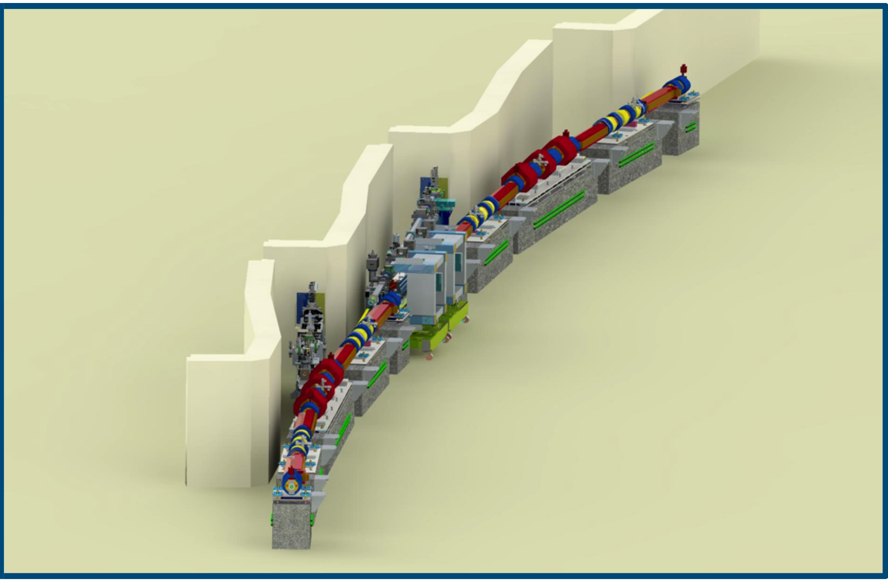
Longitudinal multi-bunch instabilities including higher-harmonic (bunch lengthening) cavities



Ryan Lindberg
Physicist
Accelerator Operations and Physics Group

Future Light Source Workshop 2018
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Outline

1. Introduction and motivation
2. Brief sketch of the theoretical derivation of our multi-bunch dispersion relation for arbitrary potentials
3. Two examples
    1. Linear oscillator: reduction to usual theory
    2. Weakly nonlinear oscillator: Landau damping
4. Dispersion relation for higher-harmonic cavity (HHC) set to ideal bunch-lengthening (quartic oscillator)
5. Comparison of HHC theory to simulation for instabilities driven by one higher-order mode (HOM) of the APS-U rf cavity
6. Conclusions and outlook

Motivation and background

- Many next-generation storage rings will employ two rf systems

Main rf cavities: Accelerate and focus bunch	+	**Higher harmonic cavity (HHC):** Lengthen bunch

  - Increase the Touschek lifetime of the ultra-low emittance beam
  - Decrease single-bunch wakefield effects (rf heating, microwave instability, TMCI)

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- APS-U plans to use a passive HHC that is tuned to maximize lifetime
  - Longitudinal potential goes from harmonic ($\sim z^2$) to approximately quartic ($\sim z^4$)
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  - Low mean synchrotron frequency increases growth rates
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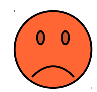
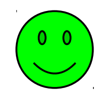
  - Low mean synchrotron frequency increases growth rates 
  - Large synchrotron frequency spread introduces Landau damping 

- We will present a theory that quantitatively addresses these competing interests
  - Growth rates are related to the matrix theory introduced by Thompson and Ruth [1]
  - Theory includes a dispersion integral which gives Landau damping similar to that predicted in [2-4], but in a fully self-consistent manner.

[1] K.A. Thompson and R.D. Ruth. “Transverse and longitudinal coupled bunch instabilities in trains of closely spaced bunches,” Proc. of 1989 PAC, pp 792; SLAC-PUB-4872 (1989).
[2] J.M. Wang. “Longitudinal symmetric coupled bunch modes,” Lab. Rep. No. BNL 51302 (1980).
[3] M.S. Zisman, S. Chattopadhyay, and J.J. Bisognano. “ZAP user's manual,” Lab. Rep. No. LBL-21270 (1986).
[4] R.A. Bosch, K.J. Kleman, and J.J. Bisognano. “Robinson instabilities with a higher-harmonic cavity,” PRSTAB **4**, 074401 (2001).

Whirlwind derivation of the theory from the linear, multi-bunch Vlasov equation

- We index each bunch by n $(0 \leq n \leq N_b - 1)$, transform to action angle variables $(\Phi, \mathcal{I})$, and linearize the set of Vlasov equations via $F_n(\Phi, \mathcal{I}; s) = \bar{F}(\mathcal{I}) + e^{-i\Omega s/c} f_n(\Phi, \mathcal{I})$ with $|f_n| \ll \bar{F}$:

$$\left[-i\Omega + \omega(\mathcal{I})\frac{\partial}{\partial \Phi}\right] f_n(\Phi, \mathcal{I}) = \frac{\partial \bar{F}}{\partial \mathcal{I}} \sum_{j=0}^{N_b-1} \frac{e^2 N_{e,j}}{\gamma m c^2 T_0} \sum_{\ell=0}^{\infty} \int d\hat{\Phi} d\hat{\mathcal{I}}\; e^{i\ell\Omega T_0} f_j(\hat{\Phi}, \hat{\mathcal{I}}) \frac{\partial}{\partial \Phi} W_{\parallel}[z - (\hat{z} + \ell c T_0 + L_{n,j})]$$

