EFFECT OF SPACE CHARGE ON TRANSVERSE INSTABILITIES

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Abstract

Transverse instability of a bunched beam is discussed in the paper with space charge effects taken into account. It is assumed that the Space Charge Impedance is a dominant part of the entire beam coupling impedance, which is a very characteristic case for high-brightness proton synchrotrons. Equation of intra-beam oscillations is derived and investigated including shape and frequency of the head-tail modes. Special attention is focused on Landau damping and threshold of possible instability.

INTRODUCTION

Transverse coherent instability of a bunched beam have been studied first by C. Pellegrini [1] and M. Sands [2] with intra-bunch degrees of freedom taken into account, but without space charge effects. A solution with these effects was presented later by F. Sacherer using boxcar model [3]. A crucial part of the space charge in Landau damping and instability threshold of bunched beams was demonstrated first in Ref. [4] for rather high synchrotron frequency. Later the problems were studied in Ref. [5-7], the last presenting most detailed study of the role of space charge impedance in the bunched beams instabilities.

Space Charge Impedance (SCI) is a part of an entire beam coupling impedance, which takes into account only local electromagnetic field carried by a beam. It is a purely imaginary value not depending on frequency and unable to cause the beam instability by itself. Real part of the impedance is just the one directly responsible for the instability. In principle, any retarding (wake) field is capable to generate such an addition. However, SCI can drastically affect intra-bunch coherent oscillations (head-tail modes) including their frequency, shape, and particularly threshold of possible instability. The effect is especially important in proton synchrotrons where SCI, typically, constitutes a significant or even dominant part of the impedance. Under these assumptions, the wake field can be treated as a small perturbation which controls mutual motion of the bunches (collective beam modes) including the instability growth rate. Just from this standpoint the problem is treated in this work. Incoherent space charge tune shift is used further as a convenient measure of the SCI.

COASTING BEAM LANDAU DAMPING

It is a well known fact that, at dominant SCI, transverse instability of a coasting beam is possible if space charge tune shift about exceeds the incoherent tune spread:

$$\Delta Q_{incoherent} > C \times \delta Q, \quad C \approx 1 \tag{1}$$

This relation has a simple physical explanation: *Self-sustaining coherent oscillations of a beam are impossible if their frequency falls within a range of incoherent betatron frequencies.*

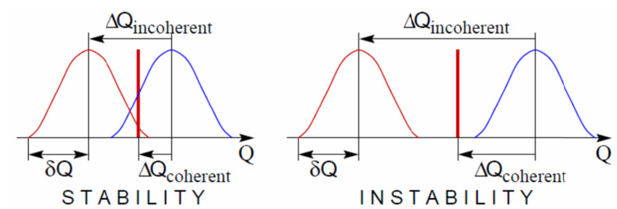

Figure 1: Landau damping origin of coasting beams.

Figure 1 is provided to illustrate the statement. Blue curve present a bare tune distribution, red one – the same distribution shifted by space charge force, and vertical red line depicts the coherent tune. Two cases are possible as it is shown in the picture. At comparably low intensity, the coherent tune could not leave the incoherent range (left-hand figure). Then the attendant electric field would excite *contra-phase* oscillations of particles, which individual tunes are located either lower or higher the coherent one. That would result a quick transfer of energy from coherent form to incoherent one, that is the beam heating and decay of the coherence. This effect is known as Landau Damping (LD). It does not arise at higher intensity when the coherent frequency leaves the incoherent range (right-hand figure). Then the coherent field excites *in-phase* oscillations of all the particles, supporting the coherence and creating conditions for instability. As it is seen from the picture, corresponding instability condition is

$$|\Delta Q_{incoherent} - \Delta Q_{coherent}| > \delta Q \tag{2}$$

This relation can be written down as Eq. (1) because $\Delta Q_{coh} \ll \Delta Q_{incoh}$ in practice. Coefficient C depends on the bunch shape being about 1.2–1.1 for Gaussian distribution truncated on the level of (3-5)σ.

BUNCHED BEAM LANDAU DAMPING

There is no doubt that above declared principle is valid for bunched beams as well. However, very different physical phenomena can be responsible for incoherent tune spread of coasting and bunched beams. In first case, the main source is, usually, momentum spread multiplied by chromaticity. However, as it was shown in Ref. [1] and [2], the instability threshold of bunched beams does not depend on chromaticity at all (though the instability growth rate can drastically depend on it). It means also that averaged in synchrotron phase tunes of the particles and corresponding tune spread only can affect coherent transverse motion of the bunch.

