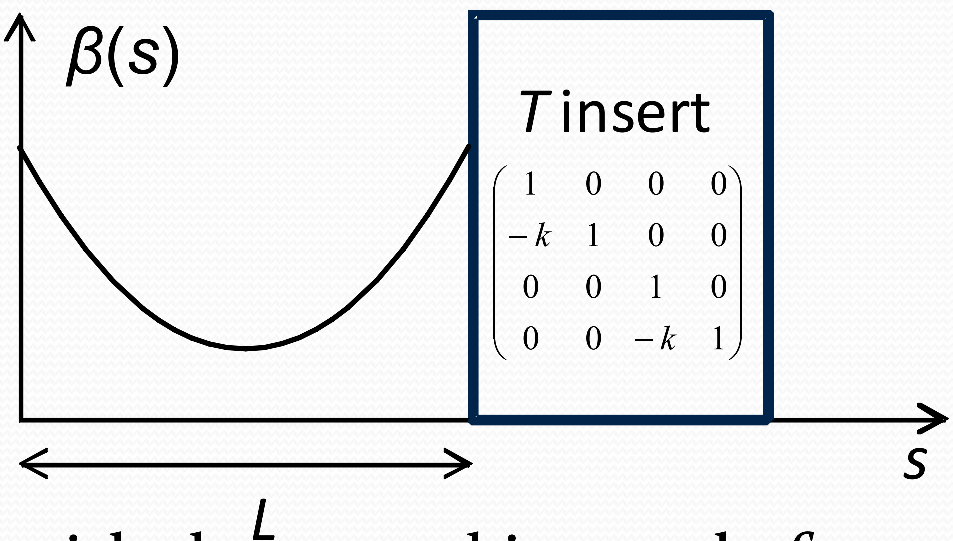
Where new "time" variable is $\psi(s) = \int_0^s \frac{ds'}{\beta(s')}$

# Four main ideas

1. Chose the potential to be time-independent in new variables $$H_N = \frac{p_{xN}^2 + p_{yN}^2}{2} + \frac{x_N^2 + y_N^2}{2} + U(x_N, y_N)$$

2. Element of periodicity

$$\beta(s) = \frac{L - sk(L-s)}{\sqrt{1 - \left(1 - \frac{Lk}{2}\right)^2}}$$




3. Find potentials *U(x, y)* with the second integral of motion

4. Convert Hamiltonian to action variables $H_0(I_1, I_2) = h(I_1, I_2)$ and check it for steepness

# Integrable 2-D Hamiltonians

- Look for second integrals quadratic in momentum
  - All such potentials are separable in some variables (cartesian, polar, elliptic, parabolic)
  - First comprehensive study by Gaston Darboux (1901)
- So, we are looking for integrable potentials such that

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{x^2 + y^2}{2} + U(x, y)$$

Second integral: $$I = Ap_x^2 + Bp_x p_y + Cp_y^2 + D(x, y)$$

$$A = ay^2 + c^2,$$

$$B = -2axy,$$

$$C = ax^2,$$

# Darboux equation (1901)

- Let $a \neq 0$ and $c \neq 0$, then we will take $a = 1$

$$xy\left(U_{xx} - U_{yy}\right) + \left(y^2 - x^2 + c^2\right)U_{xy} + 3yU_x - 3xU_y = 0$$

- General solution

$$U(x, y) = \frac{f(\xi) + g(\eta)}{\xi^2 - \eta^2}$$

$$\xi = \frac{\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2}}{2c}$$

$$\eta = \frac{\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}}{2c}$$

$\xi$ : [1, $\infty$], $\eta$ : [-1, 1], $f$ and $g$ arbitrary functions

# The second integral

- The 2nd integral

$$I\left(x, y, p_x, p_y\right) = \left(x p_y - y p_x\right)^2 + c^2 p_x^2 + 2c^2 \frac{f(\xi)\eta^2 + g(\eta)\xi^2}{\xi^2 - \eta^2}$$

