An analaytical method for longitudinal phase space backtracking

Nick Sudar (WE2A2)

SLAC National Accelerator Laboratory

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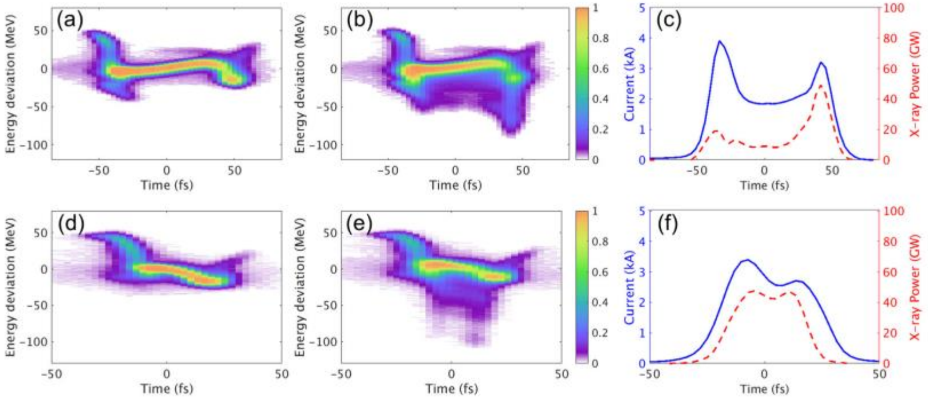
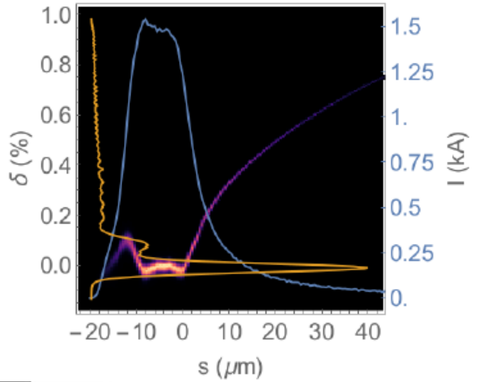
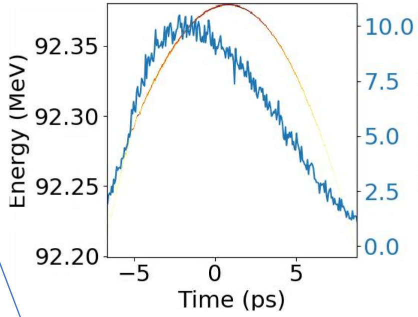
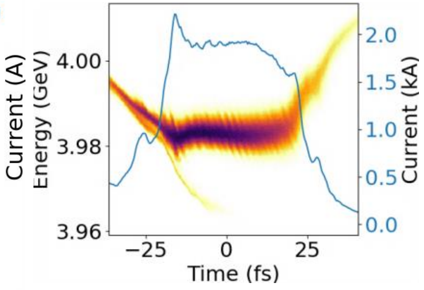
Backtracking

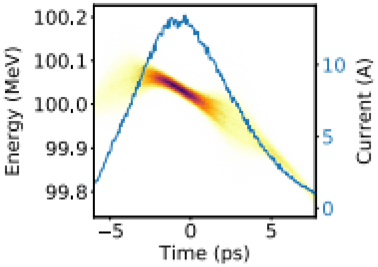
- Motivation: Improving longitudinal phase space brightness
  - Increasing the peak current while producing “flat” current profiles and minimizing correlated energy spread advantageous especially for FEL operation
  - Current horns can limit maximum compression and beam quality.
  - Available RF power and collective effects also constrain longitudinal phase space manipulations. Especially at LCLS-II which has a ~3 km bypass line after last accelerating section
  - Collimating horns can be helpful. Can’t collimate horns at LCLS-II due to beam losses
    - Y. Ding et al, PRAB 19(10):1000703 (2016)
  - Octupoles in bunch compressors to adjust U_{5666} generate losses and/or emittance growth
    - T.K. Charles et al, PRAB 20(3):030705 (2017)
    - N. Sudar et al, PRAB 23(11):112802 (2020)
  - Some success shaping temporal profile of cathode laser
    - G. Penco et al, PRL 112(4):044801 (2014)
    - R. Lemons et al. PRAB 25(1):013401 (2022)
  - Laser heater shaping can reduce horns
    - D. Cesar et al, PRAB 24:110703 (2021)
  - Want to find longitudinal phase space at injector exit that gives a desired longitudinal phase space at the undulator entrance
  - Assume this injector exit phase space can be obtained by tuning injector, shaping cathode laser, de-chirper, etc… Taking advantage of lower beam power upstream of laser heater
- Approach
  - Develop analytical method for tracking longitudinal phase space backwards to some upstream point in accelerator. Similar work:
    - K. Floettmann, et al. TESLA-FEL report 6:2001 (2001)
    - I. Zagorodnov and M. Dohlus PRSTAB 14(1):014403 (2011)
    - M. Cornacchia et al. PRSTAB 9:120701 (2006)
    - W.H. Tan PRAB 24:051303 (2021)
    - A.S. Hernandez et al. PRAB 19:090702 (2016)
    - G.P. Segurana et al. PRAB 25:021003 (2022)
    - S.K. Dutt et al AIP Conference Proceedings Vol. 25 pages 276-292 (1992)
  - Fast exploration of many parameters.
  - Find expressions for chirps to arbitrary polynomial order including collective effects
  - Convenient expressions for chirps/collective effects makes linearization easier

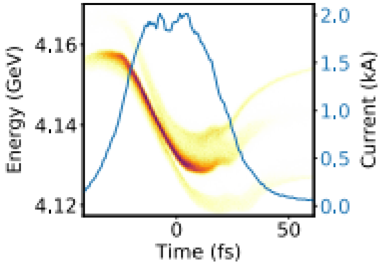












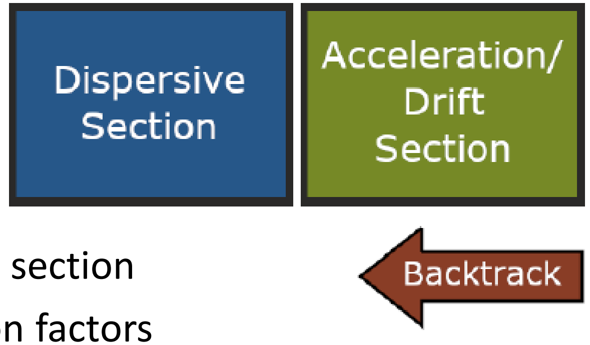
Tracking method

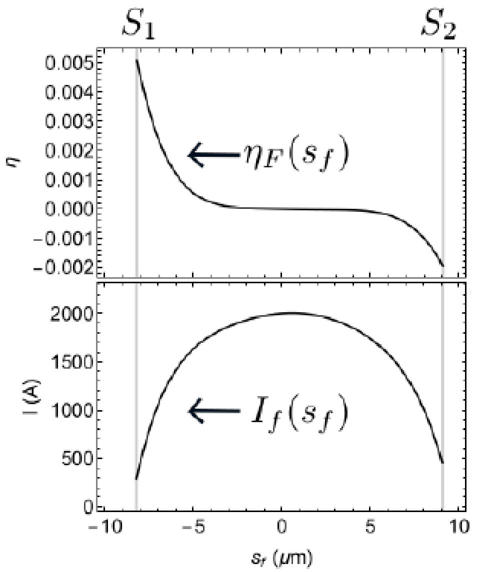
- Method: consider current profile and chirp we want to track backwards as Nth order polynomials with endpoints S1 and S2:

$$I_f(s_f) = I_{f0}(1 + I_{f1}s_f + I_{f2}{s_f}^2 + I_{f3}{s_f}^3 + \ldots + I_{fN}{s_f}^N) \quad S_1 < s_f < S_2$$

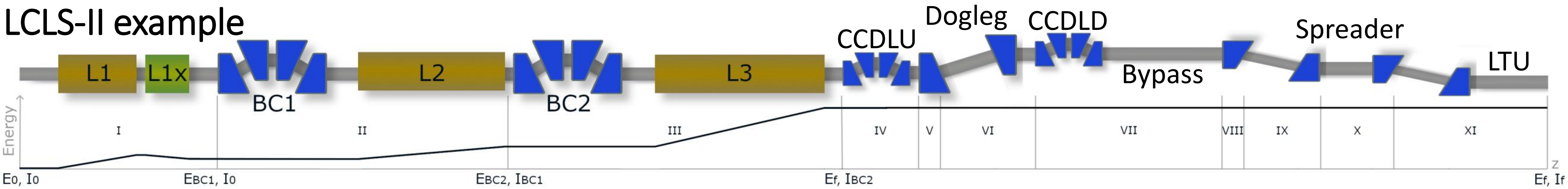
$$\eta_F(s_f) = h_{f1}s_f + h_{f2}{s_f}^2 + h_{f3}{s_f}^3 + \ldots + h_{fN}{s_f}^N$$

- s_f is coordinate along bunch, η_F is energy detuning $\eta_F = (\gamma - \gamma 0)/\gamma$. Bunch head is to the left.
- **(Assumption #1) Assume uncorrelated energy spread is negligible. Dirac delta energy distribution**
- **(Assumption #2) Assume longitudinal phase space (LPS) is piece wise continuous and single valued throughout (no fold over, no current horns)**
- **[Step 1]** Track backwards through acceleration/drift section followed by dispersive section
- In accelerator/drift section assume some energy change, $R_{66} = E_i/E_f$, and total chirp from RF curvature and collective effects described by polynomial coefficients $H_{(n)}$
- In dispersive section dispersion described by polynomial coefficients $D_{(n)}$
- Tracking backwards gives $s_I(s_f)$ and $\eta_I(s_f)$. $s_I(s_f)$ described by "decompression" factors
- Assume $s_f(s_i)$ can be described by "compression" factors to order N (2)
- **[Step 2]** Compression factors can be written in terms of decompression factors, $h_{f(n)}$, R_{66}, $H_{(n)}$. **Obtain $\eta_I(s_i)$**
- **[Step 3]** Transform current profile $I_f(s_f) \rightarrow I_i(s_i)$. Integrate over distribution (1)
- Integral simplified by assuming LPS doesn't have multiple branches (2)
- **(Assumption #3) Assume compression dominated by linear component**
- **[Step 4]** Expand $I_i(s_i)$ to order N to be of the same polynomial form as $I_f(s_f)$
- **[Step 5]** Transform endpoints with decompression factors, $(S_1, S_2) \rightarrow (S_{1i}, S_{2i})$
- Track $I_i(s_i)$, $\eta_I(s_i)$ and (S_{1i}, S_{2i}) through subsequent accelerator/drift and dispersive section
- Can be adapted to forward tracking by swapping decompression and compression factors
- No intention of exactness but hopefully useful





LCLS-II example



Backtrack from undulator entrance to injector exit
Break up LCLS-II linac into regions consisting of a dispersive section and a drift/acceleration/chirp section

- **Region XI**: Start at hard undulator exit (linac to undulator a.k.a. LTU)
  - Chirp: resistive wall (RW) wakefield, longitudinal space charge (LSC), and coherent synchrotron radiation (CSR) from bend magnets
  - Dispersion: 4th bend in spreader and 3rd drift (includes quads)
- **Region X**: Backtrack through “spreader”
  - Chirp: LSC in 3rd drift in “spreader” and CSR from 3rd spreader bend
  - Dispersion: 3rd bend and 2nd drift (rolled bends, quads and rolled sextupoles)
- **Region IX:** Backtrack through “spreader”
  - Chirp: LSC in 2nd drift in “spreader” and CSR from 2nd spreader bend and 2 small bends
  - Dispersion: 2nd bend and 1st drift (rolled bends, quads and rolled sextupoles)
- **Region VIII:** Backtrack through “spreader” kicker
  - Chirp: LSC in 1st drift in “spreader” and CSR from 1st spreader bend and 3 kicker bends
  - Dispersion: 1st bend and kicker (includes drifts between magnets)
- **Region VII**: Backtrack through bypass line and small chicane (CCDLD)
  - Chirp: RW wakefield, LSC in drifts and CSR from CCDLD bends
  - Dispersion: R56 compensating chicane CCDLD
- **Region VI**: Backtrack through Dogleg
  - Chirp: RW wake from dogleg and CSR from 2nd Dogleg bend
  - Dispersion: 2nd Dogleg bend and drift (rolled bends, quads and rolled sextupoles)
- **Region V:** Backtrack through Dogleg
  - Chirp: LSC from Dogleg drift and CSR from 1st Dogleg bend
  - Dispersion: 1st Dogleg bend
- **Region IV**: Backtrack through small chicane (CCDLU)
  - Chirp: CSR from CCDLU bends
  - Dispersion: R56 compensating chicane CCDLU
- **Region III**: Backtrack through linac section L3 and 2nd bunch compressor BC2 **(1.5 →4 GeV)**
  - Chirp: cavity wakefields from L3, LSC in L3 and drifts, and BC2 CSR (more details later)
  - Dispersion: BC2 (use desired BC1 peak current to set BC2 R56)
  - Set linac voltages based on desired beam energy at BC2
- **Region II**: Backtrack through linac section L3 and 1st bunch compressor BC1 **(250→1500 MeV)**
  - Chirp: cavity wakefields from L2, LSC in L2 and drifts, and BC1 CSR (more details later)
  - Dispersion: BC1 (use desired Laser Heater (L.H.) peak current to set BC1 R56)
  - Set linac voltage based on desired beam energy at BC1
- **Region I**: Backtrack through linac section L1 and 3rd harmonic cavities L1x **(92→250 MeV)**
  - Chirp: cavity wakefields from L1 and L1x, LSC in L1, L1x and drifts
  - Dispersion: No dispersion
  - Set linac voltages based on desired beam energy at L.H.

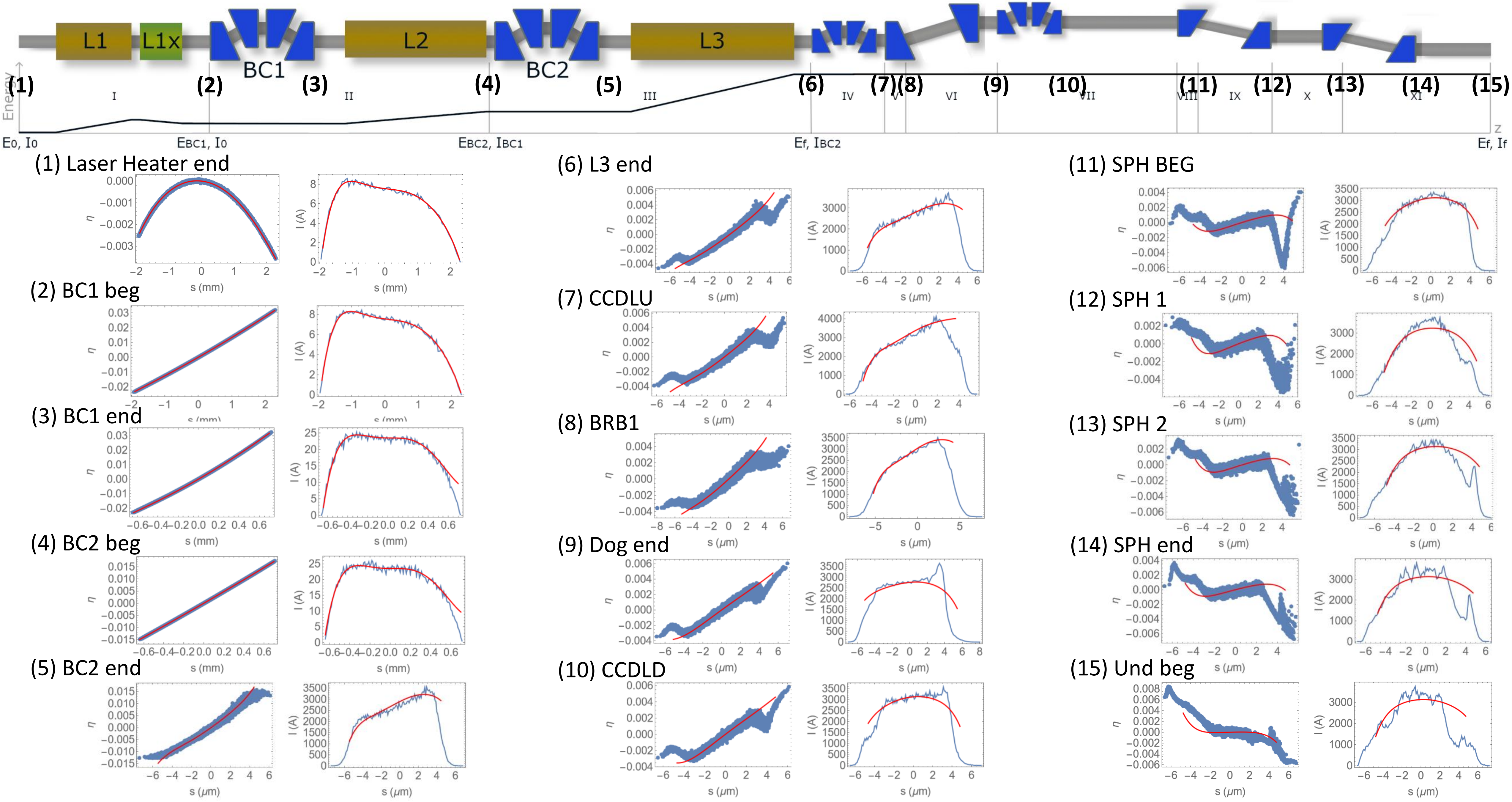
Some assumptions:
- Desired phase space doesn’t fold over
- For CSR calculations magnets are “long” compared to beam (not necessarily true)
- Consider up to 6th order
- Consider dispersive terms up to 3rd order

Free parameters:
- current profile and chirp at HXR start
- Beam energy at HXR start, BC2, BC1 and LH
- L3, L2, L1 and L1x phases
- L1 voltage
- Peak current at LH and BC1

Simulations:
- Find LPS at L.H. exit from backtracking
- Generate 6-D phase space based on this LPS and reasonable parameters for the transverse phase space ($\epsilon_x \sim \epsilon_y \sim 0.37$ mm-mrad)
- Track forward in elegant (5*10^6 particles)
- Small adjustment of BC1 and BC2 bend angles

Generating the beams:
- Integrate current profile, fs[x], normalize so fs[S_1] = -1 and fs[S_2] = 1, then invert, Fs[x]
- For energy spread Fη[x] = InvErf[x] (gaussian dist)
- Generate Hammersley sequence {hx,hy} that goes from -1 to 1 in both dimensions. Evaluate: {Fs[hx], Fη[hy]}
- Use bi-normal distributions for transverse phase space