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[5] R.D. Ruth and J.M. Wang. "Vertical fast blow-up in a single bunch, "IEEE Trans. Nucl. Science **28**, 2405 (1981).
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  4. Define the centroid $\langle z\rangle_n = \int d\Phi\, d\mathcal{I}\; f_n(\Phi,\mathcal{I}) z(\Phi,\mathcal{I})$ and approximate $e^{i\ell\Omega T_0} \approx e^{i\ell\langle\Omega\rangle T_0}$ on the right-hand side.

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Long-range wakefield coupling matrix as given in, e.g., [1,7]

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- We diagonalize the coupling matrix M by finding a matrix U such that $(\mathrm{UMU}^{-1})_{n,j} = \lambda_n\delta_{n,j}$, with $\tau_n = \sum_j \mathsf{U}_{n,j}\langle z\rangle_j$ the coupled bunch mode and λ_n the coupled bunch eigenvalue.

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- Then, each mode n satisfies the dispersion relation

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Coupled bunch dispersion relation

- The coupled set of equations for the bunch centroids is

$$\langle z\rangle_n = \frac{2\sigma_t}{\alpha_c\sigma_\delta}\sum_{j=0}^{N_b-1}\boxed{\frac{e^2\sigma_t N_{e,j}}{2\gamma mcT_0\sigma_\delta}\sum_{\ell=0}^{\infty}e^{i\ell\langle\Omega\rangle T_0}\left.\frac{dW_\parallel}{d\xi}\right|_{\xi=-(\ell cT_0+L_{n,j})}}\langle z\rangle_j\int_0^\infty d\mathcal{I}\;4\pi\bar{F}(\mathcal{I})\left|z_1(\mathcal{I})/\sigma_z\right|^2\frac{\omega(\mathcal{I})^2}{\Omega^2-\omega(\mathcal{I})^2}$$

Long-range wakefield coupling matrix as given in, e.g., [1,7]

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- Then, each mode n satisfies the dispersion relation

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Coupled-bunch matrix eigenvalue:
Can be computed from the wakefield matrix of the point-charge/SHO model that has the same revolution period T_0, momentum compaction α_c, energy spread σ_δ, and bunch length σ_t

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Landau damping part:
Integration over the action that is singular when the single-particle oscillation frequency $\omega(I)$ equals that of the collective motion Ω

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- For the simple harmonic oscillator, we have simple expressions for everything:

$$\mathcal{H}_0(\mathcal{I}) = \frac{\omega_s}{c}\mathcal{I} \quad \Rightarrow \quad \omega(\mathcal{I}) = \omega_s = \frac{\alpha_c \sigma_\delta}{\sigma_t} \qquad z_1(\mathcal{I}) = \sigma_z \sqrt{\frac{\mathcal{I}}{2\sigma_z \sigma_\delta}}, \quad \bar{F}(\mathcal{I}) = \frac{e^{-\mathcal{I}/\sigma_z \sigma_\delta}}{2\pi \sigma_z \sigma_\delta}$$

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- In general, consistency requires the approximation $\Omega^2 - \omega_s{}^2 \approx 2\omega_s\,(\Omega - \omega_s)$, because we used the approximation $\Omega \approx \langle\Omega\rangle$ in the matrix calculation for λ_n.
- However, we will see that setting $\Omega^2 - \omega_s{}^2 \approx 2\omega_s\,(\Omega - \omega_s)$ is not required for the APS long-range wakefield because of the parameters of the APS higher-order modes (HOMs).

Example 2: Weakly nonlinear oscillator

- We add a small nonlinearity to the simple harmonic potential of the single rf system:

$$\mathcal{H}_0(\mathcal{I}) = \frac{\omega_s}{c}\mathcal{I}\left(1 + b\frac{\mathcal{I}}{2\langle\mathcal{I}\rangle}\right) \quad \Rightarrow \quad \omega(\mathcal{I}) = \omega_s\left(1 + b\frac{\mathcal{I}}{\langle\mathcal{I}\rangle}\right) \quad \text{with } |b| << 1$$

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- Landau [8] showed that this dispersion relation only applies when $\text{Im}(\Omega) > 0$
- When $\text{Im}(\Omega) < 0$ we must analytically continue the dispersion relation by deforming the contour to be always below the poles $\rightarrow$ Landau damping

[8] L. Landau. "On the vibrations of the electronic plasma," J. Physics (USSR) **10**, 25 (1946).