- Example: $$U(x, y) = \frac{1}{2}\left(x^2 + y^2\right)$$

$$f_1(\xi) = \frac{c^2}{2}\xi^2\left(\xi^2 - 1\right) \qquad g_1(\eta) = \frac{c^2}{2}\eta^2\left(1 - \eta^2\right)$$

$$I\left(x, y, p_x, p_y\right) = \left(x p_y - y p_x\right)^2 + c^2 p_x^2 + c^2 x^2$$

# Laplace equation

- Now we look for potentials that also satisfy the Laplace equation (in addition to the Darboux equation):

$$U_{xx} + U_{yy} = 0$$

- We found a family with 4 free parameters (b, c, d, t):

$$f_2(\xi) = \xi\sqrt{\xi^2 - 1}\left(d + t\,\mathrm{acosh}(\xi)\right)$$

$$g_2(\eta) = \eta\sqrt{1 - \eta^2}\left(b + t\,\mathrm{acos}(\eta)\right)$$

$$U(x, y) = \frac{f(\xi) + g(\eta)}{\xi^2 - \eta^2}$$

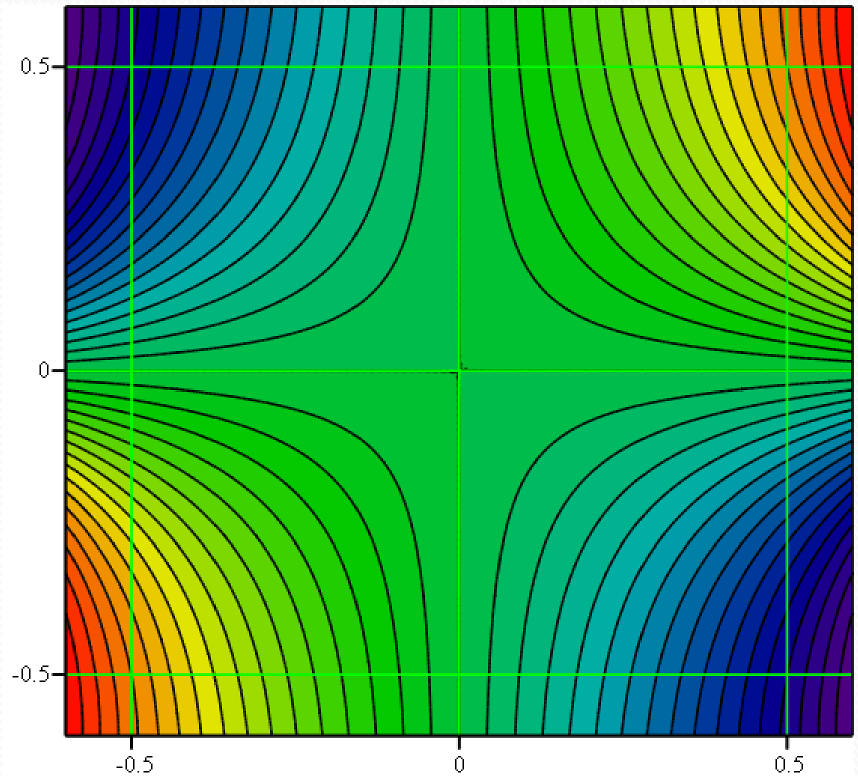
The most interesting: $c$=1, $t$=1, $d$=0, $b = -\frac{\pi}{2}$

# The integrable Hamiltonian

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{x^2 + y^2}{2} + kU(x, y)$$

Multipole expansion

$$U(x,y) \approx \mathrm{Im}\left[(x+iy)^2 + \frac{2}{3}(x+iy)^4 + \frac{8}{15}(x+iy)^6 + \frac{16}{35}(x+iy)^8 + \frac{128}{315}(x+iy)^{10} + \ldots\right]$$