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From this point of view, essentially lower instability threshold could be expected in bunched beams in comparison with coasting ones, at about the same intensity, emittance, etc. However, it would be a hasty conclusion because an additional source of the spread appears at the bunching. *It is space charge tune shift itself* because it depends on a particle position in the bunch or – after the averaging – on amplitude of synchrotron oscillations [4]. It is clear that corresponding tune spread is proportional to SCI and beam intensity – the fact which creates rather specific conditions for the Landau damping and the instability threshold.

In principle, tune spread caused by nonlinearity of betatron oscillations should be taken in the consideration as well. However, nonlinearity of external magnetic field is relatively small usually that is can be neglected in practice. As for own beam field, its nonlinearity and corresponding tune spread do not affect coherent motion at all, as it is shown in Ref. [8].

Taking into account all these circumstances, one can write down Eq. (2) for a bunched beam in the form:

$$(\Delta Q_{incoh})_{min} < \Delta Q_{coh} < (\Delta Q_{incoh})_{max} \qquad (3)$$

Right-hand part of this equation is satisfied automatically, so only two cases could be actually possible. They are sketched in Fig. 2, where both coherent tune shift and incoherent tune spread are presented. Landau damping appears if the shift is rather large (left-hand picture). Because all the tune shifts are proportional to the intensity and SCI, it could be expected that the bunch shape and synchrotron frequency are important for the instability threshold as well. Particularly, long tail of the distribution certainly should increase chances of the Landau damping. On the other hand, any bunch has a lot of eigenmodes some of them being probably unstable. Therefore additional theoretical analysis is needed to make more distinctive conclusions.

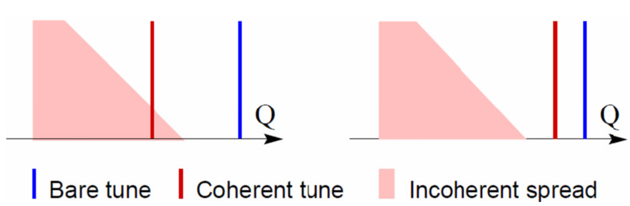

Figure 2: Landau damping origin of bunched beams. Left-hand – stability, right-hand – instability.

EQUATUIN OF BUNCHED BEAM COHERENT OSCILLATIONS

Let a bunch execute coherent oscillations with horizontal deviation $X(t,\theta,p)$ in the point (θ,p) of longitudinal phase space and $\bar{X}(t,\theta)$ in average. These variables are coupled by the relation:

$$\rho(\theta)\,\bar{X}(t,\theta) = \int F(\theta,p)X(t,\theta,p)dp \qquad (4)$$

where $F(\theta,p)$ is longitudinal distribution function of the bunch, and $\rho(\theta)$ is its linear density. Designating horizontal electric field of the *steady-state* bunch as $E(\theta,x,y)$ one can write equation of betatron oscillations of a single particle as

$$\frac{d^2x_i}{dt^2} + \Omega^2Q^2x_i = \frac{eE(\theta,x_i-\bar{X}(t,\theta),y_i)}{m\gamma^3} + (wake) \quad (5)$$

where revolution frequency Ω and bare tune Q depend on the particle momentum. Note that incoherent space charge tune shift of the particle depends on its amplitude, because $E(\theta,x,y)$ is nonlinear function of transverse coordinates. Wake field is hidden here but actually it is negligible in first approximation, by assumption.

With steady-state transverse density $\rho_t(x,y)$, coherent displacement is:

$$X(t,\theta,p) = \int \rho_t\big(x - X(t,\theta,p)\big)\,x dx \qquad (6)$$

While the deviation is small in comparison with the beam radius, it satisfies the *linear* equation:

$$\frac{d^2X}{dt^2} + \Omega^2Q^2X = 2\Omega_0^2Q_0\Delta Q(X-\bar{X}) \qquad (6)$$

with effective tune shift which depends only on θ:

$$\Delta Q(\theta) = \frac{e}{2m\gamma^3\Omega_0^2Q_0}\int\frac{\partial E}{\partial x}(\theta,x,y)\rho_t(x,y)dxdy \quad (7)$$

It proves the statement which was actually used above: *nonlinearity of space charge field and related tune spread do not affect coherent motion and cannot contribute to the instability frequency and threshold.* The effective tune shift $\Delta Q(\theta)$ is proportional to the longitudinal density $\rho(\theta)$ and depends on transverse density $\rho_t(x,y)$. It coincides with usual incoherent tune shift if the beam has elliptic cross section and constant density. For Gaussian beam, it is a half shift of small betatron oscillations [8].