Multi-bunch dispersion relation for the weakly nonlinear oscillator

- Once the Landau contour is specified, the integration can be done analytically

$$1 = -\frac{\lambda_n}{b\omega_s}\left\{1 - \zeta e^{-\zeta}\left[\mathrm{Ei}(\zeta) - i\pi\right]\right\}$$

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Strength of instability driven by a HOM

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HOM frequency difference from a revolution harmonic scaled by the instability resonance width

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HOM-driven growth rates for the weakly nonlinear oscillator

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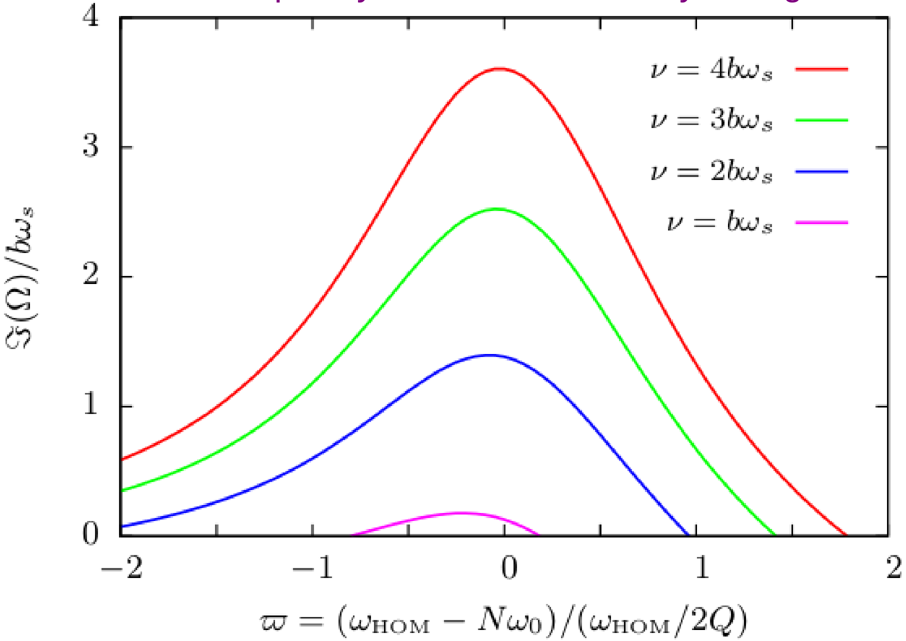

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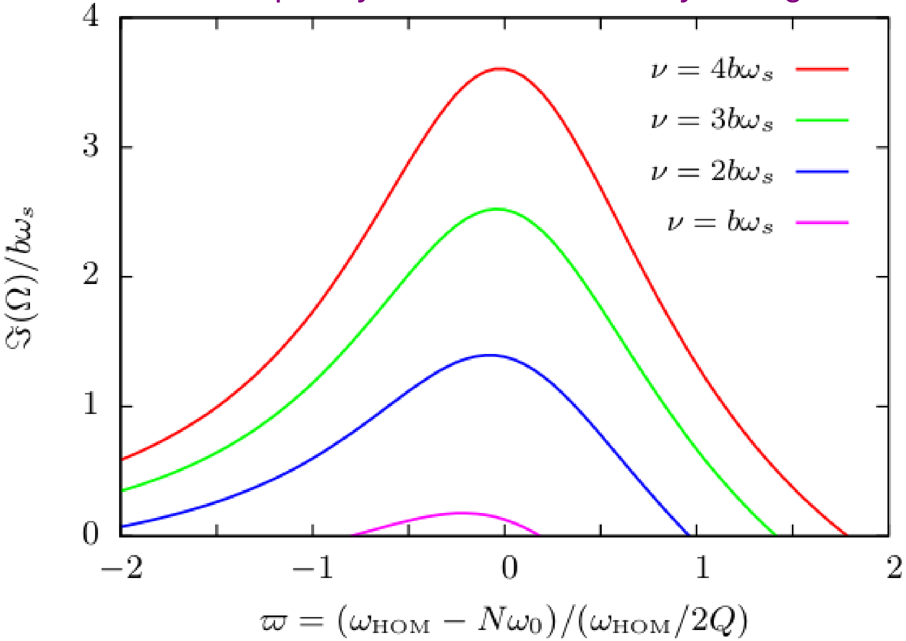
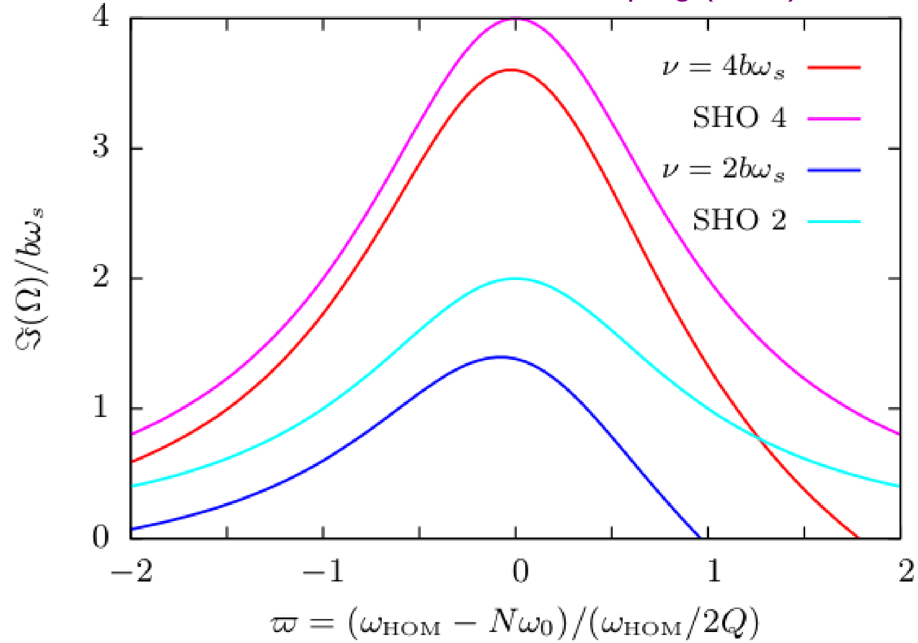