$|k| < 0.5$ to provide linear stability for small amplitudes

For $k > 0$ adds focusing in $x$

Small-amplitude tune s:

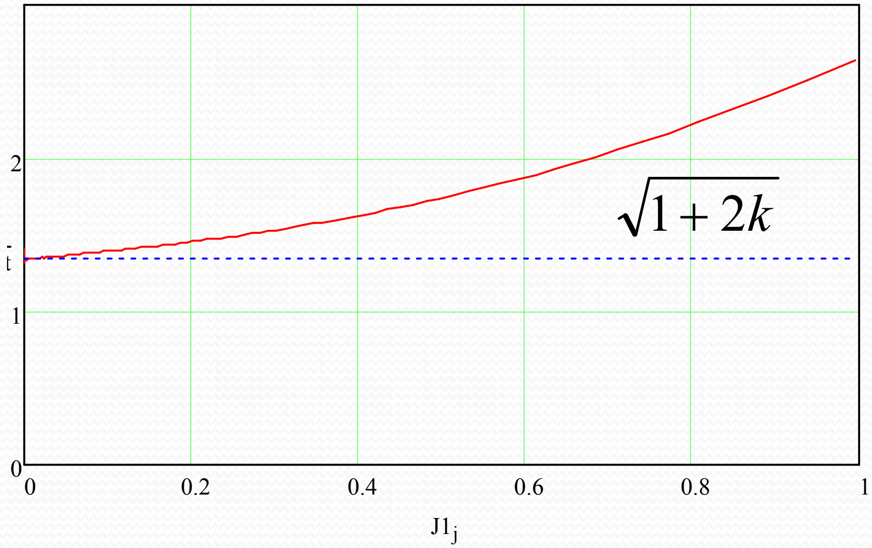
$$\nu_1 = \sqrt{1+2k}$$
$$\nu_2 = \sqrt{1-2k}$$