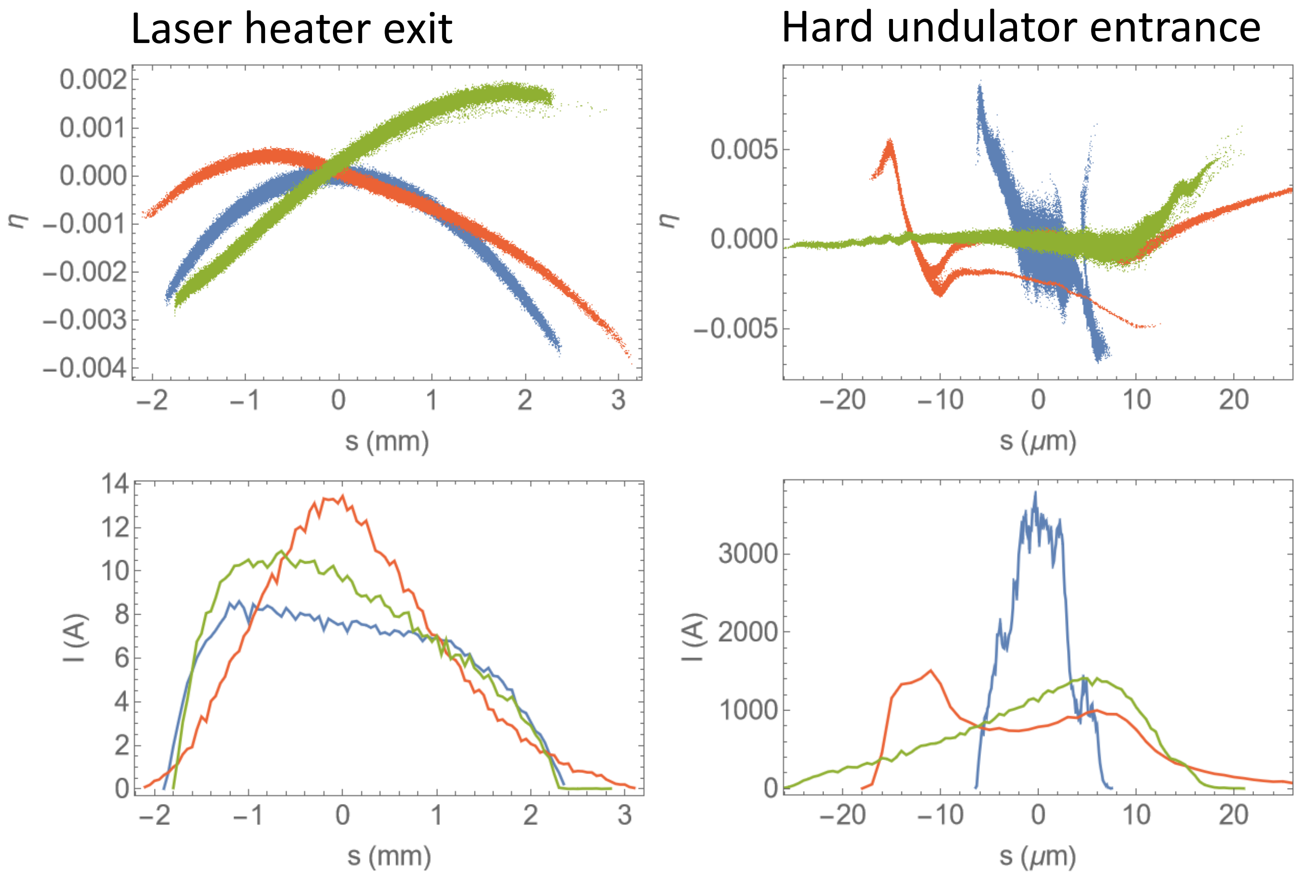
LCLS-II example: forward tracking in Elegant (blue) compared with backward tracking (red)



Phase space comparison

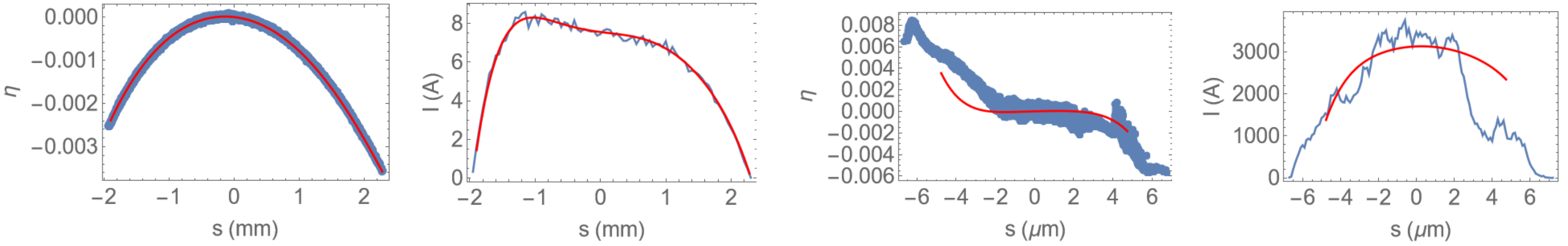
- Final beam parameters (from Elegant):
  - Bunch charge: 89 pC
  - Peak current: 3.5 kA
  - Emittance: 0.84 x 0.46 mm-mrad
  - Energy spread: 0.0011
- Emittance growth from CSR in bunch compressors compensated with quads in BC1 and BC2
- Comparison with nominal longitudinal phase space from LCLS-II injector using flat-top and gaussian temporal laser profiles
- Flat top: ~800 A, 0.42 x 0.37 mm-mrad (100 pC)
- Gaussian: ~1500 A, 0.54 x 0.59 mm-mrad (100 pC)

Flat top (Red), Gaussian (Green), Back tracking (Blue)



Linac parameters: Backtracking example

$E_0\ (MeV)$	92	$I_0\ (A)$	7.56
$E_{BC1}\ (MeV)$	250	$R_{56}^{(1)}\ (mm)$	-52.41
$E_{BC2}\ (MeV)$	1500	$I_{BC1}\ (A)$	23.4
$E_f\ (MeV)$	4000	$R_{56}^{(2)}\ (mm)$	-41.59
$V_{L1B}\ (MV)$	15.992	$\phi_{L1B}\ (^\circ)$	-25.06
$V_{L1H}\ (MV)$	4.590	$\phi_{L1H}\ (^\circ)$	-176.92
$V_{L2B}\ (MV)$	16.399	$\phi_{L2B}\ (^\circ)$	-37.23
$V_{L3B}\ (MV)$	15.67	$\phi_{L3B}\ (^\circ)$	0



Chirps and collective effects

- For all collective effects, find chirp coefficients to arbitrary order
- RF curvature
- Longitudinal space charge:
  - Consider space charge effects on order of bunch length (assume microbunching instability suppressed)
  - Need to make a guess of transverse beam size
  - LSC chirp given in terms of derivative of current profile
  - Different LSC chirp for acceleration section and drift
- Cavity wakefields:
  - Wake kernel in terms of approximate exponential form given in: *A. Novokhatski and A. Mosnier NIMA 763 (2014)*
  - Expand wake kernel in terms of x = (s-s')^0.5 to order 2N
- Resistive wall wakefields:
  - AC Resistive wall "damped oscillator model" wake kernel given in: *K. Bane and G. Stupakov LCLS-TN-04-11*
  - Given in terms of fitting parameters that depend on material and geometry
- 1D CSR:
  - Follow *Saldin et al NIM Phys. Rev. A. 398 (1997)* and *G. Stupakov and P. Emma LCLS-TN-01-12 (2001)*
  - Consider "long magnet –short bunch" case where beam experiences steady state regime (not accurate for all magnets)
  - Find analytical expression for energy modulation from steady state and entrance transient
  - Approximate exit transient integral:
    - separate integrands for each polynomial coefficient of current profile
    - Approximate integrands as quadratic with correct behavior as distant from magnet exit goes to zero and infinity and at some point in between

Example: Cavity wakefields

- Energy change from cavity wakefield given by convolution with wake kernel
  - For 1.3 GHz cavities:
    - α = 4.15e13   β = 23.973
  - For 3.9 GHz cavities:
    - α = 2.3e14   β = 34.503

$$\Delta\eta_w(s) = -\frac{eL}{mc^3\gamma_f}\int_{-S_1}^{s} ds' I(s')w(s-s')$$

$$\Delta\eta_w(s) = -\frac{eL}{mc^3\gamma_f}\int_{-S_1}^{s} ds' I_0\Big(1 + I_1 s' + I_2 s'^2 + \ldots + I_N s'^N\Big)w(s-s')$$

- Approximate wake kernel
  - Suggest M = 2N

$$w(s-s') = \alpha e^{-\beta\sqrt{s}}$$

$$w(s-s') \sim \alpha\sum_{j=0}^{M}\frac{(-1)^j\beta^j}{j!}(s-s')^{j/2}$$

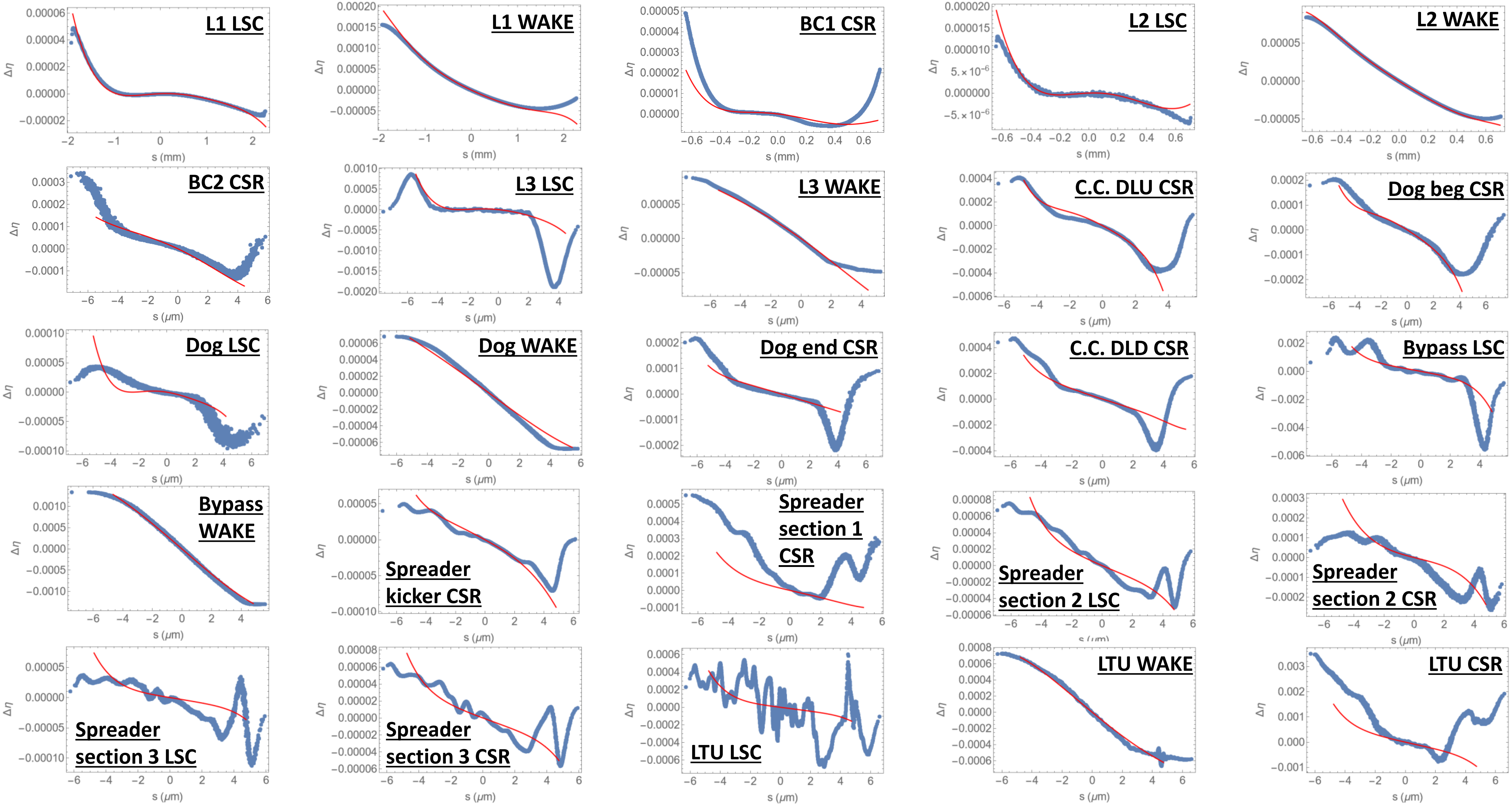
- Lc is length of each cavity, Nc is number of cavities γf is the beam energy at the exit of the acceleration section
- $\chi_0 = I_0$, $\chi_n = I_0 I_n$

$$\Delta\eta_w(s) = -\frac{eL_cN_c}{mc^3\gamma_f}\alpha\times \sum_{j=0}^{M}\frac{(-1)^j\beta^j}{j!}\sum_{k=0}^{N}\int_{S_1}^{s} ds'\chi_k s'^k (s-s')^{j/2}$$

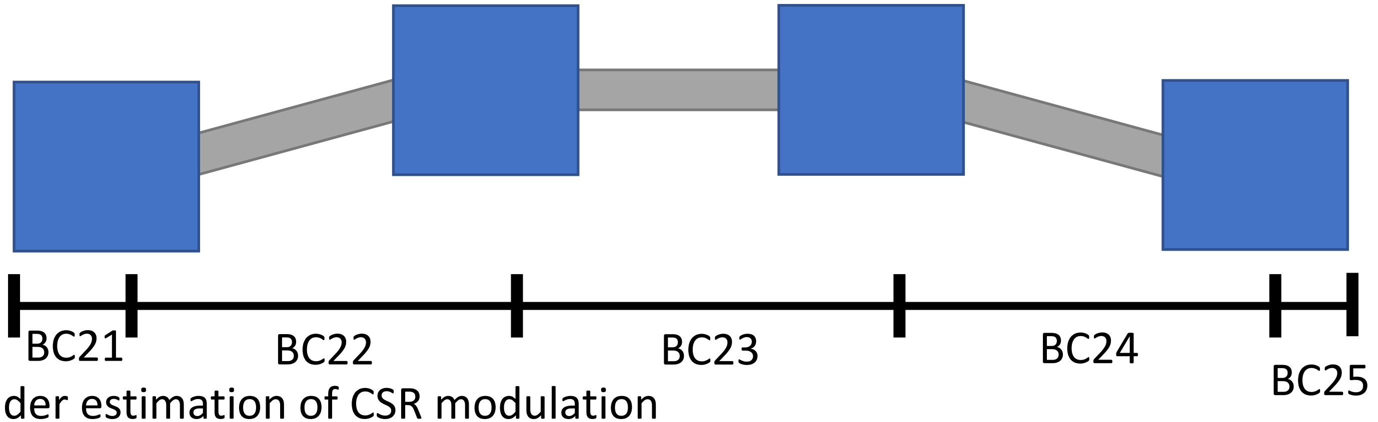
- Chirp coefficients:

$$H_{w(n)} = -\frac{eL_cN_c\alpha}{mc^3\gamma_f}\sum_{k=0}^{N}\sum_{j=0}^{M}\sum_{l=0}^{k}\frac{2(-1)^{j+l}k!\beta^j\chi_k}{j!l!(k-l)!(2l+j+2)}\times \frac{(1+\frac{j}{2}+l)!}{(n+l-k)!(1+\frac{j}{2}+k-n)!}(-S1)^{1+\frac{j}{2}+k-n}$$

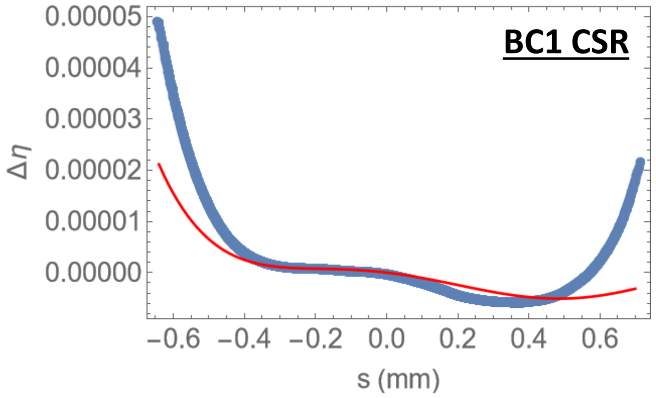
LCLS-II example collective effects: forward tracking in Elegant (blue) compared with backward tracking (red)

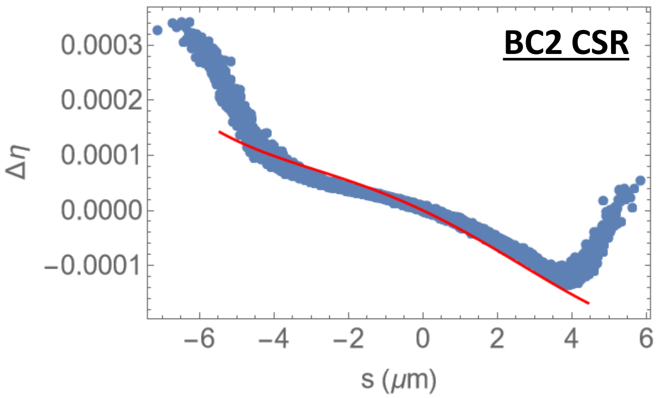


CSR in Bunch compressors



- Break up bunch compressor into multiple sections
- Evolution of current inside of bends leads to over/under estimation of CSR modulation
  - Bunch can over compress at exit of 3rd bend (backtracking method breaks down)
- Choose a point some fraction, α, inside of bend to obtain current profile to be used for CSR calculation
- For backtracking:
  - Track back (1- α)*θ into 4th bend. Calculate and add CSR modulation with new current profile.
  - Track back α*θ through 4th bend, through 3rd drift (including CSR compensating quad) and (1- α)*θ through 3rd bend. Calculate and add CSR modulation with new current profile.
  - Track back α*θ through 3rd bend, through 2nd drift and (1- α)*θ through 2nd bend. Calculate and add CSR modulation with new current profile.
  - Track back α*θ through 2nd bend, through 1st drift (including CSR compensating quad) and (1- α)*θ through 1st bend. Calculate and add CSR modulation with new current profile.
  - Track back α*θ through 1st bend
- Choose α = 0.88 (needs some more investigating)
- Longitudinal dispersion throughout calculated using expression for curvilinear path through bend (in backup slides), including edge effects keeping track of transverse dispersion throughout and including chromatic focusing in quads. (Assume no incoming dispersion)
- Compensating quads shift the maximum compression to the exit of the 4th bend, reducing CSR