Comparison of weakly nonlinear instability growth rates to those with no Landau damping (SHO)



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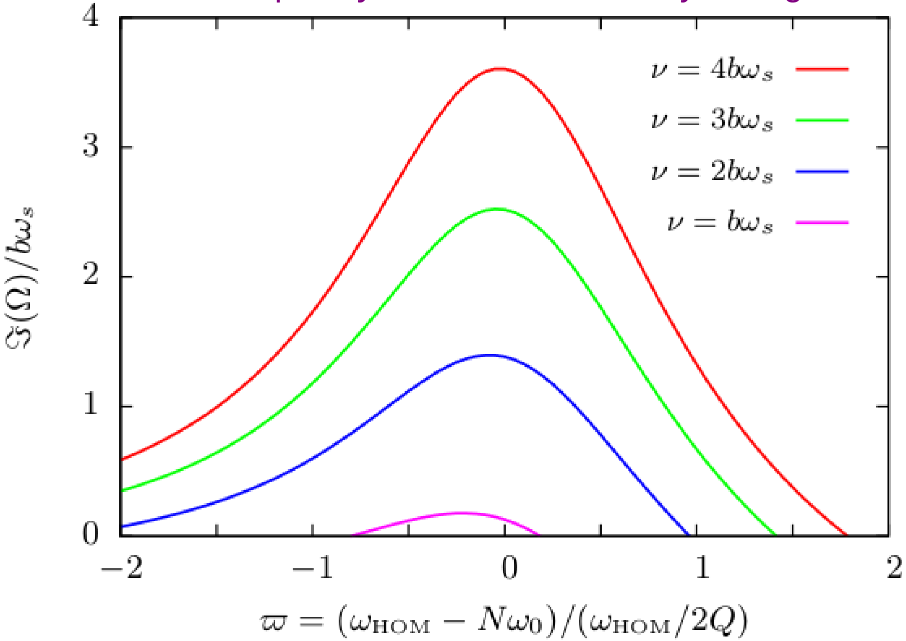

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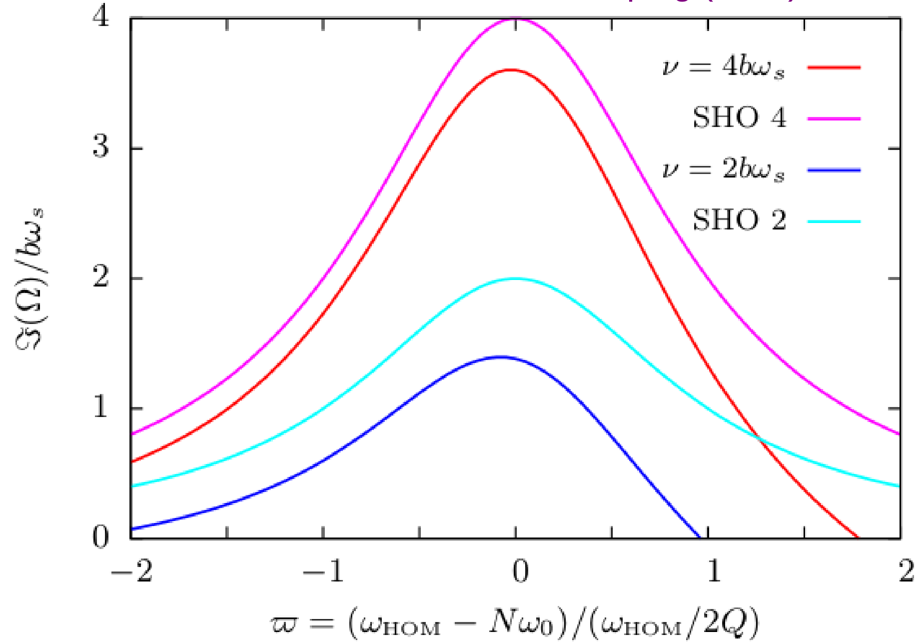

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- This implies that simply subtracting a single Landau damping rate from the instability growth rate will only approximately predict stability