B

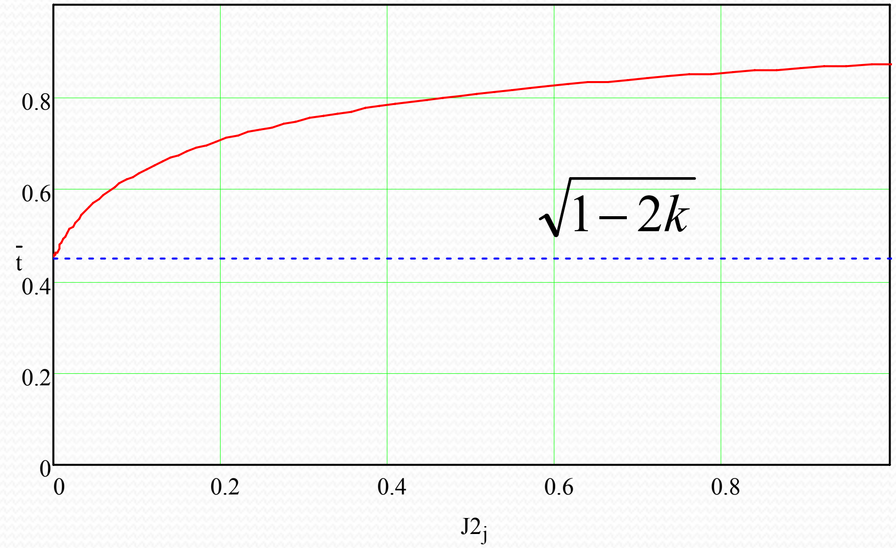
# Convex Hamiltonian

- This Hamiltonian is convex (steep)
- Example of tunes for $k$ = 0.4

$\nu_1(J_1, 0)$




$\nu_2(0, J_2)$

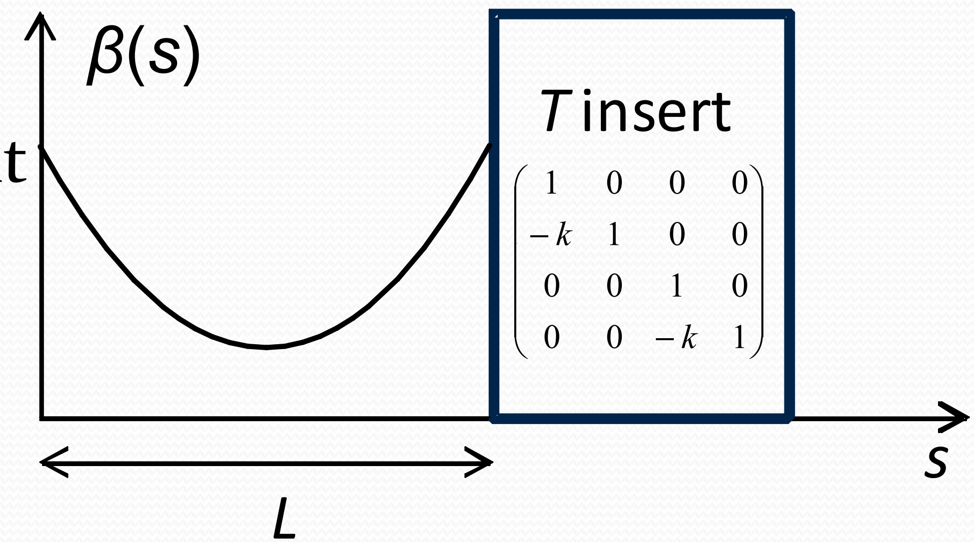



For $k$ -> 0.5 tune spreads of ~ 100% is possible

# How to realize it?

- Need to create an element of periodicity.
  - The T-insert can also be

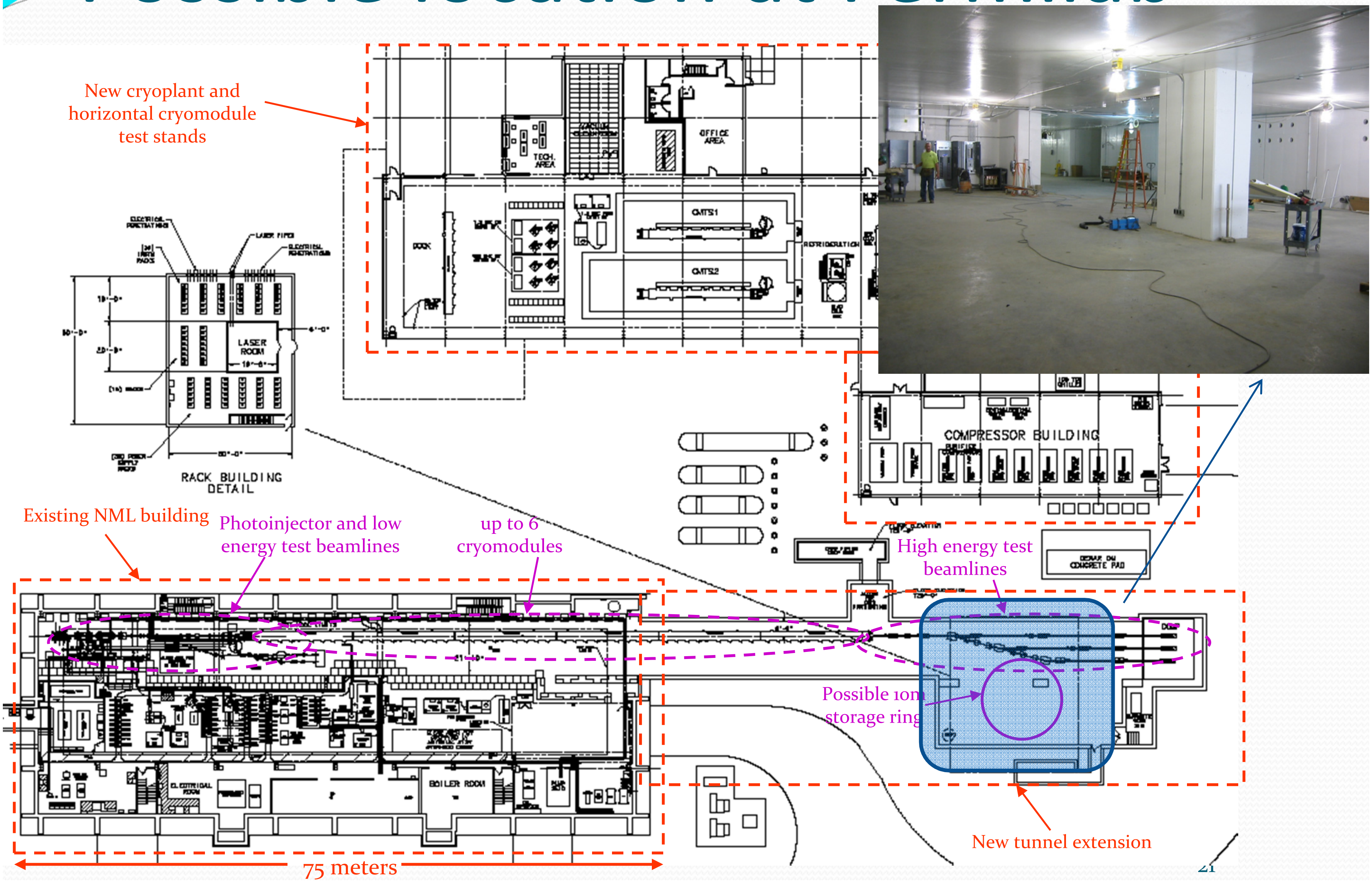
$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ k & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & k & -1 \end{pmatrix}$$