Because $\Delta Q \ll Q$, Eq. (6) can be reduced to the first order form:

$$\frac{dX}{dt} + i\Omega QX \approx i\Omega_0\Delta Q(\theta)(X-\bar{X}) \qquad (8)$$

This equation describes both coasting and bunched beams. The choice dependence on the operator d/dt which is global time derivative including longitudinal motion:

$$\text{Coasting beam:}\quad \frac{d}{dt} = \frac{\partial}{\partial t} + [\Omega(p)-\Omega_0\,]\frac{\partial}{\partial\theta} \qquad (9a)$$

$$\text{Bunched beam:}\quad \frac{d}{dt} = \frac{\partial}{\partial t} + \Omega_s\frac{\partial}{\partial\varphi} \qquad (9b)$$

where Ω_s and φ are synchrotron frequency and phase. Therefore, taking the explicit dependence of all the variables on time as $\exp(-i\omega t)$, one can represent Eq. (8) for a bunch in the form :

$$\omega X - \Omega Q X + i\Omega_s \frac{\partial X}{\partial \varphi} = -\Omega_0 \Delta Q(\theta)(X - \bar{X}) \qquad (10)$$

Because frequency of betatron oscillations ΩQ depends on momentum, chromaticity appears in this equation as a parameter. However, it influence only eigenfunctions X but not the eigenfrequencies ω. It can be shown with help of transformation:

$$X(\theta,p) = Y(\theta,p)\exp(-i\chi\theta), \quad \chi = \frac{d(\Omega Q)/dp}{d\Omega/dp} \qquad (11)$$

New variable Y satisfies an equation which looks like Eq. (10) with central frequency $\Omega_0 Q_0$ instead of $\Omega(p)Q(p)$. Therefore its solutions do not depend on chromaticity, including coherent frequency ω which is the same in both equations. Thus, *instability frequency and threshold of a bunched beam do not depend on chromaticity*. However, shape of the oscillations, wake field, and consequently the instability growth rate can depend on it [1-2].

It is more convenient further to use the parameters:

$$\nu = \frac{\omega - \Omega_0 Q_0}{\Omega_0 \Delta Q_{max}}, \qquad \mu = \frac{\Omega_s}{\Omega_0 \Delta Q_{max}}, \qquad (12)$$

Besides, we will normalize the distribution function to satisfy the condition $\rho(0)=1$. Then total set of equations for the function Y is:

$$\nu Y + i\mu \frac{\partial Y}{\partial \varphi} = -\rho(\theta)(Y - \bar{Y}) \qquad (13a)$$

$$\rho(\theta)\bar{Y}(\theta) = \int F(\theta,p) Y(\theta,p)\,dp \qquad (13b)$$

It is easy to check that all its eigenvalues are real, both with and without Landau damping (as long as wake field is not included). This statement does not contradict the possibility of Landau damping, but it means that decay of an initial perturbation caused by the LD is generally a non-exponential process.

Rigid Mode

Eq. (13) has a universal solution which does not depend at all on the distribution function, space charge tune shift, and synchrotron frequency:

$$Y(\theta,p) = \bar{Y}(\theta) = 1, \quad \nu = 0 \quad \text{that is} \quad \omega = \Omega_0 Q_0$$

It is known as rigid mode because, at zero chromaticity, the bunch oscillates as a solid without twist and rotation. With chromaticity, traveling wave is superimposed on the bunch oscillations as it follows from Eq. (9). In used approximation, this mode is never prone to Landau damping being potentially unstable at any intensity. Of course, the threshold exists in reality, at least because of the lattice nonlinearities, however, in practice it is very low in comparison with similar coasting beam.