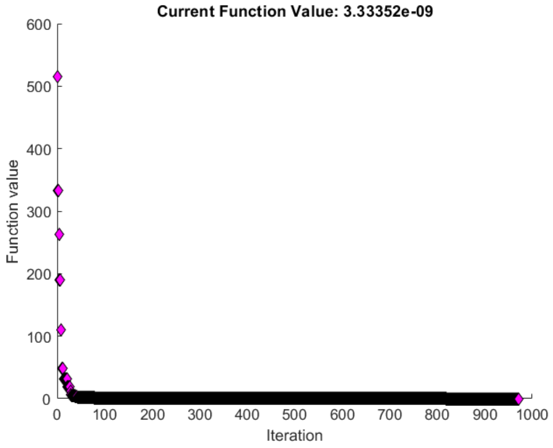
Some more chicanery

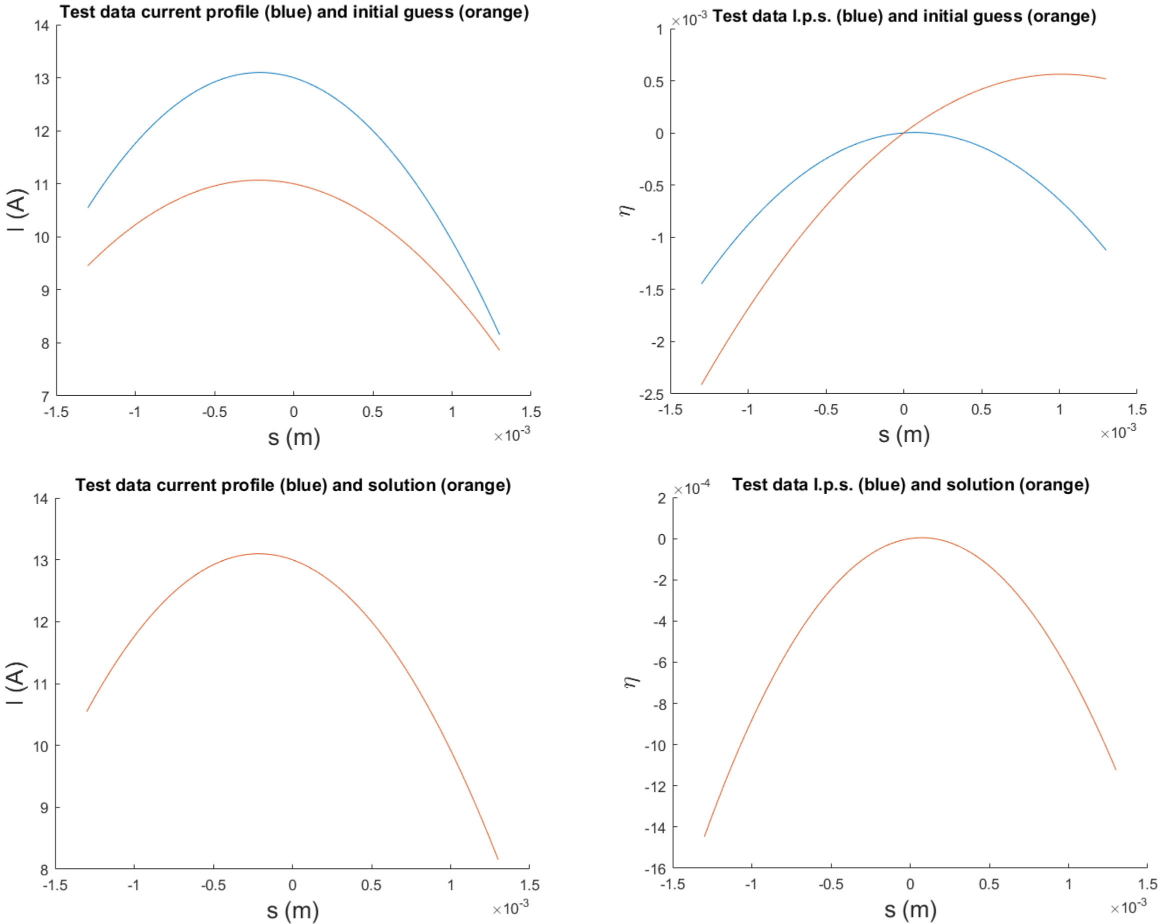
- For high peak current CSR produces x'-s kick after second bunch compressor
- Correct using quads in dispersive section that cancel x'-s kick:
  x'-x→ x'-η (x dominated by dispersion)→ x'-s (η dominated by correlated chirp)
- Find it is useful to intentionally leak dispersion out of BC1 (with quads) and correct with BC2 quads (depends on phase advance)
- In LCLS-II example In previous example: BC1 quads $K_{CQ11} = 0.1$, $K_{CQ12} = -0.1$ BC2 quads $K_{CQ21} = 0.15$, $K_{CQ22} = -0.15$
- Useful to correct CSR kick so that we don't leak dispersion into downstream dispersive regions
- Derive general expression for R56 with quads at equal and opposite strength in terms of (in the back up slides):
  - θ is bend angle
  - arc length in bend, L_m
  - drift between 1st bend and 1st quad (and 2nd quad and 4th bend), L_{d1}
  - drift between 1st quad and 2nd bend (and 3rd bend and 2nd quad), L_{d2}
  - drift between 2nd and 3rd bend, L_{d3}
  - quad length, L_q, and quad strength K_q
- For small R56 compensating chicanes calculate small correction to R56 from CSR
  - Approximate linear chirp from CSR is the same in each magnet (no significant compression):
  - $R56corr = (5 L_d + 10 * L_m/3)\alpha_h \theta^2$
  - θ is bend angle
  - L_m is arc length in bend
  - L_d is drift between 1st and 2nd (and 3rd and 4th) bends
  - α_h is ratio between linear chirp from CSR and incoming linear chirp
  - Probably not necessary

Using fast forward tracking as a diagnostic

- Use CSR signal from last dipole in BC2 assuming current profile at BC2 exit
- Calculate CSR energy based on approximate analytical expression for energy loss
- Vary linac parameters V_{L1}, ϕ_{L1}, ϕ_{L1h}, ϕ_{L2}, I_{BC1}, I_{BC2} individually and look at changing energy of CSR signal. Aim is to avoid degeneracy.
  - $E_{CSR}(V_{L1})$: $15.5 < V_{L1} < 16$ MV
  - $E_{CSR}(\phi_{L1})$: $-26 < \phi_{L1} < -23$ degrees
  - $E_{CSR}(\phi_{L1h})$: $-178 < \phi_{L1h} < -173$ degrees
  - $E_{CSR}(\phi_2)$: $-32 < \phi_2 < -28$ degrees
  - $E_{CSR}(I_{BC1})$: $35 < I_{BC1} < 45$ A
  - $E_{CSR}(I_{BC2})$: $500 < I_{BC2} < 700$ A
- In example, assume beam at injector can be described by polynomial coefficients I_0, I_1, I_2, h_1, h_2, h_3
- Use fast forward tracking, varying same linac parameters to find CSR signals as a function of initial phase space and current profile, call this $F_{CSR}(X)$
- Objective function:
  $F_{obj}(I_0, I_1, I_2, h_1, h_2, h_3) = \left([F_{CSR}(V_{L1})-E_{CSR}(V_{L1})]^2 + [F_{CSR}(\phi_{L1})-E_{CSR}(\phi_{L1})]^2 + [F_{CSR}(\phi_{L1h})-E_{CSR}(\phi_{L1h})]^2 + [F_{CSR}(\phi_{L2})-E_{CSR}(\phi_{L2})]^2 + [F_{CSR}(I_{BC1})-E_{CSR}(I_{BC1})]^2 + [F_{CSR}(I_{BC2})-E_{CSR}(I_{BC2})]^2\right)$^0.5
- Find initial beam coefficients that minimize F_{obj}, difference in data and "analytical" CSR signals
- In reality need to calculate coherent edge radiation (CER) and consider losses in optics and spectral response of detectors

Difference between test data and CSR output from forward tracking minimized by fminsearch





Conclusions/suggestions

- Backtracking method offers a quick way to find a distribution at the injector exit that approximately gives a desired final longitudinal phase space
- Easily adaptable to forward tracking
- Able to push the LCLS-II current up to 3.5 kA (provided we can generate the longitudinal phase space at the injector exit). Can probably push the current higher. Need to investigate reducing final slice energy spread.
- Method defined to arbitrary order and could possibly be used to investigate more exotic configurations
- However limited in scope to longitudinal phase spaces with no caustics
- Caustics could be included but would require some loss of “analyticity”
- Could potentially be useful as fast approximation of longitudinal phase space for virtual diagnostics
- Suggested starting point:
    - match the non-linear final chirp to cancel the non-linear chirp generated after the final bunch compressor
    - Adjust the final linear chirp and other beamline parameters to cancel the linear chirp at the laser heater exit
    - Make additional adjustments of the final non-linear chirp accordingly
    - Iterate between forward and backward tracking
    - Could probably take advantage of some machine learning techniques (maybe NSGA)
- Thanks to colleagues: Yuantao Ding, Yuri Nosochkov, Karl Bane, Zhen Zhang, David Cesar, Claudio Emma, Joe Duris, Nicole Neveu

Backup slides

Tracking method (1)

- Method: consider current profile and chirp we want to track backwards as Nth order polynomials with endpoints S1 and S2:

$$I_f(s_f) = I_{f0}(1 + I_{f1}s_f + I_{f2}{s_f}^2 + I_{f3}{s_f}^3 + \ldots + I_{fN}{s_f}^N) \quad S_1 < s_f < S_2$$

$$\eta_F(s_f) = h_{f1}s_f + h_{f2}{s_f}^2 + h_{f3}{s_f}^3 + \ldots + h_{fN}{s_f}^N$$

- s_f is coordinate along bunch, η_F is energy detuning $\eta_F = (\gamma - \gamma 0)/\gamma$ Bunch head is to the left.
- **Assume uncorrelated energy spread is negligible**
- Track backwards through acceleration/drift section followed by dispersive section
- Break up accelerator into multiple sections: accelerator/drift + dispersion sections
- In accelerator section assume some energy change, $R_{66} = E_i/E_f$, and total chirp from RF curvature and collective effects described by polynomial coefficients $H_{(n)}$
- In dispersive section assume dispersion can be described by polynomial coefficients $D_{(n)}$

(1) Track backwards:

$$s_F = s_f$$

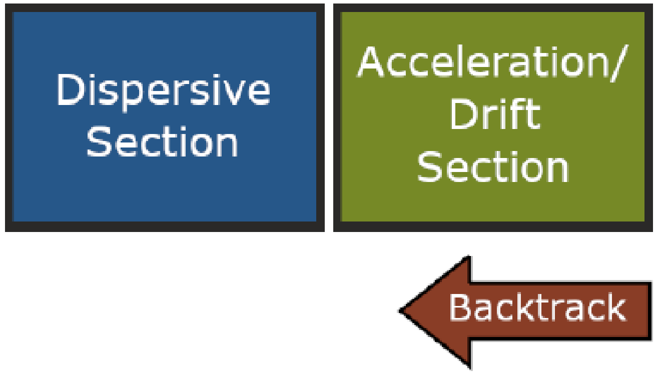
$$\eta_F = h_{f1}s_f + h_{f2}{s_f}^2 + h_{f3}{s_f}^3 + \ldots + h_{fN}{s_f}^N$$

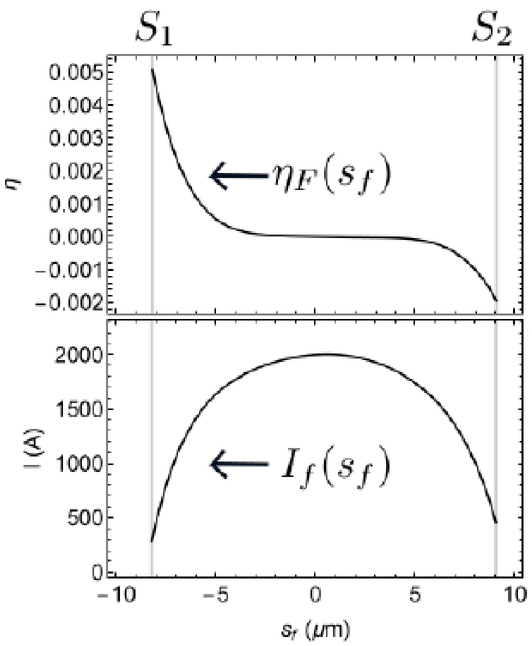
$$s_A = s_F$$

$$\eta_A = \frac{1}{R_{66}}\left[\eta_F - H_1 s_F - H_2 {s_F}^2 - H_3 {s_F}^3 - \ldots - H_N {s_F}^N\right]$$

$$s_I = s_A - D_1\eta_A - D_2{\eta_A}^2 - D_3{\eta_A}^3 - \ldots - D_N{\eta_A}^N$$

$$\eta_I = \eta_A$$





Tracking method (2)

- Tracking backwards gives $s_I(s_f)$ and $\eta_I(s_f)$
- Chirp at entrance of dispersion section given by $\eta_I(s_I) = \eta_I\big(s_f(s_i)\big)$: invert $s_I(s_f)$
- **Assume chirp at entrance of dispersion section can be described by Nth order polynomial**
- Forward tracking can be done in terms of some unknown chirp described by $h_{I(n)}$
- s_i is coordinate along bunch at dispersion section entrance
- Forward tracking gives $\boldsymbol{\eta}_F(s_I)$, Backward tracking gives $s_I(s_f)$
- Insert $s_I(s_f)$ and equate to known chirp at accelerator section exit: $\boldsymbol{\eta}_F\big(s_I(s_f)\big) = \eta_F(s_f)$
- Solve for unknown chirp coefficients, $h_{I(n)}$

(2) Track forwards:

$$\bar{s}_I = s_i$$

$$\bar{\eta}_I = h_{I1}s_i + h_{I2}{s_i}^2 + h_{I3}{s_i}^3 + \ldots + h_{IN}{s_i}^N$$

$$\bar{s}_D = s_I + D_1\bar{\eta}_I + D_2\bar{\eta}_I^2 + D_3\bar{\eta}_I^3 + \ldots + D_N\bar{\eta}_I^N$$

$$\bar{\eta}_D = \bar{\eta}_I$$

$$\bar{s}_F = \bar{s}_D$$

$$\bar{\eta}_F = R_{66}\bar{\eta}_D + H_1\bar{s}_D + H_2\bar{s}_D^2 + H_3\bar{s}_D^3 + \ldots + H_N\bar{s}_D^N$$

Tracking method (3)

- Try to write things in a "convenient" way to arbitrary order N
- Introduce vector consisting of polynomial coefficients
- Multiplication of polynomial vectors given by forming lower triangular matrix out of polynomial vector with tensor **T**
- Introduce vector **Y** that describes the chirp at the dispersive section exit
- Introduce vector **d** consisting of decompression factors $d_{(n)}$ where $s_I(s_f)=\mathbf{d}\cdot\mathbf{s_f}$
- Introduce vector **C** consisting of compression factors $C_{(n)}$ where $s_f(s_i)=\mathbf{C}\cdot\mathbf{s_i}$
- Compression factors can be written in terms of the decompression factors with complicated but compact expression from solving $\eta_F(s_I(s_f)) = \eta_F(s_f)$
- Introduce matrices that depend on the sum index, $\mathbf{M_I^{(n)}}$, tensor $\boldsymbol{\tau}$, and polynomial vector, $\mathbf{V_d}$, consisting of decompression factors scaled by the linear decompression factor and unit vectors, $\mathbf{e}_{n(i)} = \delta_{n+1,i}$
- Use compression factors to write vector of polynomial coefficients of chirp at dispersion section entrance, $\mathbf{h_I}$

(3) Solve chirp: $\eta_I\left[\bar{s}_F(s_i)\right] = \bar{\eta}_I(s_i)$

(3.1) polynomial vectors of order N:

$$P(s) = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_N \end{bmatrix} \cdot \begin{bmatrix} 1 \\ s \\ s^2 \\ \vdots \\ s^N \end{bmatrix} \equiv \vec{p}\cdot\vec{s}$$

(3.2) polynomial multiplication:

$$T_{n,m,j} = \delta_{m+j-n,1}$$

T is N+1 x N+1 x N+1 tensor

$$Q(s)P(s) = \vec{s}\cdot(\boldsymbol{T}\cdot\vec{q})\cdot\vec{p}$$

(3.3) total chirp and decompression factors:

$$\vec{Y} = \frac{1}{R_{66}}(\vec{h_f} - \vec{H})$$

$$\vec{d} = \vec{e_1} - \sum_{n=1}^{N} D_n \left(\prod_{m=1}^{n-1}(\boldsymbol{T}\cdot\vec{Y})\cdot\right)\vec{Y}$$

(3.4) solution of initial chirp:

$$\vec{C} = \frac{\vec{e_1}}{d_1} + \frac{1}{d_1}\sum_{n=1}^{N-1} \boldsymbol{M_I^{(n)}} \cdot \left(\prod_{j=1}^{n-1}(\boldsymbol{\tau}\cdot\vec{V_d})\cdot\right)\vec{V_d}$$

$$\vec{h_i} = \sum_{n=1}^{N} Y_n \left(\prod_{j=1}^{n-1}(\boldsymbol{T}\cdot\vec{C})\cdot\right)\vec{C}$$

$$M_{I(i,j)}^{(n)} = \binom{2n+j-1}{n-1}\frac{\delta_{i-j-1,n}}{n}$$

$\mathbf{M_I}$ is N-1 x N+1 matrix

$$\tau_{n,i,j} = \delta_{i+j-n,1}$$

τ is N-1 x N-1 x N-1 tensor

$$V_{d(n)} = -\frac{d_{n+1}}{{d_1}^{(n+1)}}$$

$\mathbf{V_d}$ is N-1 dimensional vector

Tracking method (4)

- **Assume delta function energy distribution**
- Transformation of current profile given by sum over roots of $s_f(s_i)$
- **Assume longitudinal phase space is single valued (no current horns). Drop sum**
- Write polynomial coefficients of $I_f(s_F(s_i))$ in terms of compression factors
- Write polynomial coefficients of denominator in terms of compression factors and Matrix, **M**$_d$, that gives derivative of polynomial vector
- Taylor expand denominator to order N in terms of polynomial coefficients of denominator and linear decompression factor
- Transformation of distribution end points given in terms of decompression factors

(4) Solve for current:

(4.1) Integrate over delta function:

$$I_i(s_i) = \int_{-\infty}^{\infty} d\eta_i I_f[\bar{s}_F(s_i,\eta_i)]\delta[\bar{\eta}_F(s_i,\eta_i)]$$

$$I_i(s_i) = \sum_j \frac{I_f[s_f]}{\left|\frac{ds_I}{ds_f}\right|}\Bigg|_{s_f=s_{f(j)}(s_i)}$$

$$I_i(s_i) = \frac{I_f[\bar{s}_F(s_i)]}{\frac{ds_I}{ds_f}[\bar{s}_F(s_i)]} \equiv \frac{\vec{I}_f\cdot\vec{s_i}}{(d_1\vec{e_0}+\vec{ds})\cdot\vec{s_i}}$$

(4.2) Numerator and denominator:

$$M_{d(n,m)} = n\delta_{m-n,1}$$

$$\vec{I}_f = I_{f0}\vec{e_0} + \sum_{n=1}^{N} I_{f0}I_{f(n)}\left(\prod_{m=1}^{n-1}(\boldsymbol{T}\cdot\vec{C})\cdot\right)\vec{C}$$

$$\vec{ds} = \sum_{n=1}^{N}\vec{e_n}\cdot(\boldsymbol{M_d}\cdot\vec{d})\left(\prod_{m=1}^{n-1}(\boldsymbol{T}\cdot\vec{C})\cdot\right)\vec{C}$$