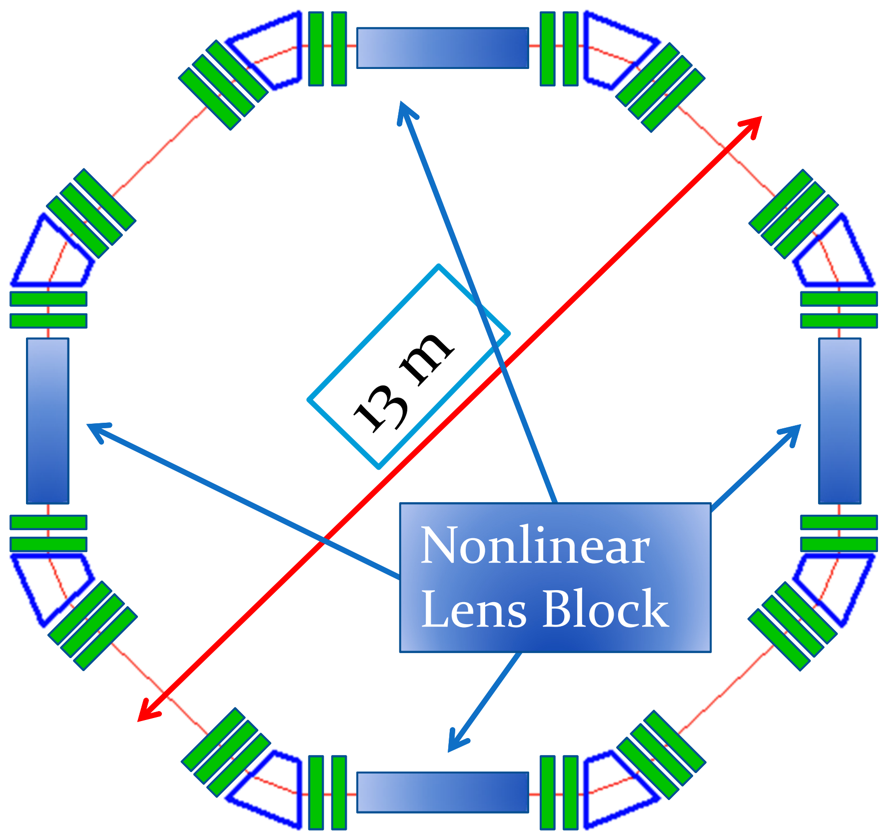



which results in a phase advance 0.5 (180 degrees) for the T-insert.