Boxcar Model

One of the simplest and widely used assumptions is so called boxcar model that is a bunch with constant linear density. It is especially productive in our case because allows to get analytical solution of Eq. (13) at any value of the parameter μ in form of Legendre polynomials [3]:

$$\bar{Y}(\theta) = P_n(\theta)$$

Substituting it to Eq. (13) one can see that, at any n, there are n+1 different functions $Y_{mn}(\theta,p)$ satisfying the conditions. Each of them is polynomial of power n+1 in space of variables (θ,p). Corresponding equations for ν are polynomial of power n+1, and all their solutions (eigentunes) are real numbers. Some of them are plotted against μ in Fig. 3. It is seen that all the curves take a start either from point ν=0 (i) or ν=-1 (ii) at μ=0. In usual units it means: either bare betatron frequency $\omega=\Omega_0 Q_0$ (i) or actual one $\omega=\Omega_0(Q_0-\Delta Q)$ (ii). If the parameter μ is rather small, the coherent oscillations are almost linearly polarized: either in θ-direction (i) or in p-direction (ii). However, the polarization becomes circular at μ » 1:

$$Y_{mn}(A,\varphi) \approx R_{mn}(A)\exp(im\varphi)$$

These eigenmodes are known as multipoles of index m. At any m there are a lot of radial modes $R_{mn}(A)$ which characterize dependence of the eigenfunctions on synchrotron amplitude. All of them are raised by different Legendre polynomials and converge at the lines $\nu \approx m\mu$ with m=0, ±1, etc.

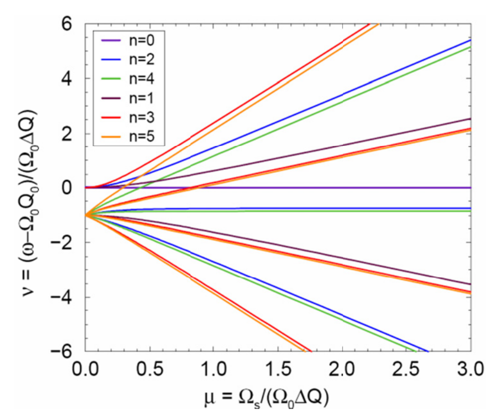

Figure 3: Eigentunes of boxcar bunch.

INSTABILITY THRESHOLD

In normalized variables, condition of Landau damping (3) could turn into $-\bar{\rho}(0) < \nu < -\bar{\rho}(1)$ where

$$\bar{\rho}(A) = \frac{1}{\pi}\int_0^{\pi} \rho(A\cos\varphi)\,d\varphi \qquad (14)$$

However, it is necessary to take into account that, at coherent frequency ω, spectrum of the force which acts on particles can carry frequencies $\omega+m\Omega_s$ with integer m.

Corresponding normalized frequencies are $\nu+m\mu$, so the instability condition should be used in the form:

$$-1 < \nu - m\mu < -\bar{\rho}(1) \qquad (15)$$

(remind that $\bar{\rho}(0) = 1$). All the multipoles presented in the eigenfunction Y_{mn} have to be examined, and Landau damping will spring up if any of them satisfies Eq. (15). Actually, the remaining multipoles at $\mu \to \infty$ are "suspicious" ones. For example, $m = 0$ is the only harmonic inherent in the rigid mode.

Transformed eigentunes of the boxcar bunch are plotted in Fig. 4. Of course, Landau damping cannot arise in this case because of zero incoherent spread. However, for better understanding of the problem, let us assume for a moment that eigentunes of a real bunch have similar behavior but there is an incoherent tune spread corresponding $\bar{\rho}(1) = 0.5$. Then stability region exists which is marked in Fig. 4 by darker colors. It is seen that the modes starting from point $\Delta\nu=\mu=0$ are potentially unstable at low μ but can become stable at higher μ. The modes starting from point $\mu = 0$, $\Delta\nu = -1$ demonstrate opposite behavior and have small (maybe zero) chance to reach unstable zones. Higher eigenmodes are more stable (prone to the Landau damping) in any case.