Double rf system with HHC tuned to produce quartic potential

- We assume that the bunch-lengthening system is tuned to produce the quartic potential

$$\mathcal{H}_0(z, p_z) = \frac{\alpha_c}{2} p_z^2 + \frac{\kappa}{4} z^4 \quad \rightarrow \quad \mathcal{H}_0(\mathcal{I}) \propto \mathcal{I}^{4/3}$$

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- The oscillation frequency scales with the (amplitude)$^{1/3}$, i.e., $\omega(\mathcal{I}) \propto \mathcal{I}^{1/3}$
- The longitudinal position $z(\Phi, \mathcal{I}) \propto \mathcal{I}^{1/3} \mathrm{cn}\left(\frac{2K(1/2)}{\pi}\Phi; 1/2\right) \approx 0.96\, \mathcal{I}^{1/3} \cos \Phi.$

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$$\mathcal{H}_0(z,p_z) = \frac{\alpha_c}{2}p_z^2 + \frac{\kappa}{4}z^4 \quad \rightarrow \quad \mathcal{H}_0(\mathcal{I}) \propto \mathcal{I}^{4/3}$$

- The oscillation frequency scales with the (amplitude)$^{1/3}$, i.e., $\omega(\mathcal{I}) \propto \mathcal{I}^{1/3}$
- The longitudinal position $z(\Phi,\mathcal{I}) \propto \mathcal{I}^{1/3}\mathrm{cn}\left(\frac{2K(1/2)}{\pi}\Phi; 1/2\right) \approx 0.96\,\mathcal{I}^{1/3}\cos\Phi.$
- After a fair bit of algebra, we can write out the dispersion relation as

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Keeps track of Landau contour, with

$$B(\zeta) = \begin{cases} 0 & \text{if } \Im(\zeta) > 0 \\ \zeta^5 e^{-\zeta^4} & \text{if } \Im(\zeta) = 0 \\ 2\zeta^5 e^{-\zeta^4} & \text{if } \Im(\zeta) < 0 \end{cases}$$

HOMs in the APS-U rf cavities

- The APS-U plans to retain between 8 and 12 main rf cavities of the present-day APS
- Five HOMs have been identified that may drive longitudinal coupled bunch modes

f_{HOM} (MHz)	R_s (kΩ)	$Q/10^3$	$1/T_0$ (kHz)	$f_{HOM}/2Q$ (kHz)	No HHC f_s (kHz)	HHC $\langle f_s \rangle$ (kHz)
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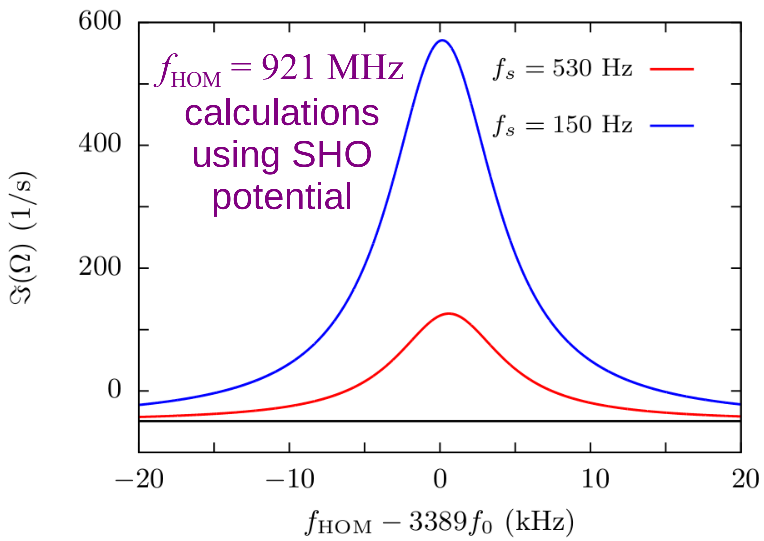

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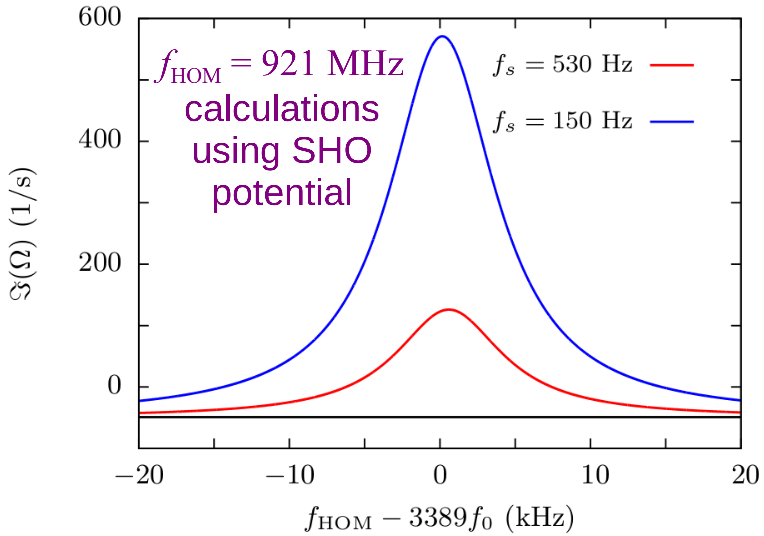