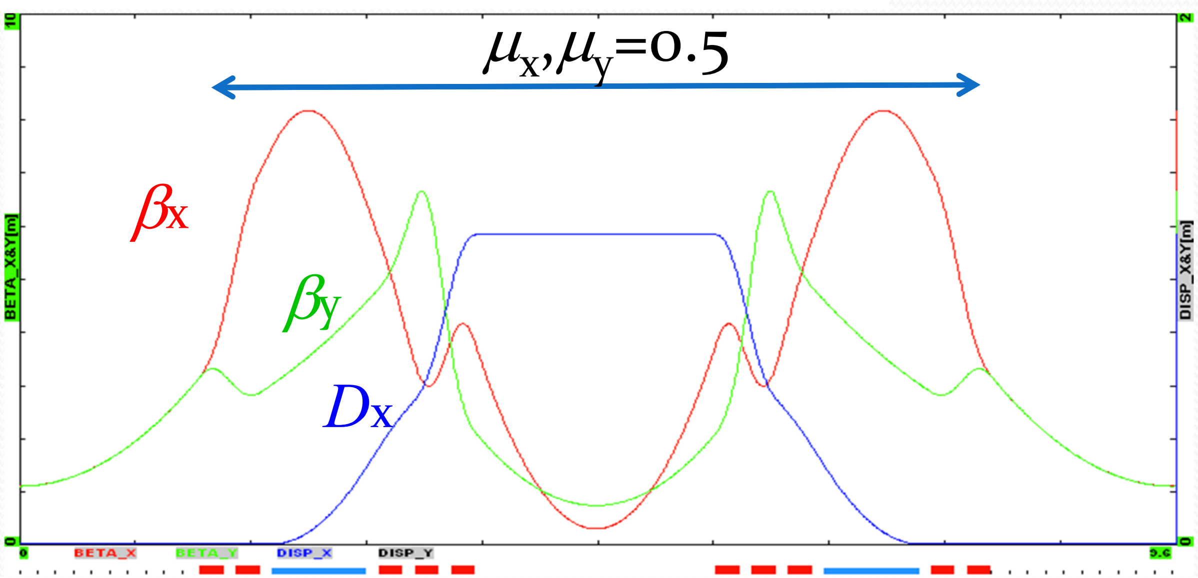
  - The drift space $L$ can give the phase advance of at most 0.5 (180 degrees).

# Possible location at Fermilab






| e- Energy | 150 MeV |
|---|---|
| Circumference | 38 m |
| Dipole field | 0.5 T |
| Betatron tunes | Qx=Qy=3.2 (2.4 to 3.6) |
| Radiation damping time | 1-2 s ($10^7$ turns) |
| Equilibrium emittance, rms, non-norm | 0.06 $\mu$m |




| Nonlinear lens block | |
|---|---|
| Length | 2.5 m |
| Number of elements | 20 |
| Element length | 0.1 m |
| Max. gradient | 1 T/m |
| Pole-to-pole distance (min) | ~ 2 cm |