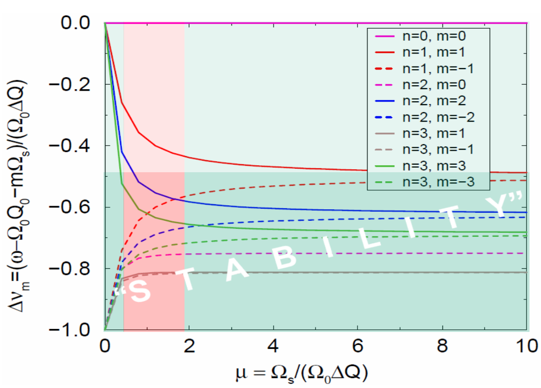

Figure 4: Transformed eigentunes of boxcar bunch.

Unfortunately, for realistic distributions, the solutions of Eq. (13) are achievable (as a rule numerically) only in extreme cases $\mu \ll 1$ and $\mu \gg 1$. These zones are rather approximately marked by green in Fig. 4. Note that Landau damping cannot be seen at $\mu \ll 1$ because of nature of this approximation. Therefore, each solution should be extrapolated from its original zone into neighboring "red" one to completed the picture. The examples are given below.

Parabolic Bunch

Linear density of a parabolic bunch is

$$\rho(\theta) = 1 - \theta^2, \qquad \rho(A) = 1 - A^2/2.$$

Landau damping should appear at $\Delta\nu < -0.5$ in this case. The calculated eigentunes are shown in Fig.5 by solid lines, both approaches $\mu \ll 1$ and $\mu \gg 1$ being plotted and extrapolated. For comparison, boxcar tunes are also presented by dashed curves. A lot of potentially unstable modes ($\Delta\nu > -0.5$) are seen at $\mu <\sim 1$, eigentune of each almost coinciding with the boxcar one. However, only three of them remain unstable at $\mu > 0.5$: lower radial modes of multipoles $m = 0, 1, 2$. Instability of these -- and only these -- modes is also confirmed by $\mu \gg 1$ approach. In this region, the eigentunes slightly exceed the boxcar ones, as if they are pushed out by incoherent tunes from their room. All other eigenmodes are singular at large μ, forming continuous spectrum at $\Delta\nu < -0.5$.

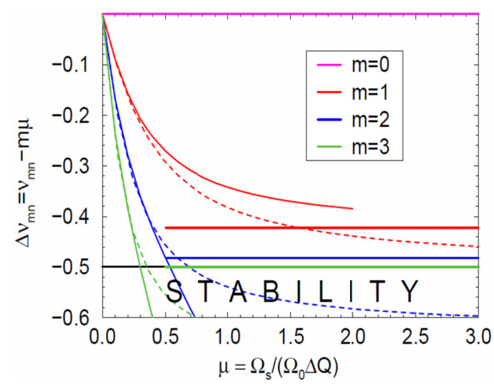

Figure 5: Transformed eigentunes and stability region of parabolic bunch.

Gaussian Bunch

Truncated Gaussian distribution is considered in this subsection:

$$F \propto \exp\left(\frac{1-A^2}{2\sigma^2}\right) - 1$$

Results of calculations in low μ approximation are presented in Fig. 6 at $\sigma = 1/3$ (3σ truncation). In this case, Landau damping should arise at $\Delta\nu < -\bar{\rho}(1) = -0.274$. Actually, a lot of unstable modes exist at $\mu <\sim 0.6$ ($\Delta Q >\sim 1.7\ Q_{syn}$), but almost all of them go down to the stable zone at more μ (Fig. 6). High μ approach confirms that all these modes are prone to Landau damping, have a singularity, and form continuous spectrum at $\Delta\nu < -0.274$. Rigid mode is the only solution which is unstable at any conditions (magenta line in Fig. 6).

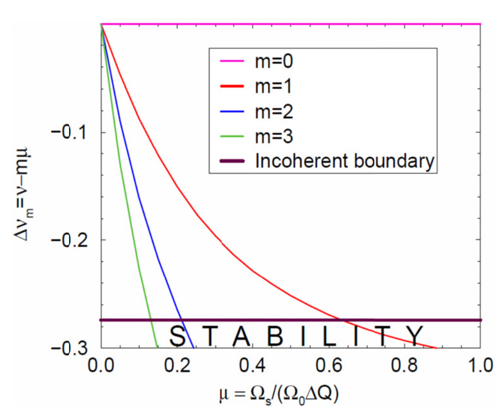

Figure 6: Transformed eigentunes of Gaussian bunch truncated on 3σ level.