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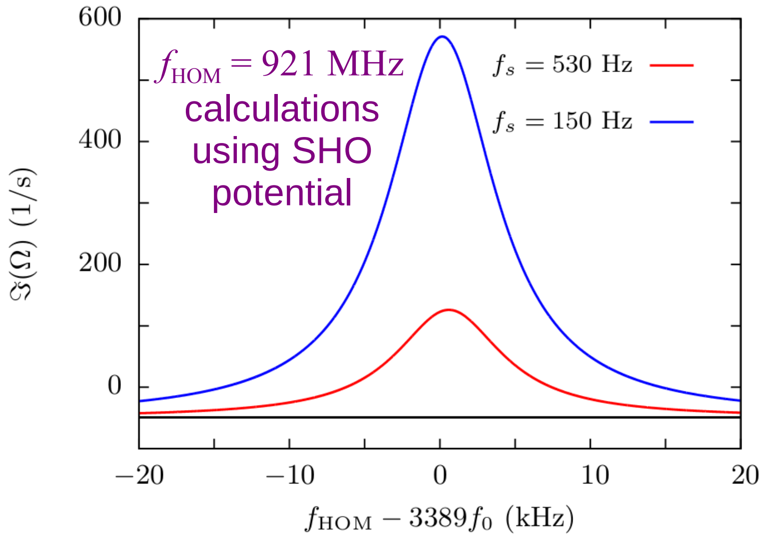

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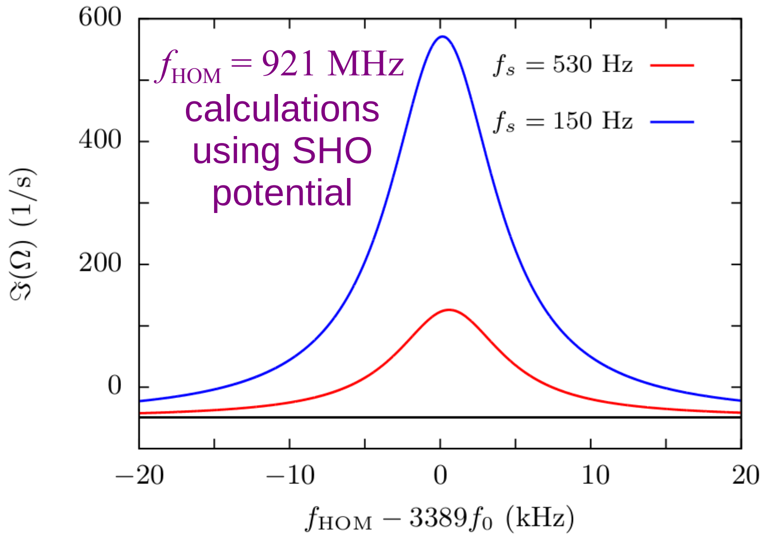

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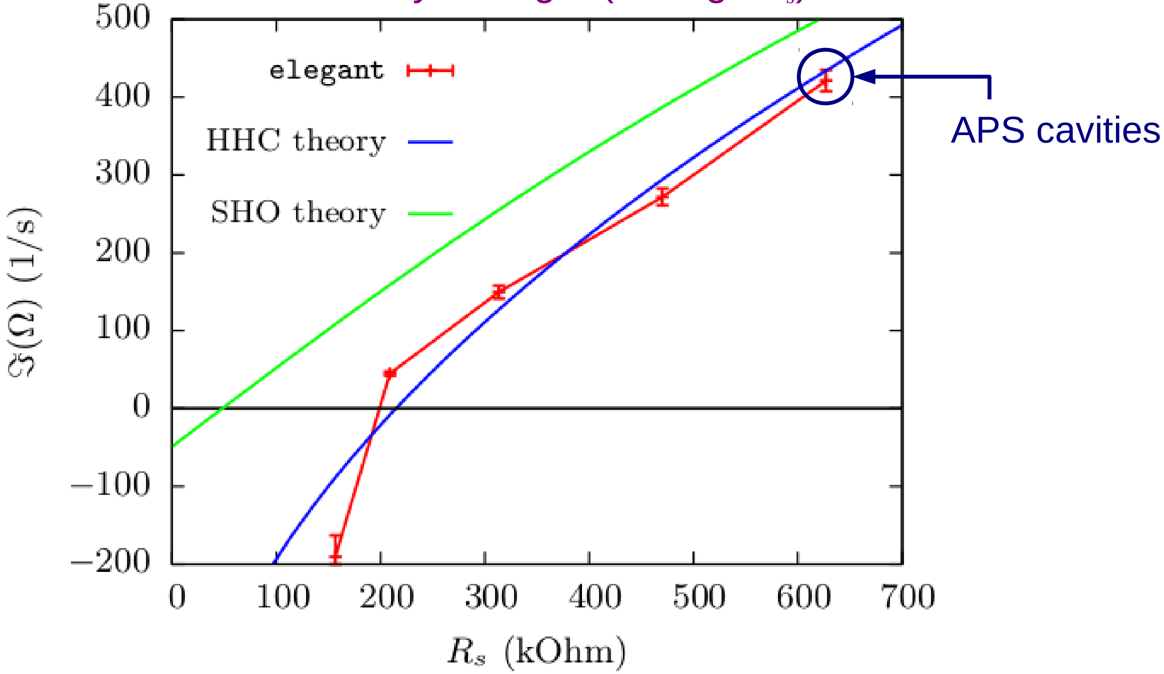
- Our theory connects growth rates including the HHC to those of a (fictitious) single rf system with the same bunch length (blue)
  - Approximate scaling comes from system with depressed synchrotron frequency $f_s \approx 150$ Hz
  - We expect maximum growth rates ~500 1/s.