# Current and Proposed Studies

## Numerical Simulations

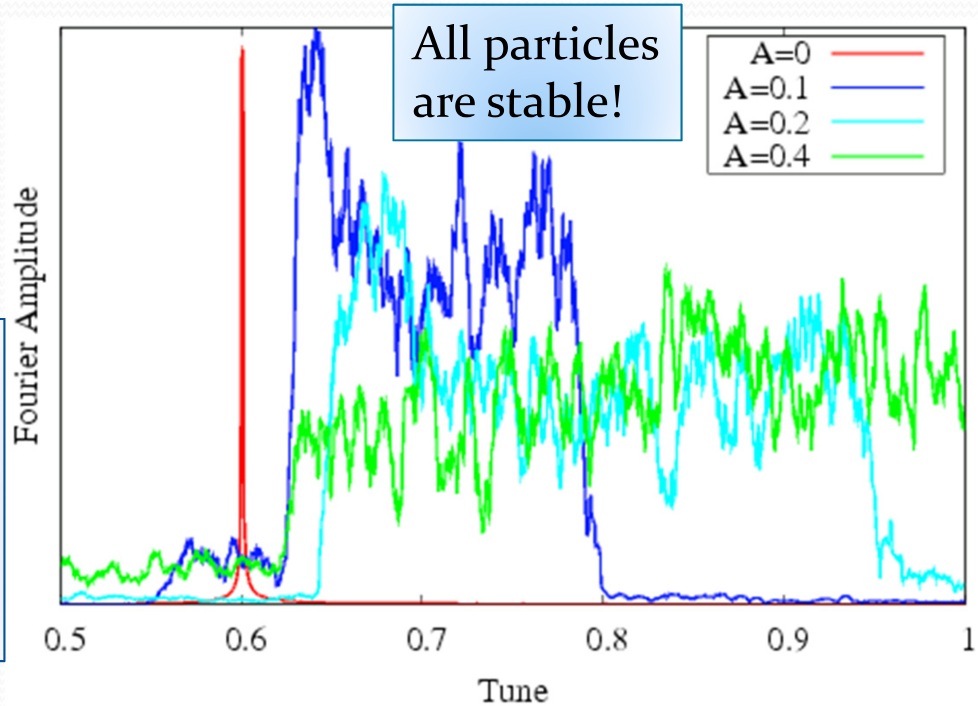
- Nonlinear lenses implemented in a multi-particle tracking code (thin kick)
- Studied particle stability
  - Effects of imperfections (phase advance, beta-functions, etc.) - acceptable
  - Synchrotron motion - acceptable
  - Number of nonlinear lenses - 20
- Simulated observable tune spread
- To Do:
  - Ring nonlinearities
  - Chromaticity

Spectrum of horizontal dipole moment
$Q_0=0.9\text{x}4=3.6$
5000 particles
8000 revolutions
But up to $10^6$ revolutions simulated

## Possible Experiments at Test Ring

- Demonstrate large betatron tune spread
- Demonstrate part of the beam crossing integer resonance
- Map phase space with pencil beam by varying an injection error

All particles are stable!



# Conclusions

- We found first examples of completely integrable non-linear optics.
  - Tune spreads of 50% are possible. In our test ring simulation we achieved tune spread of about 1.5 (out of 3.6);
- Nonlinear "integrable" accelerator optics has advanced to possible practical implementations
  - Provides "infinite" Landau damping
  - Potential to make an order of magnitude jump in beam brightness and intensity
- Fermilab is in a good position to use of all these developments for next accelerator projects
  - Rings or linacs