(4.3) Expand denominator and solve:

$$\vec{ds} = \frac{1}{(d_1\vec{e_0}+\vec{ds})\cdot\vec{s_i}} \sim \frac{\vec{e_0}}{d_1} + \sum_{n=1}^{N}\frac{(-1)^n}{{d_1}^{n+1}}\left(\prod_{m=1}^{n-1}(\boldsymbol{T}\cdot\vec{ds})\cdot\right)\vec{ds}$$

$$\vec{I_i} = (\boldsymbol{T}\cdot\vec{I_f})\cdot\vec{ds}$$

(4.4) Transform endpoints:

$$S_{1i} = \vec{d}\cdot\vec{S_1} \qquad S_{2i} = \vec{d}\cdot\vec{S_2}$$

Chirps and collective effects: RF curvature

- Chirp from acceleration
- Nc is number of cavities
- V is voltage in cavities
- γf is the beam energy at the exit of the acceleration **section**
- Chirp coefficients →

$$\Delta\eta_a(s) = \frac{eVN_c}{mc^2\gamma_f}\cos(ks+\phi)$$

$$H_{a(n)} = \frac{eVN_c}{mc^2\gamma_f}\frac{k^n}{n!}\cos(\phi+n\frac{\pi}{2})$$

Chirps and collective effects: cavity wakefield

- Energy change from cavity wakefield given by convolution with wake kernel
  - For 1.3 GHz cavities:
    - $\alpha = 4.15e13$ $\beta = 23.973$
  - For 3.9 GHz cavities:
    - $\alpha = 2.3e14$ $\beta = 34.503$
- Approximate wake kernel
  - Suggest M = 2N
- Lc is length of each cavity, Nc is number of cavities γf is the beam energy at the exit of the acceleration section
- $\chi_0 = I_0$, $\chi_n = I_0 I_n$
- Chirp coefficients

$$\Delta\eta_w(s) = -\frac{eL}{mc^3\gamma_f}\int_{-S_1}^{s} ds' I(s')w(s-s')$$

$$\Delta\eta_w(s) = -\frac{eL}{mc^3\gamma_f}\int_{-S_1}^{s} ds' I_0\Big(1 + I_1 s' + I_2 s'^2 + \ldots + I_N s'^N\Big) w(s-s')$$

$$w(s-s') = \alpha e^{-\beta\sqrt{s}}$$

$$w(s-s') \sim \alpha \sum_{j=0}^{M} \frac{(-1)^j \beta^j}{j!}(s-s')^{j/2}$$

$$\Delta\eta_w(s) = -\frac{eL_cN_c}{mc^3\gamma_f}\alpha \times \sum_{j=0}^{M}\frac{(-1)^j\beta^j}{j!}\sum_{k=0}^{N}\int_{S_1}^{s} ds' \chi_k s'^k (s-s')^{j/2}$$

$$H_{w(n)} = -\frac{eL_cN_c\alpha}{mc^3\gamma_f}\sum_{k=0}^{N}\sum_{j=0}^{M}\sum_{l=0}^{k}\frac{2(-1)^{j+l}k!\beta^j\chi_k}{j!l!(k-l)!(2l+j+2)} \times \frac{(1+\frac{j}{2}+l)!}{(n+l-k)!(1+\frac{j}{2}+k-n)!}(-S1)^{1+\frac{j}{2}+k-n}$$

Chirps and collective effects: RW wakefield

- AC Resistive wall wake kernel given in:
  K. Bane and G. Stupakov LCLS-TN-04-11
- kr and Qr are fitting parameters
  - 17.4 mm Cu pipe kr = 6.0423e4   Qr = 1.6949
  - 24.5 mm S.S. pipe kr = 1.3748e4   Qr = 1.1388
- L is length of pipe, γf is the beam energy at the exit of the acceleration section
- Chirp coefficients

$$w(s-s') = \frac{Z_0 c}{\pi r^2} e^{-\frac{k_r}{2Q_r}(s-s')} \cos\left[k_r(s-s')\right]$$

$$\Delta\eta_{rw}(s) = -\frac{eL}{mc^3\gamma_f}\frac{Z_0 c}{\pi r^2}\times$$

$$\sum_{k=0}^{N}\int_{S_1}^{s} ds'\chi_k {s'}^k e^{-\frac{k_r}{Q_r}(s-s')}\cos\left[k_r(s-s')\right]$$

$$F_j = \sum_{k=0}^{j}\frac{(-1)^k (j-1)!}{(2k)!(j-2k-1)!(2k+1)}{Q_r}^{2k} \quad j>0 \qquad F_j = (1+{Q_r}^2)^j F_{|j|} \quad j<0 \qquad F_0 = 0$$

$$G_j = \sum_{k=0}^{j}\frac{(-1)^k j!}{(2k)!(j-2k)!}{Q_r}^{2k} \quad j \geq 0 \qquad G_j = (1+{Q_r}^2)^j G_{|j|} \quad j<0$$

$$H_{rw(n)} = -\frac{eL}{mc^3\gamma_f}\frac{Z_0 c}{\pi r^2}\frac{1}{n!{k_r}^{N-n+1}{Q_r}^{n-1}(1+{Q_r}^2)^{N-n+1}}\Bigg(e^{k_r S_1/Q_r}\sum_{k=0}^{N}\sum_{j=0}^{k}\Big[\sin(k_r S_1)Q_r(k-n-j+1)F_{n-k+j-1}+$$

$$\cos(k_r S_1)G_{n-k+j-1}\Big](-1)^{n+k+j+1}\frac{k!}{j!}{k_r}^{N-k+j}{Q_r}^{k-j}(1+{Q_r}^2)^{N-n+1}{S_1}^j\chi_k$$

$$+\sum_{j=0}^{N-n}(-1)^j(n+j)!{k_r}^{N-n-j}{Q_r}^{j+n}(1+{Q_r}^2)^{N-n-j}G_{j+1}\chi_{n+j}\Bigg)$$

Chirps and collective effects: Longitudinal space charge

- Approximate LSC impedance
- Change in beam energy as a function of wavenumber given in terms of fourier transform of current profile
- Consider only long range space charge effects on order of bunch length
  - Define kc = 4π I0/(Qc) **λc ~ 0.5 bunch length**
  - Ignore k dependence in log term
  - Inverse fourier transform gives derivative of I(s)
- Define impedance in drift
- Define impedance in acceleration section integrating over constant gradient
  - γ1 and γ2 are initial and final energies in section (important distinction for L1 and L1x)
- L is length of section, γf is the beam energy at the exit of the total acceleration section, IA is alfven current
- Chirp coefficients

$$Z_{LSC}(k) = \frac{ikZ_0}{\pi\gamma^2}\frac{1-\zeta K_1(\zeta)}{\zeta^2}$$

$$Z_{LSC}(k) \sim \frac{ikZ_0}{4\pi\gamma^2}[1.232 - 2\log(\zeta)]$$

$$d\gamma_{LSC}(k) = \frac{4\pi}{I_A}\tilde{I}(k)\frac{Z_{LSC}}{Z_0}$$

$$= \frac{ik\tilde{I}(k)}{I_A\gamma^2}[1.232 + 2\log(\frac{\gamma}{k\sigma})]dz$$

$$d\gamma_{LSC}(s) = \frac{1}{I_A\gamma^2}[1.232 + 2\log(\frac{\gamma}{k_c\sigma})]\int_{-\infty}^{\infty} dke^{iks}ik\tilde{I}(k)$$

$$d\gamma_{LSC}(s) = \frac{1}{I_A\gamma^2}[1.232 + 2\log(\frac{\gamma}{k_c\sigma})]\frac{\partial I(s)}{\partial s}dz$$

$$\mu_D = \frac{L}{{\gamma_1}^2}[1.232 + 2\log(\frac{\gamma_1}{k_c\sigma})]$$

$$\mu_A = \int_0^L dz\frac{1}{\gamma^2}[1.232 + 2\log(\frac{\gamma}{k_c\sigma})]$$

$$= 2L\left(\frac{1+\log(\gamma_1)}{(\gamma_2-\gamma_1)\gamma_1} - \frac{1+\log(\gamma_2)}{(\gamma_2-\gamma_1)\gamma_2} + \frac{1.232 + 2\log(\frac{1}{k_c\sigma})}{2\gamma_1\gamma_2}\right)$$

$$H_{LSC(n)} = \frac{1}{I_A\gamma_f}\mu_x(n+1)\chi_{n+1}$$

Chirps and collective effects: CSR

$$\Xi_{n,k,i} = \frac{8^{\frac{1}{3}} k (\frac{2}{3})^{(\overline{k-1-i})} (1-k)^{(\overline{k-1-i})}}{(k-n)!(n-i)!(\frac{5}{3})^{(\overline{k-1-i})}}$$

Rising factorial X*(X+1)*(X+2)*…*(X+k-1-i-1)

$$\Lambda^0_{n,k} = \frac{(-1)^k 4}{n!(24)^{\frac{1}{3}}} \left(-\frac{1}{3}\right)^{(\underline{n})}$$

Falling factorial X*(X-1)*(X-2)*…*(X-n+1)

$$\Lambda^1_{n,m,k,i} = \frac{3^{\frac{2}{3}} k (\frac{2}{3})^{(\underline{n-m})} (\frac{2}{3})^{(\overline{k-1-i})} (1-k)^{(\overline{k-1-i})}}{(n-m)!(m-i)!(k-m-i)!(\frac{5}{3})^{(\overline{k-1-i})}}$$

$$\Lambda^2_{n,k,i} = \frac{6k(k-i)(\frac{2}{3})^{(\overline{k-1-i})} (1-k)^{(\overline{k-1-i})}}{(n-i)!(k-n)!(\frac{5}{3})^{(\overline{k-1-i})}}$$

$$\Upsilon^0_{n,m,k} = \frac{3(11 - 2^{\frac{1}{3}} 10 + 2^{\frac{2}{3}})(-1)^{k+m}(k+m)!(k+n)!}{(2^{\frac{1}{3}} 4 - 5)(3(k+m)-1)k!m!(k+m-1)!(n-m)!}$$

$$\Upsilon^1_{n,i,m,k} = (-1)^{k+m-i} \times$$

$$\frac{2^{\frac{1}{3}} 12^{m+k-1} 4[2i(4+\log(27)) + (k+m)(10+\log(27))]}{2^{6(m+k)-1} 3^i i! m! k!} \times$$

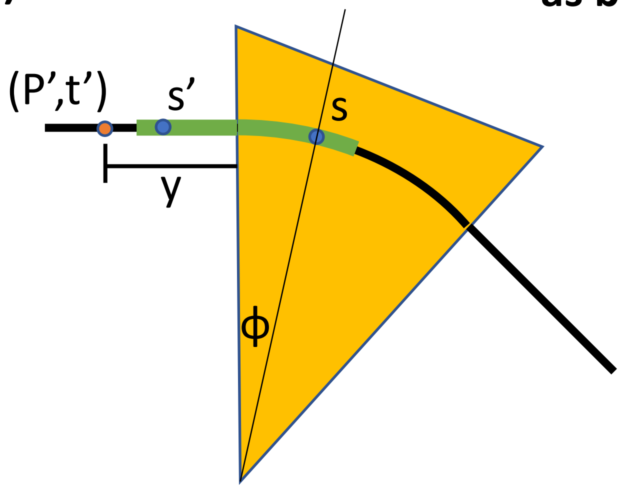
$$\frac{(k+n)!(m+k)!(3(k+m)-1)!}{(k+m-1-i)!(n-m)!(k+m-i)!(i+2(k+m))!}$$

$$\Upsilon^2_{n,l,i,m,k} = \frac{(-1)^{l-1}}{(n-m)!} \left(\frac{3^{i+l-k-m} - 1}{i+l-k-m} + \frac{3^{i+l-k-m+1} - 1}{i+l-k-m+1} \right) \times$$

$$\frac{2^{\frac{1}{3}} 12^{m+k-1} 4(k+n)!(k+m)!(3(k+m)-1)!}{2^{6(m+k)-1} 3^{i-1} i! l! m! k! (k+m-1-i)!(3(k+m)-1-l)!}$$

$$H_{CSR(n)} = -\frac{e(-1)^n}{mc^3 4\pi\epsilon_0\gamma} \times$$
$$\Bigg(\frac{4}{3} \log(4)(-1)^n \chi_n + \sum_{k=n}^{N} \sum_{i=0}^{n-1} \Xi_{n,k,i} (-1)^k \chi_k {S_2}^{k-n} +$$
$$\sum_{k=n+1}^{N} \Xi_{n,k,n} (-1)^k \chi_k {S_2}^{k-n}$$
$$+ \sum_{k=0}^{N} \Lambda^0_{n,k} \Phi \rho^{\frac{1}{3}} (-1)^k \chi_k {S_2}^{k-n-\frac{1}{3}} +$$
$$\sum_{m=0}^{n} \sum_{k=m+1}^{N} \sum_{i=0}^{m} \Lambda^1_{n,m,k,i} \Phi \rho^{\frac{1}{3}} (-1)^k \chi_k {S_2}^{k-n-\frac{1}{3}} -$$
$$\sum_{k=n+1}^{N} \sum_{i=0}^{m} \Lambda^2_{n,k,i} (-1)^k \chi_k {S_2}^{k-n} - \sum_{i=0}^{n-1} \Lambda^2_{n,n,i} (-1)^n \chi_n +$$
$$\sum_{k=0}^{N-n} \sum_{m=0}^{n} (-1)^{k+n} \chi_{k+n} {S_2}^k \Bigg[\Upsilon^0_{n,m,k} + \sum_{i=0}^{m+k-1} \Upsilon^1_{n,i,m,k} -$$
$$\sum_{i=0}^{m+k-1} \sum_{l=0}^{m+k-i-2} \Upsilon^2_{n,l,i,m,k} - \sum_{i=0}^{m+k-1} \sum_{l=m+k-i+1}^{3(m+k)-1} \Upsilon^2_{n,l,i,m,k} \Bigg] \Bigg)$$

(A)



Find wakefield for different cases as beam passes through magnet

- Source point at position **P'** at time **t'**
- Observation point at position **s** along the beam at time **t**
- Source point at position **s'** in the beam at time **t**
- Note: electron bunch in green

For observation point inside magnet and source point outside of magnet entrance:

Flip current profile so head is on right

$$\bar{I}(s) = I_0(1 - I_1 s + I_2 s^2 - I_3 s^3 + \ldots + (-1)^N I_N s^N) \quad -S_2 < s < -S_1$$
$$\bar{I}(s) = 0 \quad s < -S_2 \quad \& \quad s > -S_1$$

Wake kernel/energy change from source point, s'

$$w_A[s - s'] = -\frac{4}{\rho^2 \phi} \delta\left[\frac{s - s'}{\rho} - \frac{\phi^3}{6}\right]$$

Integrate over source points

$$\frac{1}{\rho}\frac{d\eta_A}{d\phi} = -\frac{e}{mc^3 4\pi\epsilon_0 \gamma} \int ds' w_A[s - s'] \bar{I}[s']$$
$$= \frac{e}{mc^3 4\pi\epsilon_0 \gamma} \frac{4}{\rho\phi} \bar{I}\left[s - \frac{\rho\phi^3}{6}\right]$$

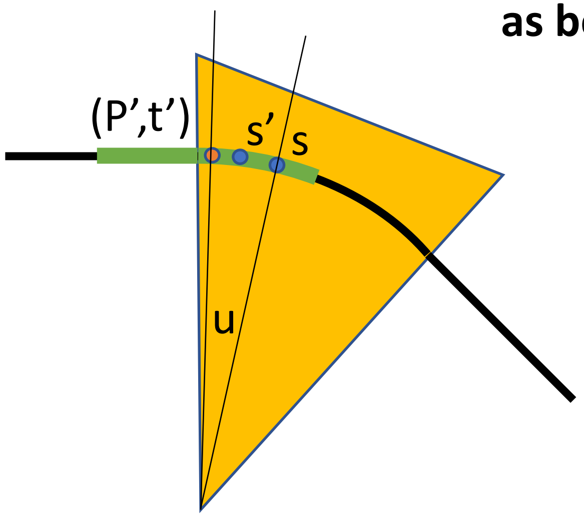
Total energy change over path through magnet

$$\Delta\eta_A = \frac{e}{mc^3 4\pi\epsilon_0 \gamma} \int_0^{\phi_f} d\phi \frac{4}{\phi} \bar{I}\left[s - \frac{\rho\phi^3}{6}\right]$$

Integration limit defined by endpoint of beam tail

$$\phi_f = \left[\frac{6}{\rho}(s + S_2)\right]^{1/3}$$

(B)



Find wakefield for different cases as beam passes through magnet

- Source point at position **P’** at time **t’**
- Observation point at position **s** along the beam at time **t**
- Source point at position **s’** in the beam at time **t**
- Note: electron bunch in green

For observation point inside magnet and source point inside of magnet:

Wake kernel/energy change from source point, s’

$$w_B[s-s'] = -\frac{2}{(3\rho^2)^{1/3}}\frac{d}{ds'}\frac{1}{(s-s')^{1/3}}$$

Integrate over source points

$$\frac{1}{\rho}\frac{d\eta_B}{d\phi} = -\frac{e}{mc^3 4\pi\epsilon_0\gamma}\int ds' w_B[s-s']\bar{I}[s']$$

$$= -\frac{e}{mc^3 4\pi\epsilon_0\gamma}\left(\frac{4}{\rho\phi}\bar{I}\left[s-\frac{\rho\phi^3}{24}\right]+ \frac{2}{(3\rho^2)^{1/3}}\int_{s-\frac{\rho\phi^3}{24}}^{s} ds'\frac{1}{(s-s')^{1/3}}\frac{d\bar{I}[s']}{ds'}\right)$$

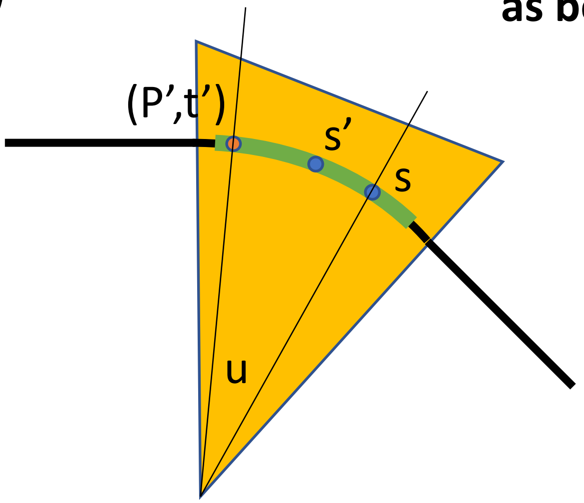
Total energy change over path through magnet

$$\Delta\eta_B = -\frac{e}{mc^3 4\pi\epsilon_0\gamma}\int_0^{\phi_f} d\phi\left(\frac{4}{\phi}\bar{I}\left[s-\frac{\rho\phi^3}{24}\right]+ \frac{2}{(3\rho^2)^{1/3}}\int_{s-\frac{\rho\phi^3}{24}}^{s} ds'\frac{1}{(s-s')^{1/3}}\frac{d\bar{I}[s']}{ds'}\right)$$