Theory and simulation [9] agree well for the maximum predicted growth rate

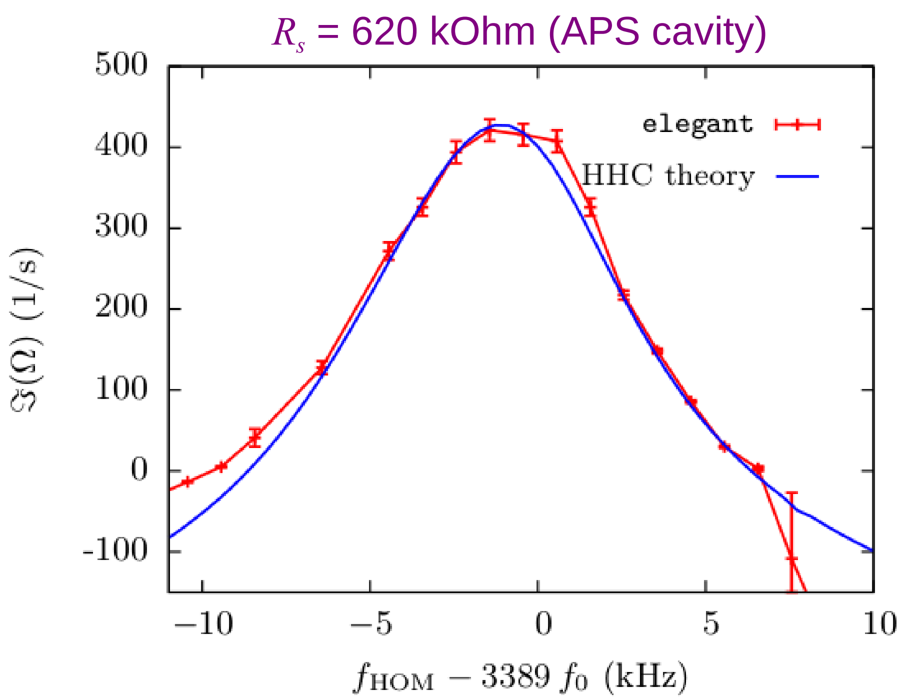
Maximum growth rate as a function of instability strength (through R_s)



- Theory has subtracted off the synchrotron radiation damping rate = 49 1/s.
- Difference between SHO and HHC give damping rate between 75 1/s and 100 1/s, in reasonable agreement with Bosch, *et al*.'s Landau damping rate of 80 1/s [4]

[4] R.A. Bosch, K.J. Kleman, and J.J. Bisognano. "Robinson instabilities with a higher-harmonic cavity," PRSTAB **4**, 074401 (2001).
[9] M. Borland. "`Elegant`: A flexible SDDS-compliant code for accelerator simulation," APS LS-287 (2000)

Growth rate as we vary the HOM frequency



- Curves are asymmetric with larger growth rates at negative detuning
  - Synchrotron frequency ~ 150 Hz << asymmetry ~ 1 kHz
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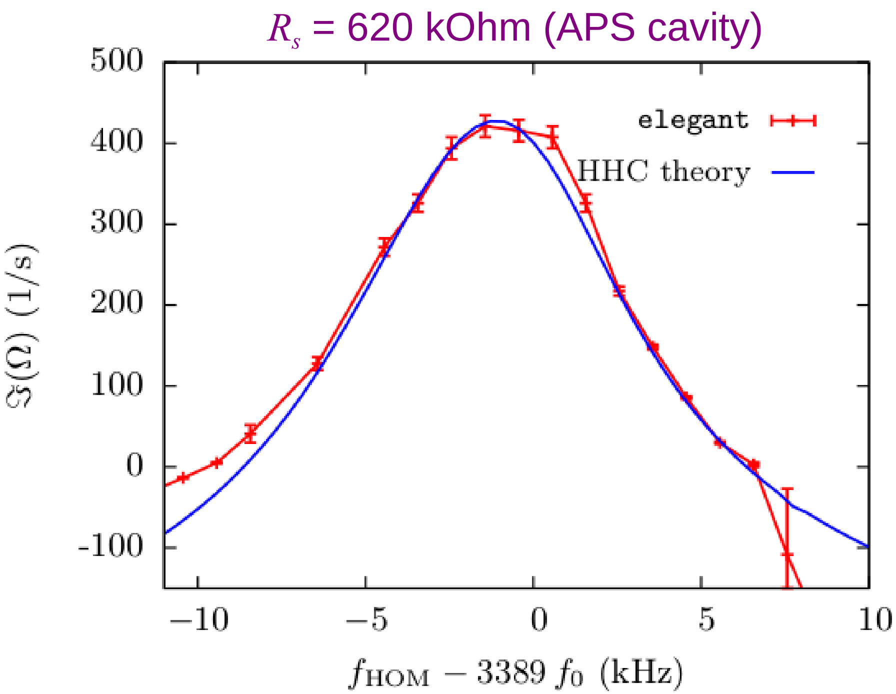