Some eigenfunctions of the Gaussian bunch are plotted in Fig. 7. Low μ approximation is used to plot the left-hand graph, actually at μ=1E-4 and at μ=1. It is seen that the bunch edges are more excited at higher μ (lower space charge) when coherent frequency comes nearer to the incoherent boundary. However, Landau damping is excluded in this approximation, so the deviation remains finite. High μ approximation should be used to make this effect clearly visible. Related results are presented in the righ-hand picture where the bunch dipole moment is plotted vs θ. Magenta curve presents the rigid mode which coincides with the bunch linear density in this format. Next modes demonstrate infinite growth in the bunch tails what is certainly a sign of Landau damping.

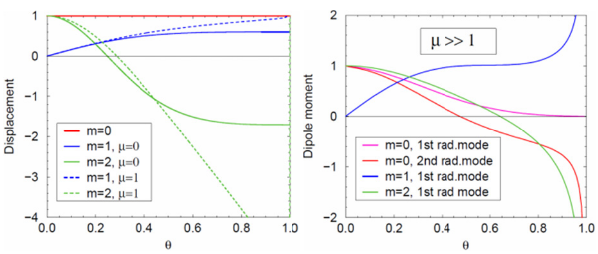

Figure 7: Eigenfunctions of Gaussian bunch. Left/righ graphs – low/high μ approximations.

BUNCHED BEAM COLLECTIVE MODES

If some intra-bunch mode $\bar{Y}(\theta)$ is not prone to Landau damping, collective bunch oscillations are possible with linear density of dipole moment in the form:

$$D(t,\theta) = \sum_j C_j N_j \rho(\Theta_j)\, \bar{Y}(\Theta_j)\, \rho(\Theta_j) \exp(-i\chi\Theta_j - i\omega t)$$

where C_j and N_j are amplitude and intensity of j-th bunch, $\Theta_j=\theta-\theta_j-\Omega_0 t$ is current position of its center. Wake field of this beam should be found and substituted to initial equations as a small perturbation. It will result in a set of linear equations for coefficients C_j at eigenfrequency ω. As a rule, it is a complex number describing growth or decay rate of obtained collective mode. Concrete result depends on specific wake shape, but some common features can be referred.

If the wake is short and does not reach neighboring bunches, the formal solution is $A_j=\delta_{jJ}$, that is all bunches oscillate separately and independently. Another known case is symmetric beam consisting of K alike and equidistant bunches. Then, independently on the wake field nature, all collective modes are the waves:

$$A = \exp\left(2\pi i \frac{jk}{K}\right), \qquad k = 1,2,\dots,K$$

In all the cases the field rather strongly depends on chromaticity which influences the instability growth rate and can transform unstable collective modes to stable one (or vise versa). Usually it is used to control stability of the rigid based collective modes. As it was shown, LD is an effective way to prevent instability other rigid modes.

However, the perturbation method is unsuitable for very low μ when all basic eigentunes converge in the point ν=0 as it obvious from in Fig. 3-6. The same follows also directly from Eq. (13) which can be satisfied by any function $Y(\theta)$ at μ=ν=0. Wake field should be included in consideration from very beginning onder these conditions. In the extreme case $\mu = 0$, corresponding equation is:

$$\nu\bar{Y} = \widehat{W}\bar{Y}$$

with a wake field operator in the right-hand part. It is somewhat unexpectedly that the space charge drops out this equation at all, though the condition $\mu \approx 0$ can mean not only very low synchrotron frequency but extremely large space charge tune shift as well.

SUMMARY

Transverse coherent instability of a bunched beam is considered in the paper with space charge effects taken into account. The basic assumption is a domination of the space charge impedance in the entire impedance budget. In such conditions, SCI determines parameters of intra-bunch coherent oscillations including their frequency, shape, and threshold of possible instability. A wake field can be treated as a small perturbation which determines parameters of collective modes including relative bunch amplitudes and the instability growth rate.

It is shown that SCI is crucially affects threshold of Landau damping determining both coherent tune and incoherent tune spread of the bunch. As a result, Landau damping suppresses almost all intra-bunch modes if the space charge tune shift does not exceed about synchrotron tune. However, several modes can be unstable at higher tune shift. Furthermore, there is a rigid mode which is not sensitive to space charge and synchrotron oscillations at any distribution, and therefore not vulnerable to the Landau damping at all.

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