Integration limit defined by endpoint of beam tail

$$\phi_f = \left[\frac{24}{\rho}(s+S_2)\right]^{1/3}$$

(SS)



Find wakefield for different cases as beam passes through magnet

- Source point at position **P'** at time **t'**
- Observation point at position **s** along the beam at time **t**
- Source point at position **s'** in the beam at time **t**
- Note: electron bunch in green

For observation point inside magnet and source point inside of magnet (steady state):

Wake kernel/energy change from source point, s'

$$w_{SS}[s-s'] = -\frac{2}{(3\rho^2)^{1/3}}\frac{d}{ds'}\frac{1}{(s-s')^{1/3}}$$

Integrate over source points

$$\frac{1}{\rho}\frac{d\eta_{SS}}{d\phi} = -\frac{e}{mc^3 4\pi\epsilon_0\gamma}\int ds' w_{SS}[s-s']\bar{I}[s']$$

$$= -\frac{e}{mc^3 4\pi\epsilon_0\gamma}\left(\frac{4}{\rho\left[\frac{24}{\rho}(s+S_2)\right]^{1/3}}\bar{I}[-S_2] + \frac{2}{(3\rho^2)^{1/3}}\int_{-S_2}^{s} ds' \frac{1}{(s-s')^{1/3}}\frac{d\bar{I}[s']}{ds'}\right)$$

Total energy change over path through magnet **constant over path**

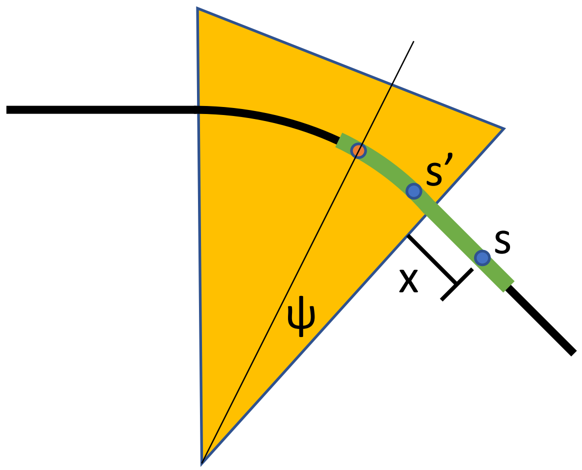
$$\Delta\eta_{SS} = -\frac{e}{mc^3 4\pi\epsilon_0\gamma}\rho(\phi_f-\phi_i)\left(\frac{4}{\rho\left[\frac{24}{\rho}(s+S_2)\right]^{1/3}}\bar{I}[-S_2] + \frac{2}{(3\rho^2)^{1/3}}\int_{-S_2}^{s} ds' \frac{1}{(s-s')^{1/3}}\frac{d\bar{I}[s']}{ds'}\right)$$

Integration limit defined by endpoint of beam tail and full magnet angle, Φ

$$\phi_i = \left[\frac{24}{\rho}(s+S_2)\right]^{1/3} \qquad \phi_f = \Phi$$

(D)

For observation point outside magnet exit and source point inside of magnet:



Relationship between s, s', angle ψ, and distance from exit, x

$$s - s' = \frac{\rho\psi^3}{24}\frac{\psi + 4x}{\psi + x}$$

Wake kernel/energy change from source point, s'

$$w_D[s - s'] = -\frac{4}{\rho}\frac{d}{ds'}\frac{1}{\psi[s'] + 2x}$$

Integrate over source points

$$\frac{1}{\rho}\frac{d\eta_D}{d\phi} = -\frac{e}{mc^3 4\pi\epsilon_0\gamma}\int ds' w_D[s - s']\bar{I}[s']$$

$$= -\frac{e}{mc^3 4\pi\epsilon_0\gamma}\frac{4}{\rho}\left(\frac{\bar{I}[-S_2]}{\psi[-S_2] + 2x} + \int_{-S_2}^{s} ds' \frac{1}{\psi[s'] + 2x}\frac{d\bar{I}[s']}{ds'}\right)$$

Maximum angle ψ0 is given by distance between observation point and bunch tail

$$s + S_2 = \frac{\rho{\psi_0}^3}{24}\frac{\psi_0 + 4x}{\psi_0 + x}$$

$$x[\psi_0] = \frac{\rho{\psi_0}^4 - 24(s + S_2)\psi_0}{24(s + S_2) - 4\rho{\psi_0}^3}$$

Change variables to ψ and write distance from exit, x, in terms of ψ0

$$\frac{1}{\rho}\frac{d\eta_D}{d\phi} = -\frac{e}{mc^3 4\pi\epsilon_0\gamma}\frac{4}{\rho}\left(\frac{\bar{I}[-S_2]}{\psi_0 + 2x[\psi_0]} - \int_0^{\psi_0} d\psi \frac{1}{\psi + 2x[\psi_0]}\frac{d}{d\psi}\bar{I}\left[s - \frac{\rho\psi^3}{24}\frac{\psi + 4x[\psi_0]}{\psi + x[\psi_0]}\right]\right)$$

Total energy change over tail's path through magnet

$$\Delta\eta_D = \frac{4e}{mc^3 4\pi\epsilon_0\gamma}\int_{\psi_{0i}}^{\psi_{0f}} d\psi_0 \left(\frac{\bar{I}[-S_2]}{\psi_0 + 2x[\psi_0]} - \int_0^{\psi_0} d\psi \frac{1}{\psi + 2x[\psi_0]}\frac{d}{d\psi}\bar{I}\left[s - \frac{\rho\psi^3}{24}\frac{\psi + 4x[\psi_0]}{\psi + x[\psi_0]}\right]\right)$$

Ψ0i is max angle as x goes to infinity

Ψ0f is max angle as x goes to 0

$$\psi_{0i} = \left[\frac{6}{\rho}(s + S_2)\right]^{1/3} \qquad \psi_{0f} = \left[\frac{24}{\rho}(s + S_2)\right]^{1/3}$$

Case D approximation

- For integral term in case D wake, integration over ψ0 can't be done explicitly
- Approximate integrand associated with each polynomial order as quadratic
- Quadratic approximation determined by observation of integrand behavior
- This approximation is therefore independent of the current profile
- Choose A(n), B(n) and C(n) such that:

$$A_{(n)} = F_{D(n)}\left[\left(\frac{24}{\rho}(s+S_2)\right)^{1/3}\right]$$

$$f_{D(n)}\left[\left(\frac{6}{\rho}(s+S_2)\right)^{1/3}\right] = 0$$

$$f_{D(n)}\left[\left(\frac{12}{\rho}(s+S_2)\right)^{1/3}\right] = F_{D(n)}\left[\left(\frac{12}{\rho}(s+S_2)\right)^{1/3}\right]$$

$$F_{D(n)}[\psi_0] = -\int_0^{\psi_0} d\psi(-1)^n n\chi_n \times$$

$$\left(s - \frac{\rho\psi^3}{24}\frac{\psi+4x[\psi_0]}{\psi+x[\psi_0]}\right)^{n-1}\frac{\rho\psi^2(\psi+2x[\psi_0])}{8(\psi+x[\psi_0])^2}$$

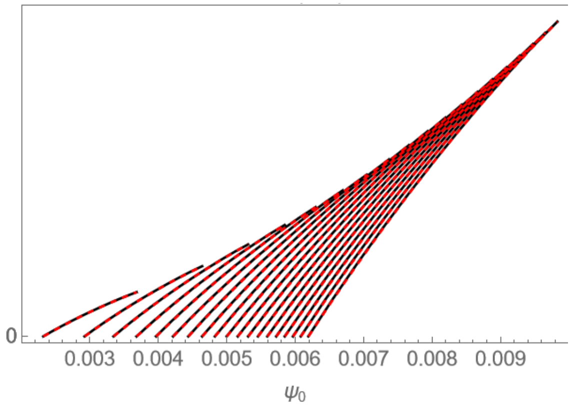
$$\sim f_{D(n)}[\psi_0] = A_{(n)} + B_{(n)}\left(\psi_0 - \left[\frac{24}{\rho}(s+S_2)\right]^{1/3}\right) +$$

$$C_{(n)}\left(\psi_0 - \left[\frac{24}{\rho}(s+S_2)\right]^{1/3}\right)^2$$

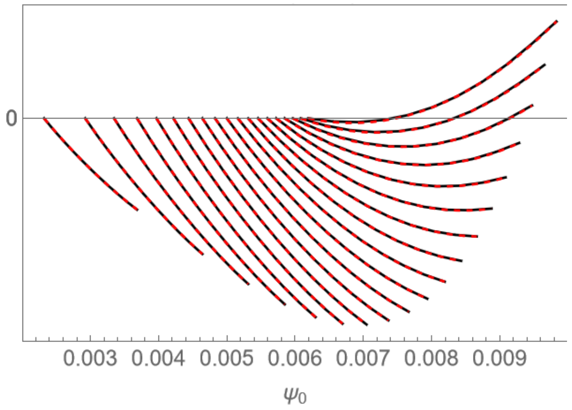
Case D approximation comparison

- Compare analytical integrals (black) with quadratic approximation of integrals, (red, dashed) in **arbitrary units**
- Use scaling $(\zeta + \zeta_2) = (s+S_2)/R$ and choose bunch length such that $\zeta_2 = \Phi^3/48$
- Scale $F_{D(n)}$ such that Φ is only independent variable, choose $\Phi = 0.01$
- Plot comparison for $\zeta = -0.9\ \zeta_2$ to $0.9\ \zeta_2$ in steps of $0.1\ \zeta_2$

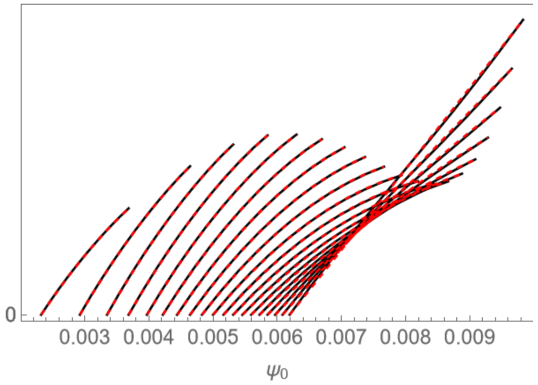
$F_{D(1)}(\zeta,\psi 0)$ (black) $f_{D(1)}(\zeta,\psi 0)$ (red dashed)



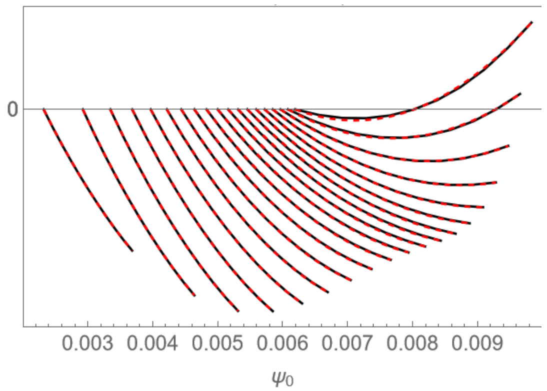
$F_{D(2)}(\zeta,\psi 0)$ (black) $f_{D(2)}(\zeta,\psi 0)$ (red dashed)



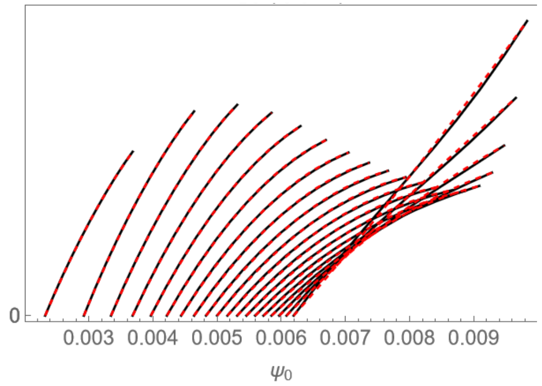
$F_{D(3)}(\zeta,\psi 0)$ (black) $f_{D(3)}(\zeta,\psi 0)$ (red dashed)



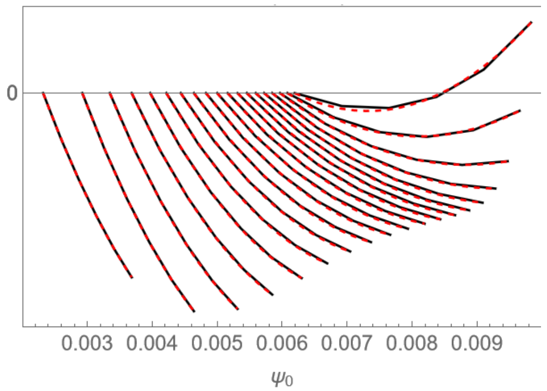
$F_{D(4)}(\zeta,\psi 0)$ (black) $f_{D(4)}(\zeta,\psi 0)$ (red dashed)



$F_{D(5)}(\zeta,\psi 0)$ (black) $f_{D(5)}(\zeta,\psi 0)$ (red dashed)



$F_{D(6)}(\zeta,\psi 0)$ (black) $f_{D(6)}(\zeta,\psi 0)$ (red dashed)



CSR chirp integral results

Flip chirp for head on left

$$H_{A+B(n)} = -\frac{e(-1)^n}{mc^3 4\pi\epsilon_0\gamma}\times$$
$$\left(\frac{4}{3}\log(4)(-1)^n\chi_n + \sum_{k=n}^{N}\sum_{i=0}^{n-1}\Xi_{n,k,i}(-1)^k\chi_k {S_2}^{k-n} + \sum_{k=n+1}^{N}\Xi_{n,k,n}(-1)^k\chi_k {S_2}^{k-n}\right)$$

$$H_{SS(n)} = -\frac{e(-1)^n}{mc^3 4\pi\epsilon_0\gamma}\times$$
$$\left(\sum_{k=0}^{N}\Lambda^0_{n,k}\Phi\rho^{\frac{1}{3}}(-1)^k\chi_k {S_2}^{k-n-\frac{1}{3}} + \sum_{m=0}^{n}\sum_{k=m+1}^{N}\sum_{i=0}^{m}\Lambda^1_{n,m,k,i}\Phi\rho^{\frac{1}{3}}(-1)^k\chi_k {S_2}^{k-n-\frac{1}{3}} - \sum_{k=n+1}^{N}\sum_{i=0}^{m}\Lambda^2_{n,k,i}(-1)^k\chi_k {S_2}^{k-n} - \sum_{i=0}^{n-1}\Lambda^2_{n,n,i}(-1)^n\chi_n\right)$$

$$H_{D(n)} \sim -\frac{e(-1)^n}{mc^3 4\pi\epsilon_0\gamma}\times$$
$$\left(\sum_{k=0}^{N-n}\sum_{m=0}^{n}(-1)^{k+n}\chi_{k+n}{S_2}^k\left[\Upsilon^0_{n,m,k} + \sum_{i=0}^{m+k-1}\Upsilon^1_{n,i,m,k} - \sum_{i=0}^{m+k-1}\sum_{l=0}^{m+k-i-2}\Upsilon^2_{n,l,i,m,k} - \sum_{i=0}^{m+k-1}\sum_{l=m+k-i+1}^{3(m+k)-1}\Upsilon^2_{n,l,i,m,k}\right]\right)$$

$$\Xi_{n,k,i} = \frac{8^{\frac{1}{3}}k\left(\frac{2}{3}\right)^{\overline{(k-1-i)}}(1-k)^{\overline{(k-1-i)}}}{(k-n)!(n-i)!\left(\frac{5}{3}\right)^{\overline{(k-1-i)}}}$$

$$\Lambda^0_{n,k} = \frac{(-1)^k 4}{n!(24)^{\frac{1}{3}}}\left(-\frac{1}{3}\right)^{\underline{(n)}}$$

$$\Lambda^1_{n,m,k,i} = \frac{3^{\frac{2}{3}}k\left(\frac{2}{3}\right)^{\underline{(n-m)}}\left(\frac{2}{3}\right)^{\overline{(k-1-i)}}(1-k)^{\overline{(k-1-i)}}}{(n-m)!(m-i)!(k-m-i)!\left(\frac{5}{3}\right)^{\overline{(k-1-i)}}}$$

$$\Lambda^2_{n,k,i} = \frac{6k(k-i)\left(\frac{2}{3}\right)^{\overline{(k-1-i)}}(1-k)^{\overline{(k-1-i)}}}{(n-i)!(k-n)!\left(\frac{5}{3}\right)^{\overline{(k-1-i)}}}$$

$$\Upsilon^0_{n,m,k} = \frac{3(11-2^{\frac{1}{3}}10+2^{\frac{2}{3}})(-1)^{k+m}(k+m)!(k+n)!}{(2^{\frac{1}{3}}4-5)(3(k+m)-1)k!m!(k+m-1)!(n-m)!}$$

$$\Upsilon^1_{n,i,m,k} = (-1)^{k+m-i}\times$$
$$\frac{2^{\frac{1}{3}}12^{m+k-1}4[2i(4+\log(27))+(k+m)(10+\log(27))]}{2^{6(m+k)-1}3^i i!m!k!}\times$$
$$\frac{(k+n)!(m+k)!(3(k+m)-1)!}{(k+m-1-i)!(n-m)!(k+m-i)!(i+2(k+m))!}$$

$$\Upsilon^2_{n,l,i,m,k} = \frac{(-1)^{l-1}}{(n-m)!}\left(\frac{3^{i+l-k-m}-1}{i+l-k-m} + \frac{3^{i+l-k-m+1}-1}{i+l-k-m+1}\right)\times$$
$$\frac{2^{\frac{1}{3}}12^{m+k-1}4(k+n)!(k+m)!(3(k+m)-1)!}{2^{6(m+k)-1}3^{i-1}i!l!m!k!(k+m-1-i)!(3(k+m)-1-l)!}$$

Coordinate transformation through bend

(***)

$$xf = \frac{Lm\,(-1+\mathrm{Cos}[\theta])}{\theta} + xi\,\mathrm{Cos}[\theta] - \frac{Lm\,(1+\eta)\,\mathrm{Cos}\left[xpi+\theta+\frac{xi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right]}{\theta\sqrt{1+\left(ypi-\frac{yi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right)^2}} + \frac{Lm\,(1+\eta)}{\theta\sqrt{1+\left(ypi-\frac{yi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right)^2}}\sqrt{1-\frac{\left(1+\left(ypi-\frac{yi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right)^2\right)}{Lm^2\,(1+\eta)^2}\left(Lm\,\mathrm{Sin}[\theta]+xi\,\theta\,\mathrm{Sin}[\theta]-\frac{Lm\,(1+\eta)\,\mathrm{Sin}\left[xpi+\theta+\frac{xi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right]}{\sqrt{1+\left(ypi-\frac{yi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right)^2}}\right)^2}$$