# Extra slides

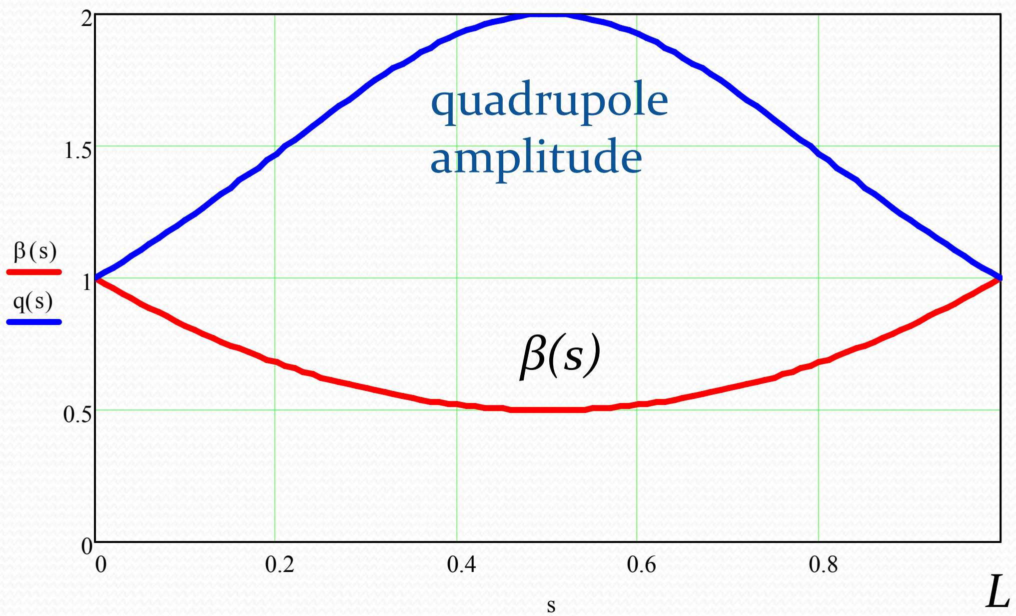
# Examples of time-independent Hamiltonians

- Quadrupole

$$V(x,y,s)=\frac{q}{\beta(s)^2}\left(x^2-y^2\right)$$

$$U(x_N,y_N)=q\left(x_N^2-y_N^2\right)$$

$$H_N=\frac{p_{xN}^2+p_{yN}^2}{2}+\frac{x_N^2+y_N^2}{2}+q\left(x_N^2-y_N^2\right)$$




Integrable but still linear…

Tunes: $\nu_x^2=\nu_0^2(1+2q)$

$\nu_y^2=\nu_0^2(1-2q)$

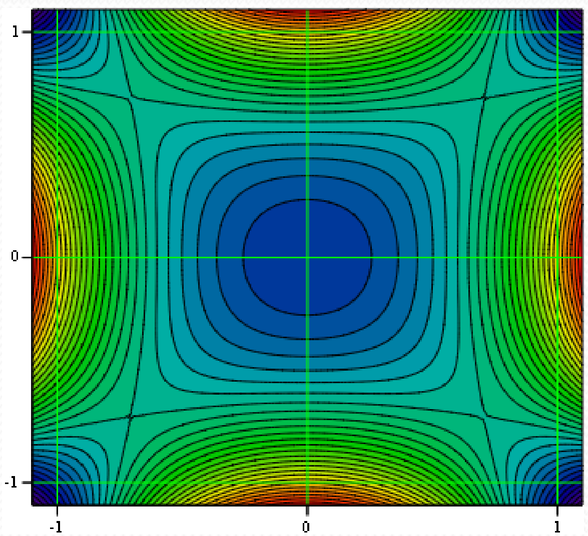
Tune spread: zero

# Examples of time-independent Hamiltonians

- Octupole

$$V(x,y,s)=\frac{\kappa}{\beta(s)^3}\left(\frac{x^4}{4}+\frac{y^4}{4}-\frac{3x^2y^2}{2}\right)$$

$$U=\kappa\left(\frac{x_N^4}{4}+\frac{y_N^4}{4}-\frac{3y_N^2x_N^2}{2}\right)$$




$$H=\frac{1}{2}(p_x^2+p_y^2)+\frac{1}{2}(x^2+y^2)+\frac{k}{4}\left(x^4+y^4-6x^2y^2\right)$$

This Hamiltonian is NOT integrable

Tune spread (in both x and y) is limited to ~12%

B

Spectrum of vertical dipole moment. $Q_o$=0.905x4=3.62