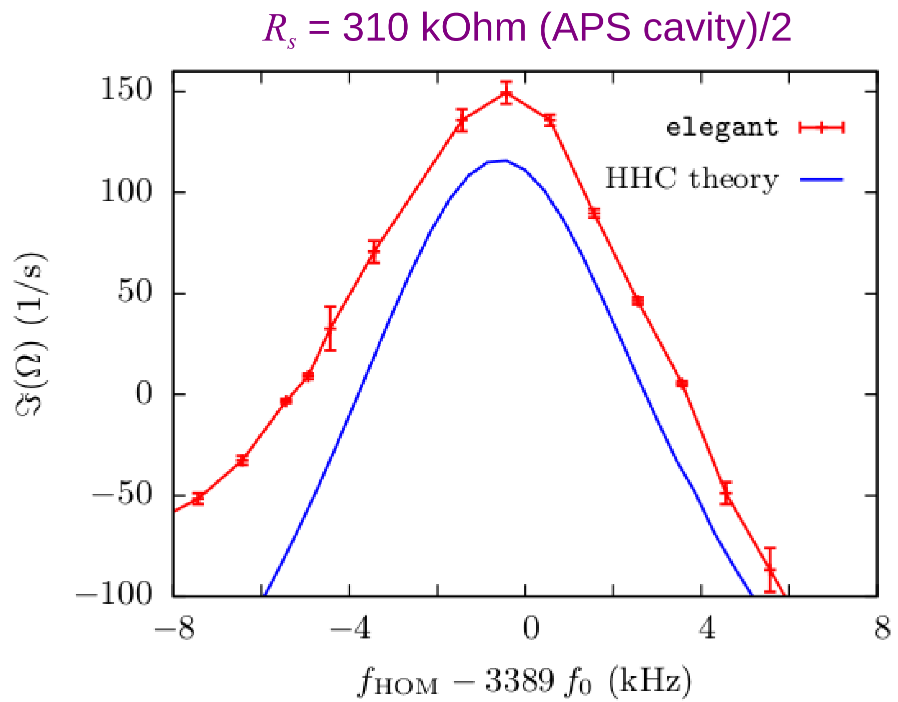

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Conclusions and outlook

- We have developed a self-consistent theory of multi-bunch instabilities for an arbitrary longitudinal potential
  - The resulting expression is easy and fast to solve numerically
  - The theory naturally joins the coupled-bunch matrix analysis developed for harmonic potentials with a dispersion integral that includes Landau damping
  - Theory is restricted to centroid oscillations due to the expansion of the long-range wakefield (valid if the bunch length << distance over which the wakefield varies)

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Additional slide

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- Theory can be used to quickly assess stability for many HOMs, and can incorporate wakefield and damping models of longitudinal feedback systems
- I would like to see how well the theory works for “overstretched” bunches like those that might be used for APS-U, and to what extent the passive HHC changes these results (initial simulations seem to show little difference, but more work is needed...)

Additional slide

Some details on `elegant` simulations [9]

- Lattice dynamics are simulated using `ILMATRIX` that includes
  - Transverse linear motion and lowest order nonlinear tune shift with amplitudes
  - First through third order chromatic effects
  - Momentum compaction at first through third order
- Damping and diffusion from synchrotron emission simulated with `SREFFECTS`
- Prescribed main rf cavity at 352 MHz fundamental simulated with `RFCA`
- Prescribed fourth harmonic cavity tuned to flatten potential simulated with `RFCA`
- Long range wakefield from the 921 MHz HOM simulated with `RFMODE`
- Tracking proceeds is as follows:
  1. Track 50k macroparticles over 20000 turns to get equilibrium bunch (ONCE)
  2. Make 47 copies to evenly populate the ring with 48 bunches
  3. Ramp HOM over 5000-10000 passes
  4. Track particles over 25k-55k turns, use exponetial fit to determine growth rate
  5. Obtain damping rate by first driving coupled-bunch motion with an unstable HOM, then shift the HOM frequency toward stability and measure damping rate

[9] M. Borland. "`Elegant`: A flexible SDDS-compliant code for accelerator simulation," APS LS-287 (2000)