(***)

$$xpf = \mathrm{ArcSin}\left[\mathrm{Sin}\left[xpi+\theta+\frac{xi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right]-\frac{\mathrm{Sin}[\theta]\sqrt{1+\left(ypi-\frac{yi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right)^2}}{1+\eta}-\frac{xi\,\theta\,\mathrm{Sin}[\theta]\sqrt{1+\left(ypi-\frac{yi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right)^2}}{Lm+Lm\,\eta}\right]+\frac{(-1+\mathrm{Cos}[\theta])\,\mathrm{Tan}[\theta e2]}{1+\eta}+\frac{xi\,\theta\,\mathrm{Cos}[\theta]\,\mathrm{Tan}[\theta e2]}{Lm+Lm\,\eta}-\frac{\mathrm{Cos}\left[xpi+\theta+\frac{xi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right]\mathrm{Tan}[\theta e2]}{\sqrt{1+\left(ypi-\frac{yi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right)^2}}$$

$$+\frac{\mathrm{Tan}[\theta e2]}{\sqrt{1+\left(ypi-\frac{yi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right)^2}}\sqrt{1-\frac{\left(1+\left(ypi-\frac{yi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right)^2\right)}{Lm^2\,(1+\eta)^2}\left(Lm\,\mathrm{Sin}[\theta]+xi\,\theta\,\mathrm{Sin}[\theta]-\frac{Lm\,(1+\eta)\,\mathrm{Sin}\left[xpi+\theta+\frac{xi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right]}{\sqrt{1+\left(ypi-\frac{yi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right)^2}}\right)^2};$$

(***)

$$yf = yi+\frac{(Lm\,ypi\,(1+\eta)-yi\,\theta\,\mathrm{Tan}[\theta e1])}{Lm\,(1+\eta)\,\theta\sqrt{1+\left(ypi-\frac{yi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right)^2}}\left(Lm\,(1+\eta)\,(xpi+\theta)-Lm\,(1+\eta)\,\mathrm{ArcSin}\left[\mathrm{Sin}\left[xpi+\theta+\frac{xi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right]-\frac{\mathrm{Sin}[\theta]\sqrt{1+\left(ypi-\frac{yi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right)^2}}{1+\eta}-\frac{xi\,\theta\,\mathrm{Sin}[\theta]\sqrt{1+\left(ypi-\frac{yi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right)^2}}{Lm+Lm\,\eta}\right]+xi\,\theta\,\mathrm{Tan}[\theta e1]\right)$$

(***)

$$ypf = ypi-\frac{yi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}-\frac{yi\,\theta\,\mathrm{Tan}[\theta e2]}{Lm+Lm\,\eta}-\frac{(Lm\,ypi\,(1+\eta)-yi\,\theta\,\mathrm{Tan}[\theta e1])\,\mathrm{Tan}[\theta e2]}{Lm^2\,(1+\eta)^2\sqrt{1+\left(ypi-\frac{yi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right)^2}}\left(Lm\,(1+\eta)\,(xpi+\theta)-Lm\,(1+\eta)\,\mathrm{ArcSin}\left[\mathrm{Sin}\left[xpi+\theta+\frac{xi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right]-\frac{\mathrm{Sin}[\theta]\sqrt{1+\left(ypi-\frac{yi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right)^2}}{1+\eta}-\frac{xi\,\theta\,\mathrm{Sin}[\theta]\sqrt{1+\left(ypi-\frac{yi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right)^2}}{Lm+Lm\,\eta}\right]+xi\,\theta\,\mathrm{Tan}[\theta e1]\right)$$

(***)

$$sf = si+\frac{1}{\theta}\left(Lm\,(xpi+xpi\,\eta+\eta\,\theta)-Lm\,(1+\eta)\,\mathrm{ArcSin}\left[\mathrm{Sin}\left[xpi+\theta+\frac{xi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right]-\frac{\mathrm{Sin}[\theta]\sqrt{1+\left(ypi-\frac{yi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right)^2}}{1+\eta}-\frac{xi\,\theta\,\mathrm{Sin}[\theta]\sqrt{1+\left(ypi-\frac{yi\,\theta\,\mathrm{Tan}[\theta e1]}{Lm+Lm\,\eta}\right)^2}}{Lm+Lm\,\eta}\right]+xi\,\theta\,\mathrm{Tan}[\theta e1]\right)$$

(***)

R56 including quads

$$
\begin{aligned}
R_{56} = {} & \frac{\theta^2}{\sqrt{Kq}} \Bigg(\sqrt{Kq} \Big(\mathrm{Ld3} + \frac{5}{3}\mathrm{Lm} + (-2\,\mathrm{Ld1} - 2\,\mathrm{Ld2} - \mathrm{Ld3} - 3\,\mathrm{Lm})\,\mathrm{Cos}[\sqrt{Kq}\,\mathrm{Lq}] \Big) + \Big(-2 + Kq\,\mathrm{Lm}\Big(\mathrm{Ld2} + \frac{1}{2}\mathrm{Ld3} + \mathrm{Lm}\Big) + Kq\,\mathrm{Ld1}\,(2\,\mathrm{Ld2} + \mathrm{Ld3} + 2\,\mathrm{Lm}) \Big)\,\mathrm{Sin}[\sqrt{Kq}\,\mathrm{Lq}] + \\
& \quad \mathrm{Cosh}[\sqrt{Kq}\,\mathrm{Lq}] \Bigg(\sqrt{Kq}\Big(\mathrm{Ld1} + \mathrm{Ld2} + \frac{1}{2}\mathrm{Ld3} + \frac{3}{2}\mathrm{Lm}\Big)\big(-2 + 2\,\mathrm{Cos}[\sqrt{Kq}\,\mathrm{Lq}]\big) + \Big(1 + Kq\Big(-1\,\mathrm{Ld1}^2 + \mathrm{Ld1}\,(-2\,\mathrm{Ld2} - \mathrm{Ld3} - 3\,\mathrm{Lm}) + \Big(-\mathrm{Ld2} - \frac{1}{2}\mathrm{Ld3} - \frac{5}{4}\mathrm{Lm}\Big)\mathrm{Lm}\Big)\Big)\,\mathrm{Sin}[\sqrt{Kq}\,\mathrm{Lq}] \Bigg) + \\
& \quad \Bigg(-2 + Kq\,\mathrm{Ld1}\,(-2\,\mathrm{Ld2} - \mathrm{Ld3} - 2\,\mathrm{Lm}) + Kq\Big(-\mathrm{Ld2} - \frac{1}{2}\mathrm{Ld3} - \mathrm{Lm}\Big)\mathrm{Lm} + \Big(1 + Kq\Big(1.\grave{}\ \mathrm{Ld1}^2 + \mathrm{Lm}\Big(\mathrm{Ld2} + \frac{1}{2}\mathrm{Ld3} + \frac{5}{4}\mathrm{Lm}\Big) + \mathrm{Ld1}\,(2\,\mathrm{Ld2} + \mathrm{Ld3} + 3\,\mathrm{Lm})\Big)\Big)\,\mathrm{Cos}[\sqrt{Kq}\,\mathrm{Lq}] + \\
& \qquad Kq^{3/2}\Big(\mathrm{Ld1}^2\,(-2\,\mathrm{Ld2} - \mathrm{Ld3} - 2\,\mathrm{Lm}) + \mathrm{Ld1}\,(-2\,\mathrm{Ld2} - \mathrm{Ld3} - 2\,\mathrm{Lm})\,\mathrm{Lm} + \Big(-\frac{1}{2}\mathrm{Ld2} - \frac{1}{4}\mathrm{Ld3} - \frac{1}{2}\mathrm{Lm}\Big)\mathrm{Lm}^2 \Big)\,\mathrm{Sin}[\sqrt{Kq}\,\mathrm{Lq}] \Bigg)\,\mathrm{Sinh}[\sqrt{Kq}\,\mathrm{Lq}] \Bigg) + \\
& \frac{\theta^4}{120} \Bigg(-40\,\mathrm{Ld3} - 34\,\mathrm{Lm} - 10\,(16\,\mathrm{Ld1} + 16\,\mathrm{Ld2} - 4\,\mathrm{Ld3} + 3\,\mathrm{Lm})\,\mathrm{Cos}[\sqrt{Kq}\,\mathrm{Lq}] - 10\,(16\,\mathrm{Ld1} + 16\,\mathrm{Ld2} - 4\,\mathrm{Ld3} + 3\,\mathrm{Lm})\,\mathrm{Cosh}[\sqrt{Kq}\,\mathrm{Lq}] + \\
& \quad 10\,(16\,\mathrm{Ld1} + 16\,\mathrm{Ld2} - 4\,\mathrm{Ld3} + 3\,\mathrm{Lm})\,\mathrm{Cos}[\sqrt{Kq}\,\mathrm{Lq}]\,\mathrm{Cosh}[\sqrt{Kq}\,\mathrm{Lq}] + \frac{5\,(-32 + 8\,Kq\,\mathrm{Ld1}\,(4\,\mathrm{Ld2} - \mathrm{Ld3}) + Kq\,(6\,\mathrm{Ld2} - 9\,\mathrm{Ld3} - 10\,\mathrm{Lm})\,\mathrm{Lm})\,\mathrm{Sin}[\sqrt{Kq}\,\mathrm{Lq}]}{\sqrt{Kq}} - \\
& \quad \frac{5\,\big(-16 + Kq\,\big(16\,\mathrm{Ld1}^2 + (6\,\mathrm{Ld2} - 9\,\mathrm{Ld3} - 11\,\mathrm{Lm})\,\mathrm{Lm} + \mathrm{Ld1}\,(32\,\mathrm{Ld2} - 8\,\mathrm{Ld3} + 6\,\mathrm{Lm})\big)\big)\,\mathrm{Cosh}[\sqrt{Kq}\,\mathrm{Lq}]\,\mathrm{Sin}[\sqrt{Kq}\,\mathrm{Lq}]}{\sqrt{Kq}} - \\
& \quad \frac{5\,(32 + 8\,Kq\,\mathrm{Ld1}\,(4\,\mathrm{Ld2} - \mathrm{Ld3}) + Kq\,(6\,\mathrm{Ld2} - 9\,\mathrm{Ld3} - 10\,\mathrm{Lm})\,\mathrm{Lm})\,\mathrm{Sinh}[\sqrt{Kq}\,\mathrm{Lq}]}{\sqrt{Kq}} + \\
& \quad \frac{5\,\big(16 + Kq\,\big(16\,\mathrm{Ld1}^2 + (6\,\mathrm{Ld2} - 9\,\mathrm{Ld3} - 11\,\mathrm{Lm})\,\mathrm{Lm} + \mathrm{Ld1}\,(32\,\mathrm{Ld2} - 8\,\mathrm{Ld3} + 6\,\mathrm{Lm})\big)\big)\,\mathrm{Cos}[\sqrt{Kq}\,\mathrm{Lq}]\,\mathrm{Sinh}[\sqrt{Kq}\,\mathrm{Lq}]}{\sqrt{Kq}} - 160\,Kq\,\mathrm{Ld1}^2\,\mathrm{Ld2}\,\mathrm{Sin}[\sqrt{Kq}\,\mathrm{Lq}]\,\mathrm{Sinh}[\sqrt{Kq}\,\mathrm{Lq}] + \\
& \quad 40\,Kq\,\mathrm{Ld1}^2\,\mathrm{Ld3}\,\mathrm{Sin}[\sqrt{Kq}\,\mathrm{Lq}]\,\mathrm{Sinh}[\sqrt{Kq}\,\mathrm{Lq}] + 5\,Kq\,\mathrm{Lm}^2\,(2\,\mathrm{Ld2} + 7\,\mathrm{Ld3} + 10\,\mathrm{Lm})\,\mathrm{Sin}[\sqrt{Kq}\,\mathrm{Lq}]\,\mathrm{Sinh}[\sqrt{Kq}\,\mathrm{Lq}] + 10\,Kq\,\mathrm{Ld1}\,\mathrm{Lm}\,(-6\,\mathrm{Ld2} + 9\,\mathrm{Ld3} + 10\,\mathrm{Lm})\,\mathrm{Sin}[\sqrt{Kq}\,\mathrm{Lq}]\,\mathrm{Sinh}[\sqrt{Kq}\,\mathrm{Lq}] \Bigg)
\end{aligned}
$$

LCLS-II HE 4 kA emittance *corrected*

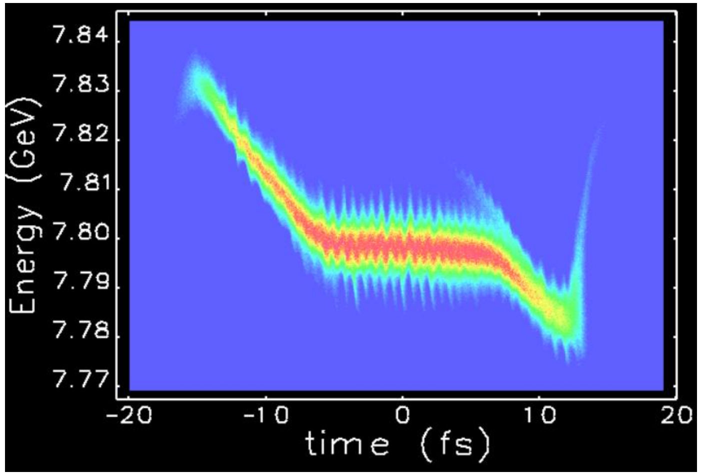
Changes:

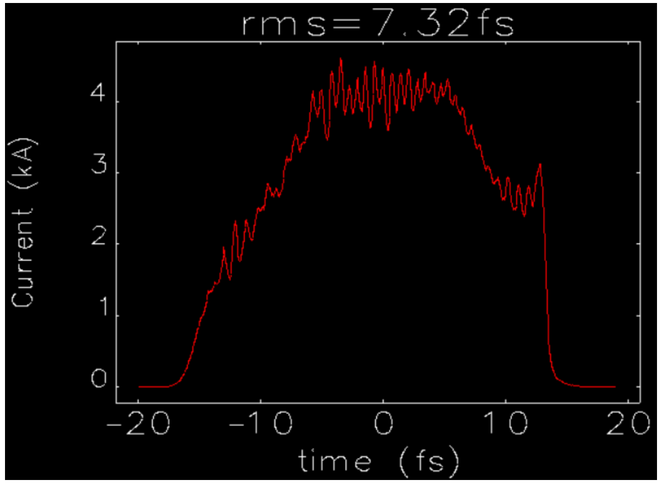
BC2 bend = 0.0457434

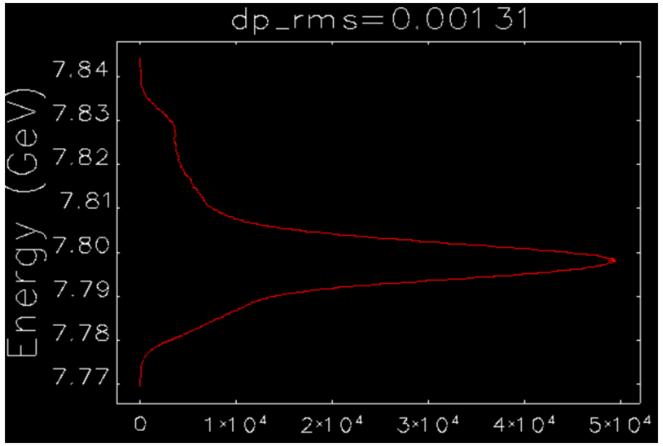
L2 voltage = 16.6 MV L2 phase = -38.1324

CQ11 = 0.2 CQ12 = -0.2

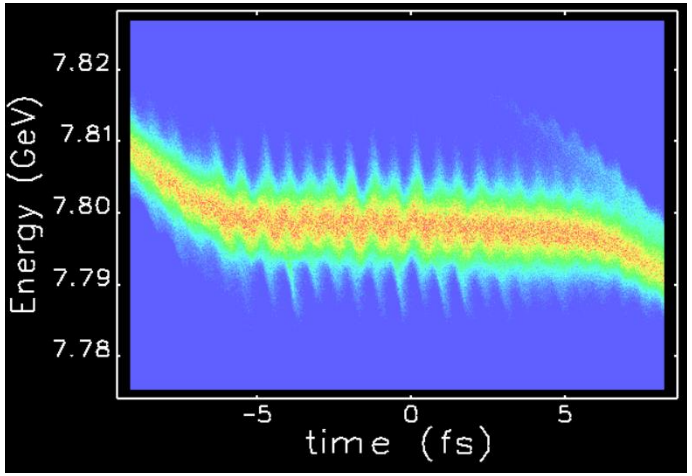
CQ21 = 0.1 CQ22 = -0.8

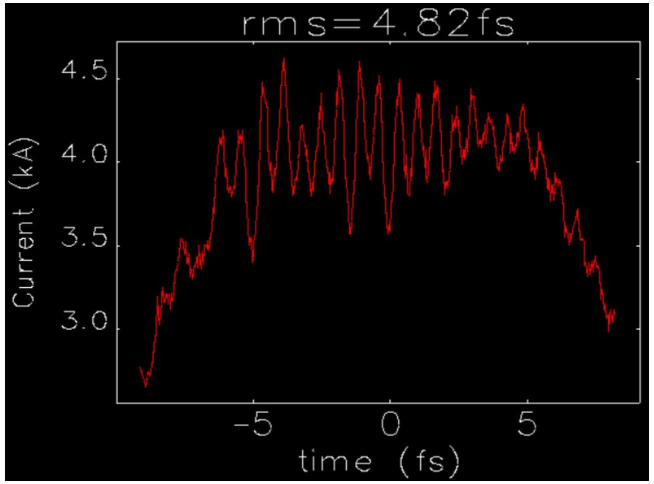


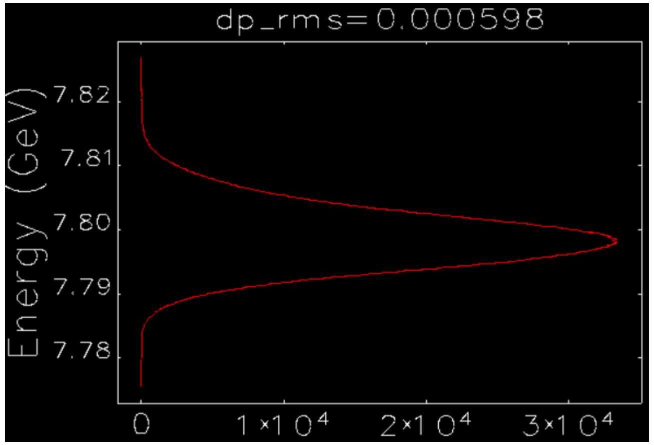




ε = 0.55 mm-mrad







LCLS-II HE 6 kA emittance *corrected*

Changes:

BC2 bend = 0.0454634

L2 voltage = 16.7 MV

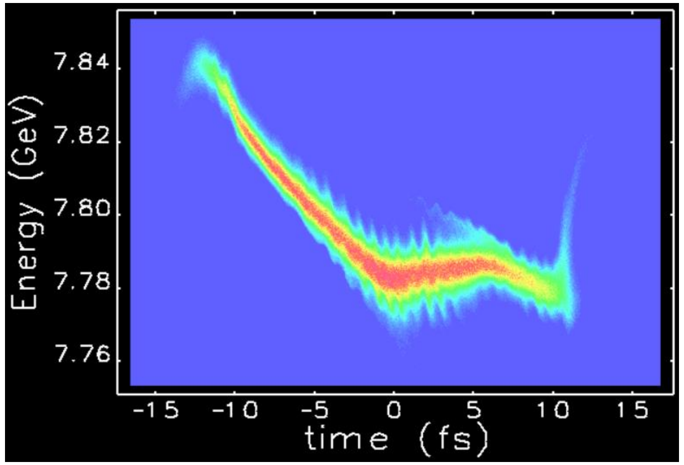
L2 phase = -38.6324

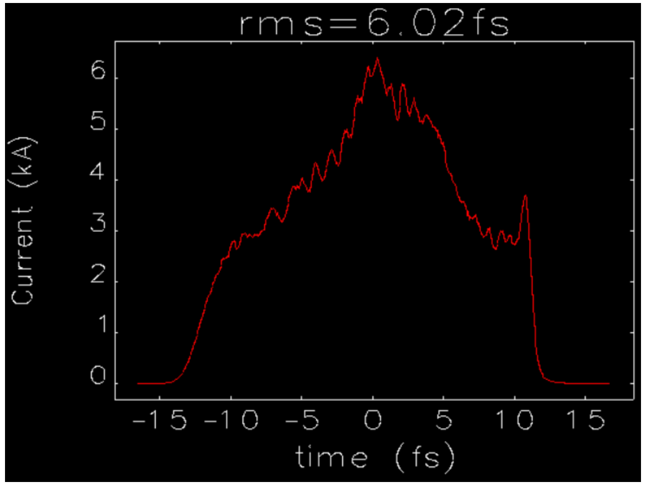
CQ11 = 0.2

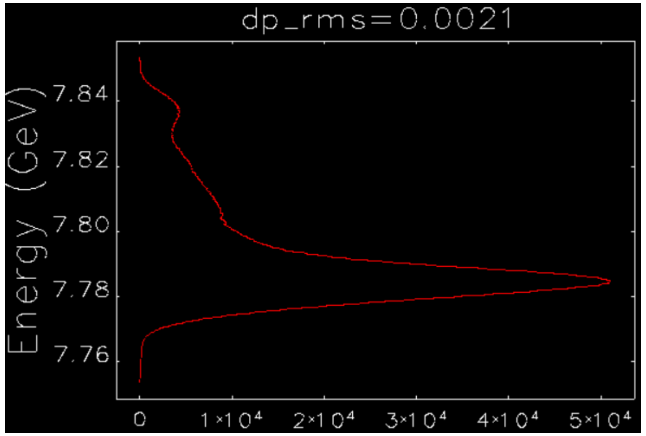
CQ12 = -0.2

CQ21 = 0.1

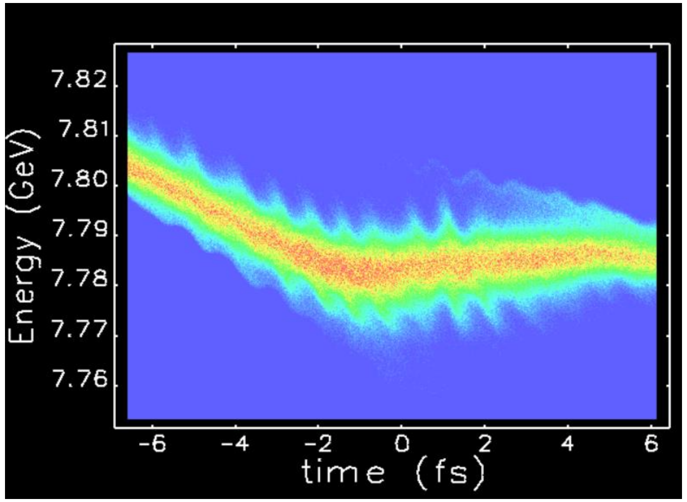
CQ22 = -0.8

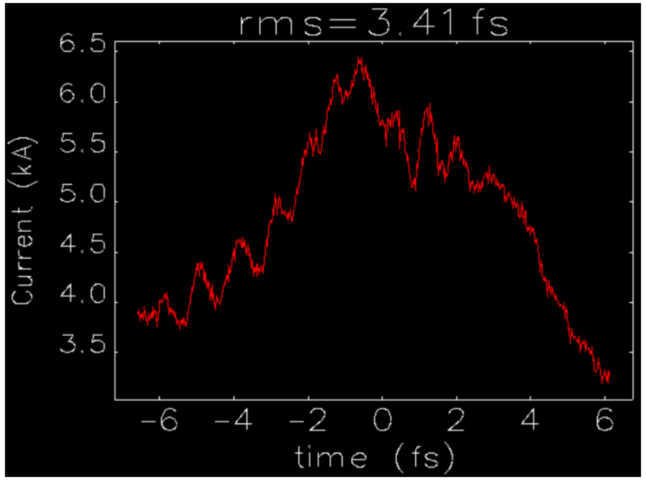


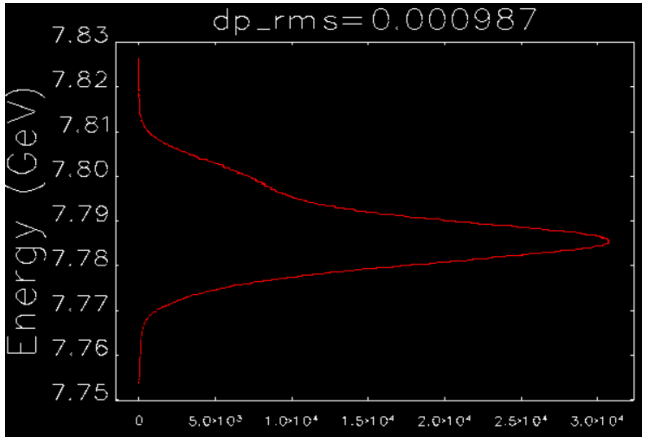




ε = 0.6 mm-mrad







LCLS-II HE 7 kA emittance *corrected*

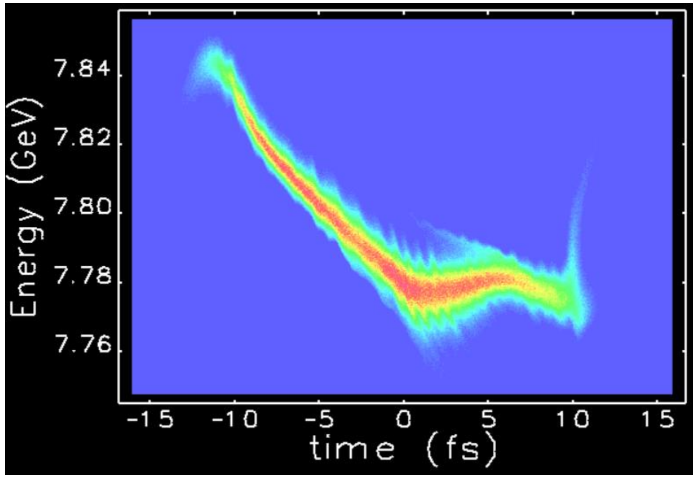
Changes:

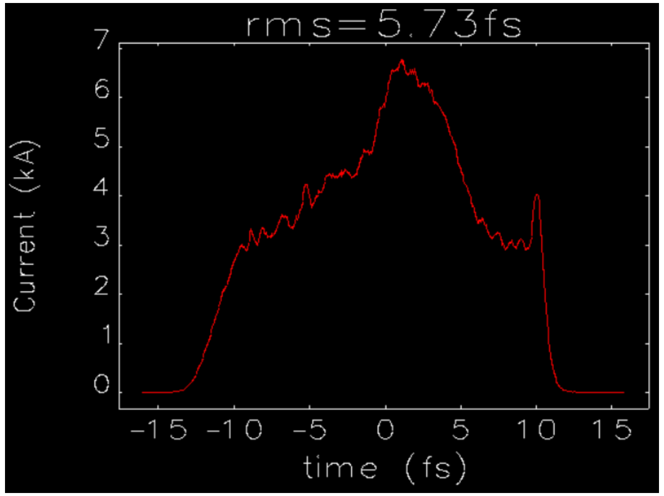
BC2 bend = 0.0454734

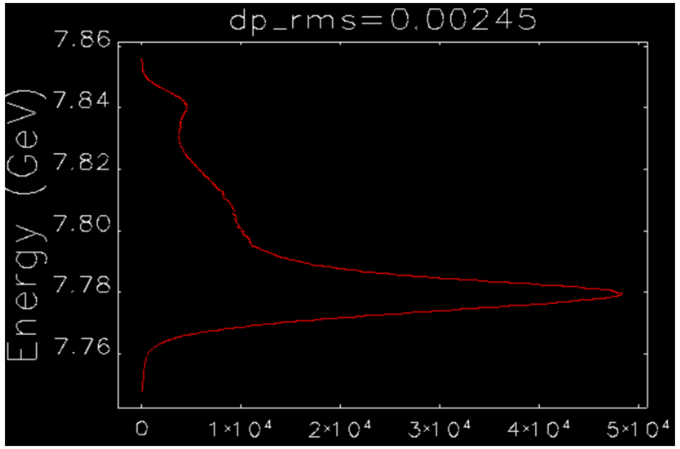
L2 voltage = 16.7 MV   L2 phase = -38.6324

CQ11 = 0.2   CQ12 = -0.2

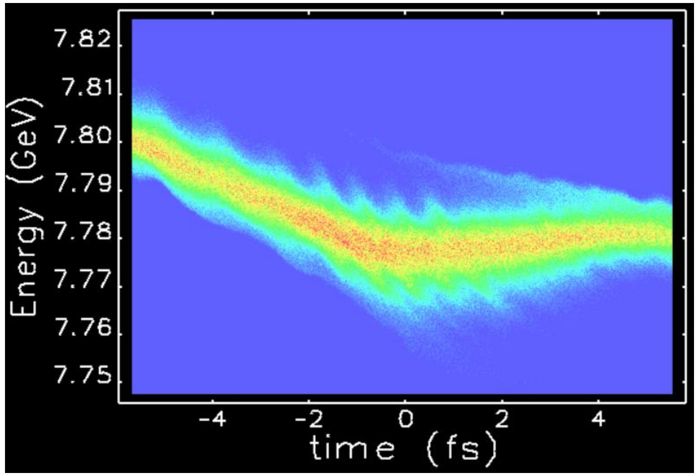
CQ21 = 0.1   CQ22 = -0.8

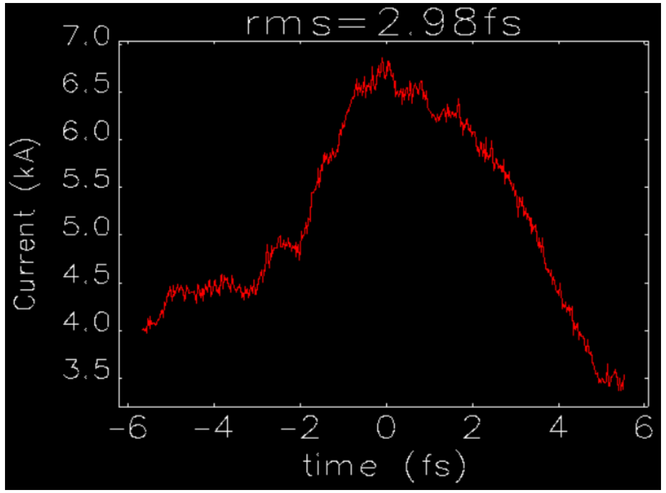


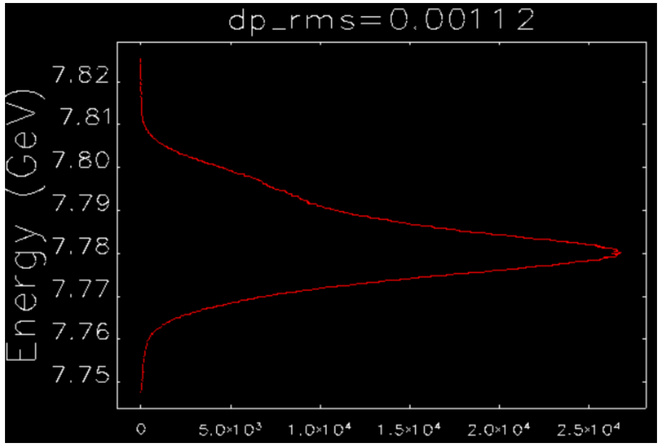




ε = 0.7 mm-mrad

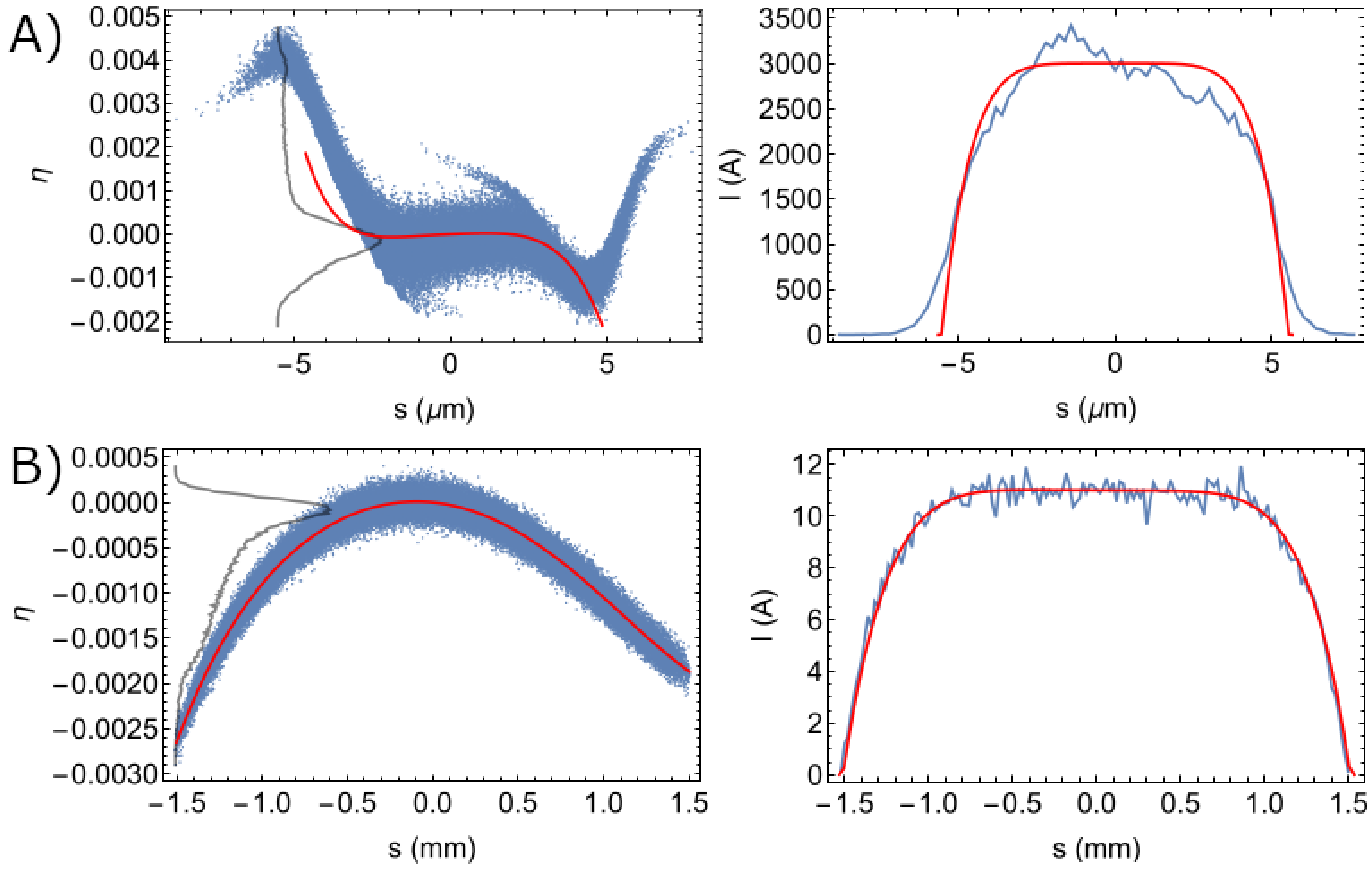






LCLS-HE backtracking

- 3 kA example for LCLS-HE
- Matches well with forward tracking in Elegant
- Evolution of current profile downstream of L4 less significant for high current
- Energy modulation of bunch head still greater in Elegant
- Have been assuming 0.37 mm-mrad should switch to 0.1 mm-mrad

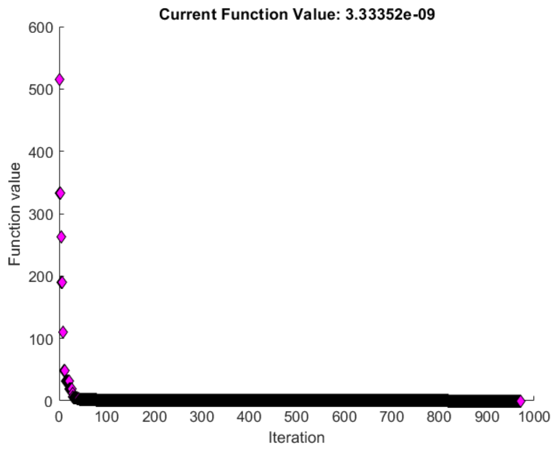


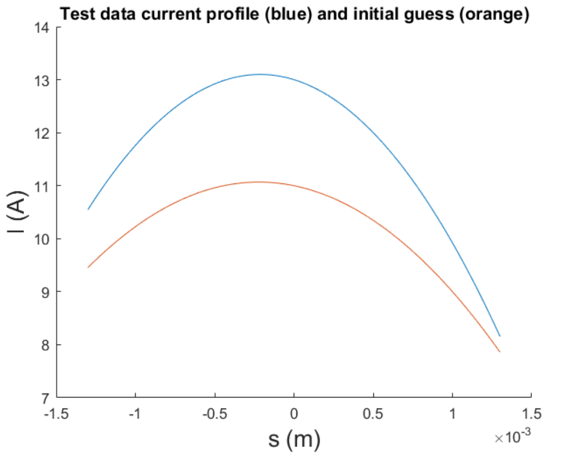
Using fast forward tracking as a diagnostic

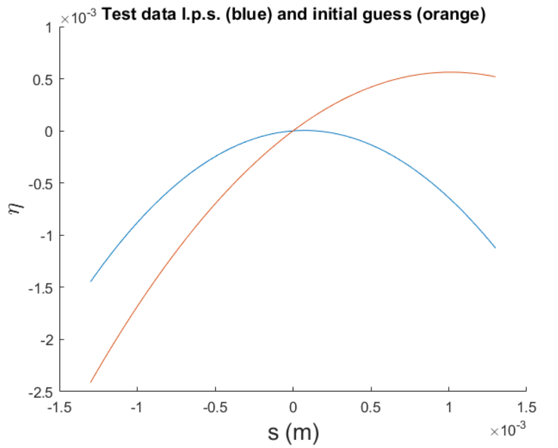
- Use CSR signal from last dipole in BC2 assuming current profile at BC2 exit
- Calculate CSR energy based on approximate analytical expression for energy loss
- Vary linac parameters V_{L1}, ϕ_{L1}, ϕ_{L1h}, ϕ_{L2}, I_{BC1}, I_{BC2} individually and look at changing energy of CSR signal. Aim is to avoid degeneracy.
  - $E_{CSR}(V_{L1})$: $15.5 < V_{L1} < 16$ MV
  - $E_{CSR}(\phi_{L1})$: $-26 < \phi_{L1} < -23$ degrees
  - $E_{CSR}(\phi_{L1h})$: $-178 < \phi_{L1h} < -173$ degrees
  - $E_{CSR}(\phi_2)$: $-32 < \phi_2 < -28$ degrees
  - $E_{CSR}(I_{BC1})$: $35 < I_{BC1} < 45$ A
  - $E_{CSR}(I_{BC2})$: $500 < I_{BC2} < 700$ A
- In example, assume beam at injector can be described by polynomial coefficients I_0, I_1, I_2, h_1, h_2, h_3
- Use fast forward tracking, varying same linac parameters to find CSR signals as a function of initial phase space and current profile, call this $F_{CSR}(X)$
- Objective function:

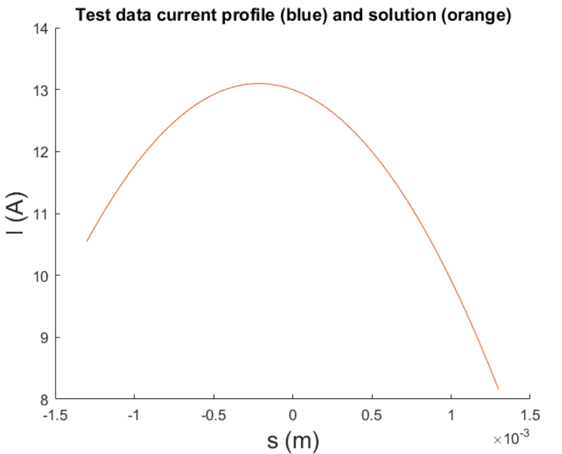
  $F_{obj}(I_0, I_1, I_2, h_1, h_2, h_3) = \left([F_{CSR}(V_{L1})-E_{CSR}(V_{L1})]^2 + [F_{CSR}(\phi_{L1})-E_{CSR}(\phi_{L1})]^2 + [F_{CSR}(\phi_{L1h})-E_{CSR}(\phi_{L1h})]^2 + [F_{CSR}(\phi_{L2})-E_{CSR}(\phi_{L2})]^2 + [F_{CSR}(I_{BC1})-E_{CSR}(I_{BC1})]^2 + [F_{CSR}(I_{BC2})-E_{CSR}(I_{BC2})]^2\right)$^0.5
- Find initial beam coefficients that minimize F_{obj}, difference in data and "analytical" CSR signals

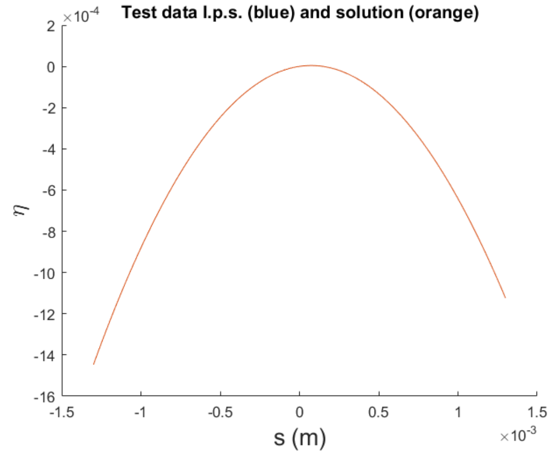
Difference between test data and CSR output from forward tracking minimized by fminsearch







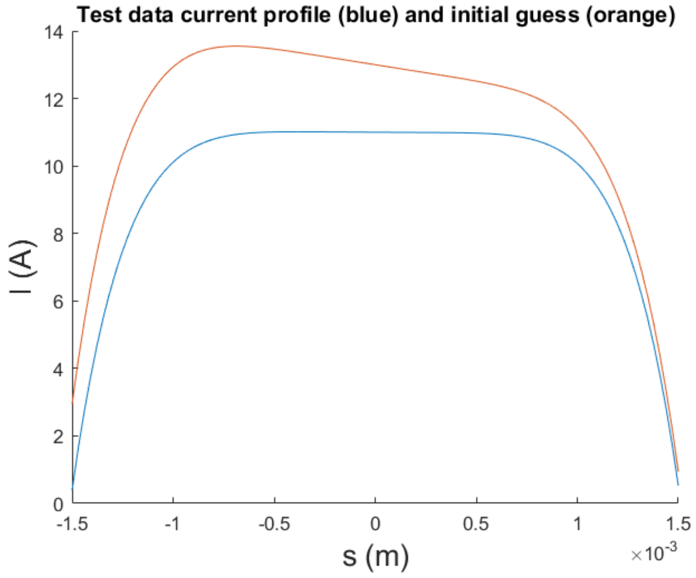


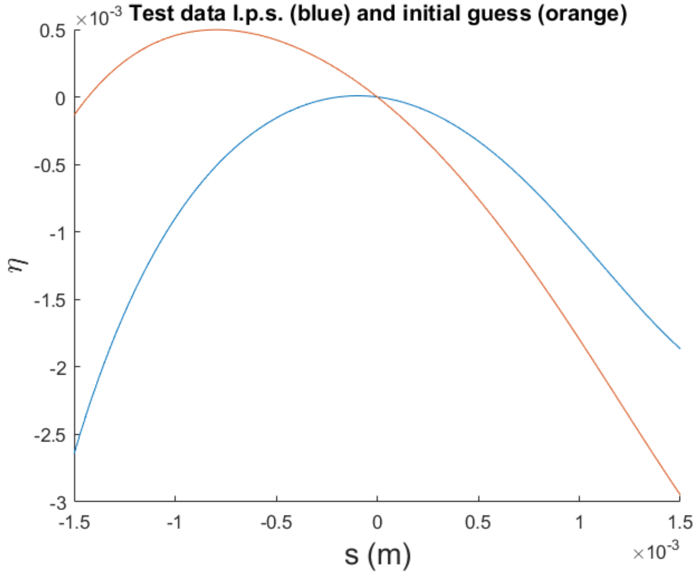


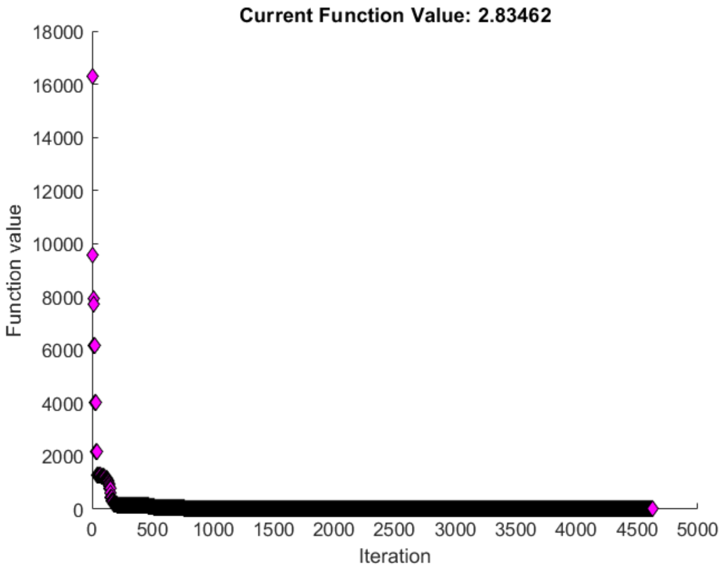
Another example up to 6^{th} order

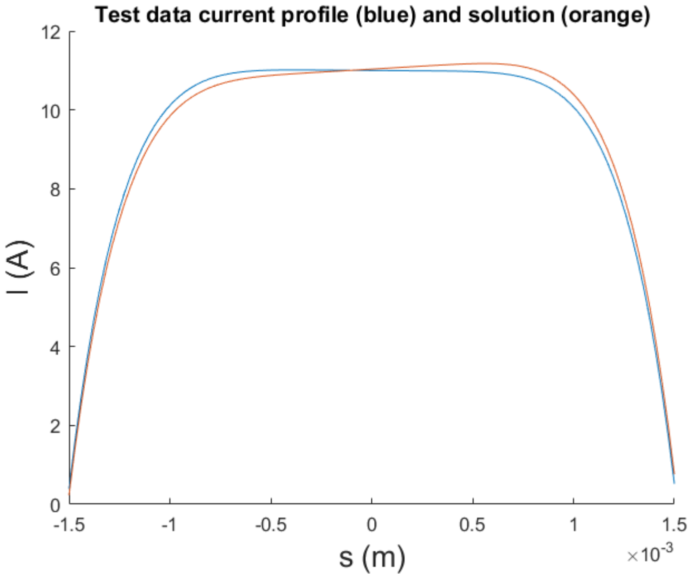
Another example: assume beam at injector can be described by polynomial coefficients I_0, I_1, I_2, I_3, I_4, I_5, I_6, h_1, h_2, h_3, h_4, h_5, h_6

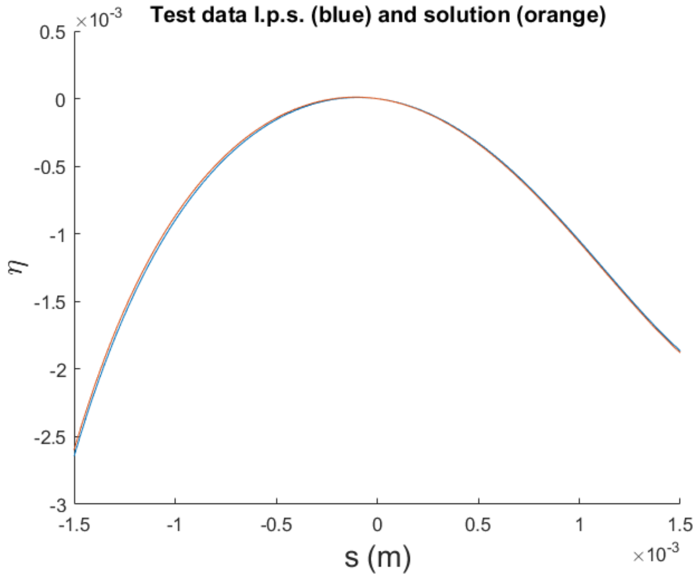
Not quite as effective but still potentially useful. Can perhaps be improved by scanning additional linac parameters











Including octupoles in backtracking studies

- Include octupoles in BC1 and BC2 to modify the U566

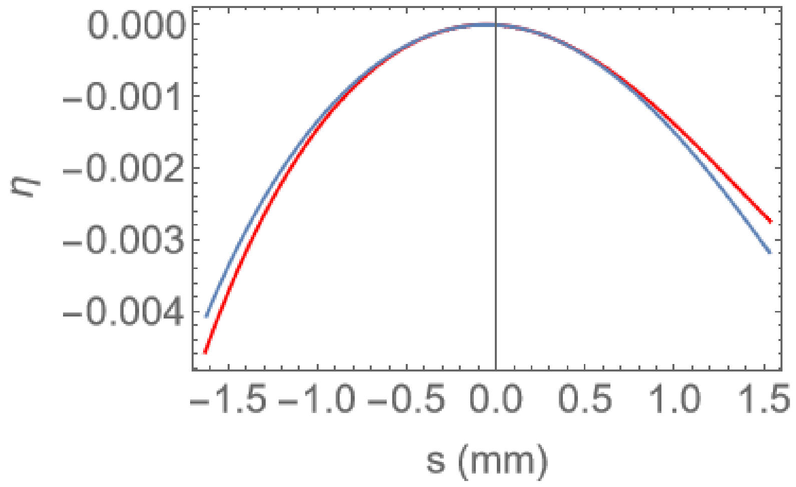
$$U_{5666} \approx -\frac{1}{6} K_3 L \theta^4 (l_b + l_d)^4 + 2R_{56}$$

- Adjust octupole strengths to compensate for emittance growth following Phys. Rev. Accel. Beams **23**, 112802

$$\alpha_K \equiv \frac{K_3^{(2)} L_2}{K_3^{(1)} L_1} = \left(\frac{(l_{d1} + l_{b1})(l_{d2} + \frac{2}{3} l_{b2})}{(l_{d1} + \frac{2}{3} l_{b1})(l_{d2} + l_{b2})} \right)^3 \left(\frac{\theta_2}{\theta_1} \right)^3 \times \left(\frac{C_2(C_1 - 1)}{C_2 - C_1} \right)^3 \sqrt{\frac{\beta_1 E_1}{\beta_2 E_2}}$$

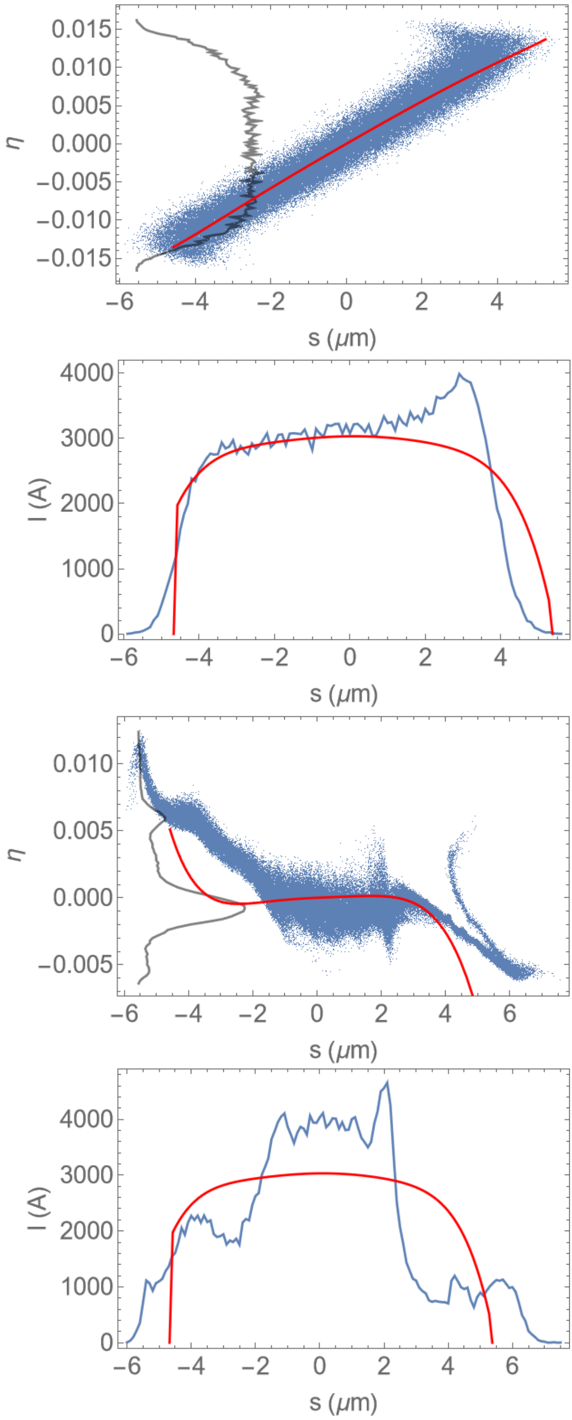
- Results in a small change in the third order chirp in the initial phase space
- May be useful for finding a solution that can be realized by the injector

Backtracking example 3 kA case: Longitudinal phase space at the laser heater exit with octupoles turned off (blue) and octupoles split between BC1 and BC2 to produce a total U5666 = 6 m

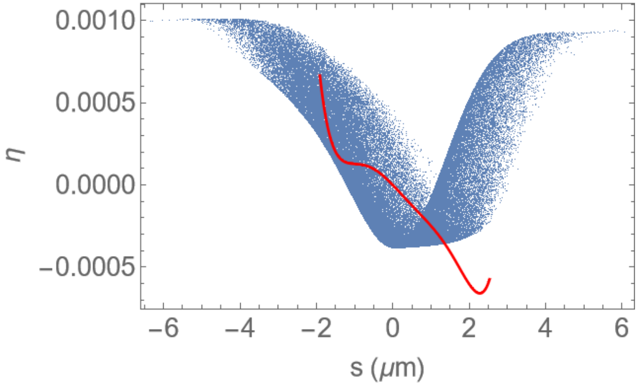


CSR in BC2 (3 kA case) problems

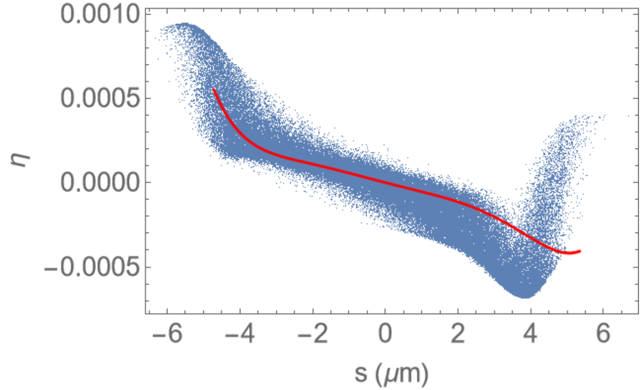
- For high current case evolution of current profile in 3rd dipole is significant (50 A to 7 kA)
- Beam over-compresses in 3rd dipole (multiple roots)
- Beam near full compression at 4th dipole entrance (expansion of denominator in current profile transformation not as good. Current can go negative.)



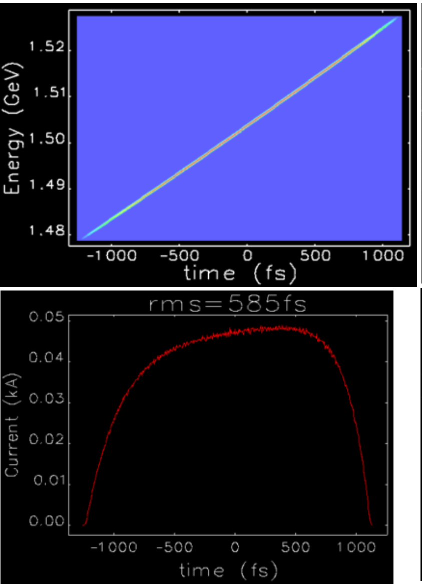
CSR chirp 3rd dipole



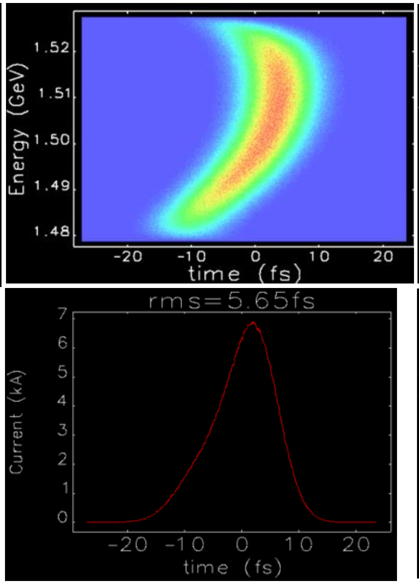
CSR chirp 4th dipole



Entrance of BC2 3rd dipole



Exit of BC2 3rd dipole



Entrance of BC2 4th dipole