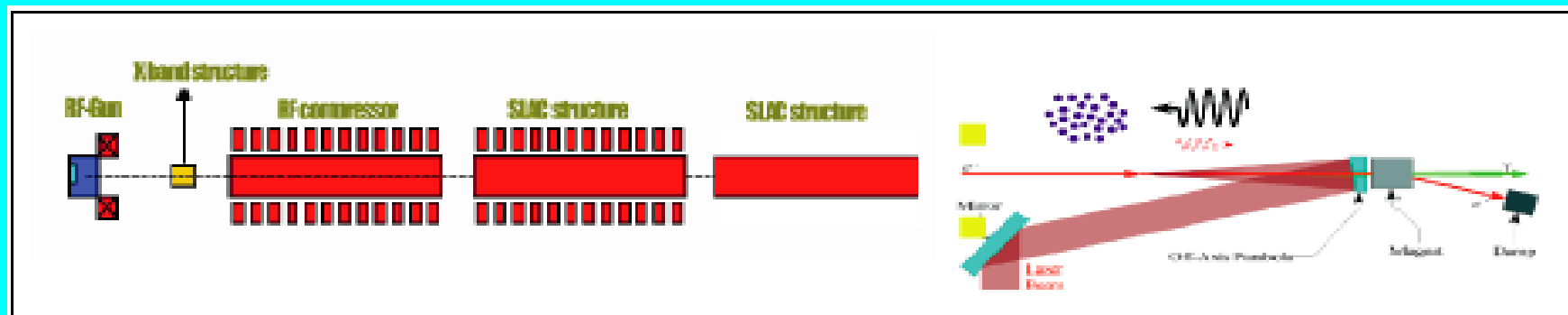


Design of the Thomson Source at SPARC/ PLASMONX for incoherent and coherent X-rays

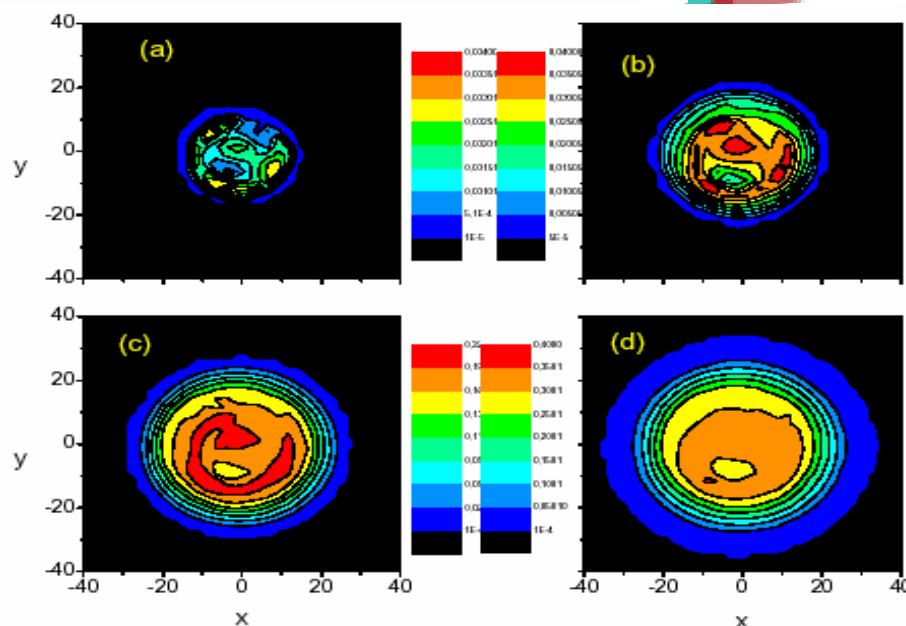
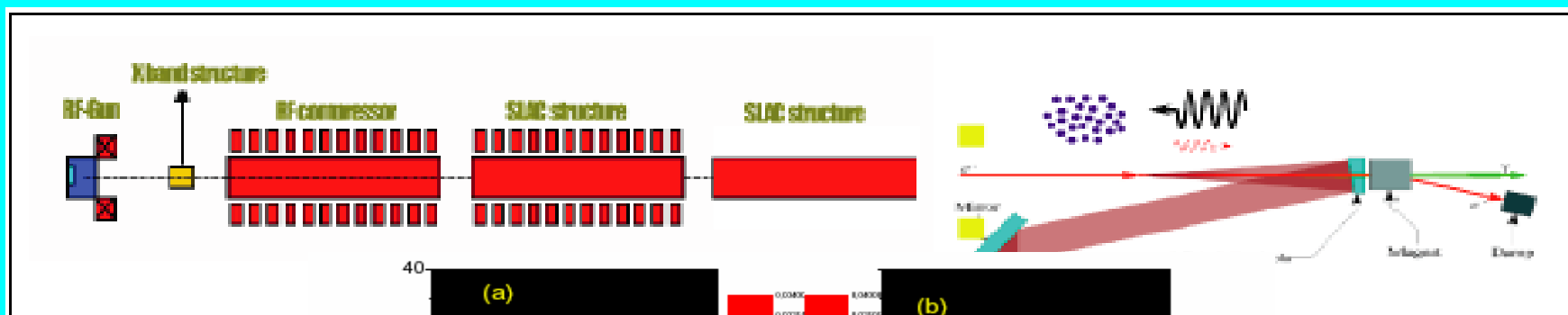
Luca Serafini - INFN/MI - on behalf of SPARC & PLASMONX Team

- **The PLASMON-X Project: a marriage between the SPARC high brightness electron beam and a high intensity laser beam**

Design and acquisition of a 200 TW Ti:Sa laser system for Plasma and IFEL acceleration exp. and a Thomson Source for monochromatic X-rays



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- The PLASMON X high brightness laser beam design for the SPARC IFEL accelerator

The SPARC high brightness laser beam design for the SPARC IFEL accelerator

- Coherent X-Rays at 1 Å from Thomson Sources as classical *SASE-FELs* or *Quantum FELs* (see also Maroli's and Piovella's talks in WG3)

(PLasma Acceleration at Sparc & MONochromatic X-rays)
IS BASED ON THE MARRIAGE BETWEEN:

*High Brightness
Electron Beams
(from 10's fs to a few
ps bunch length)*

$$B_n \equiv \frac{cQ_b}{\sigma_z \varepsilon_{nx} \varepsilon_{ny}} > 10^{15} \left[\frac{A}{m^2 \text{rad}^2} \right]$$

*High Intensity
Laser Beams
(30-100 fs pulses)*

$$I > 10^{19} \left[\frac{W}{cm^2} \right]$$



IS THE FIRST INGREDIENT

Under INFN responsibility

(see also Daniele Filippetto's talk in WG4)

Under ENEA responsibility





IS THE FIRST INGREDIENT

Under INFN responsibility

(see also Daniele Filippetto's talk in WG4)

Under ENEA responsibility

GOALS

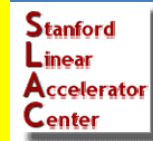
Generation of 30-150 MeV e^- beams

$(Q, \sigma_t, \varepsilon_n, \Delta\gamma/\gamma)$

Phase 1) 1 nC, 3 ps, 1 μm , 10^{-3}

Phase 2) 1 nC, 300 fs, 2 μm , $2 \cdot 10^{-3}$

PLASMONX) 20 pC, 60 fs, 0.3 μm , $2 \cdot 10^{-3}$

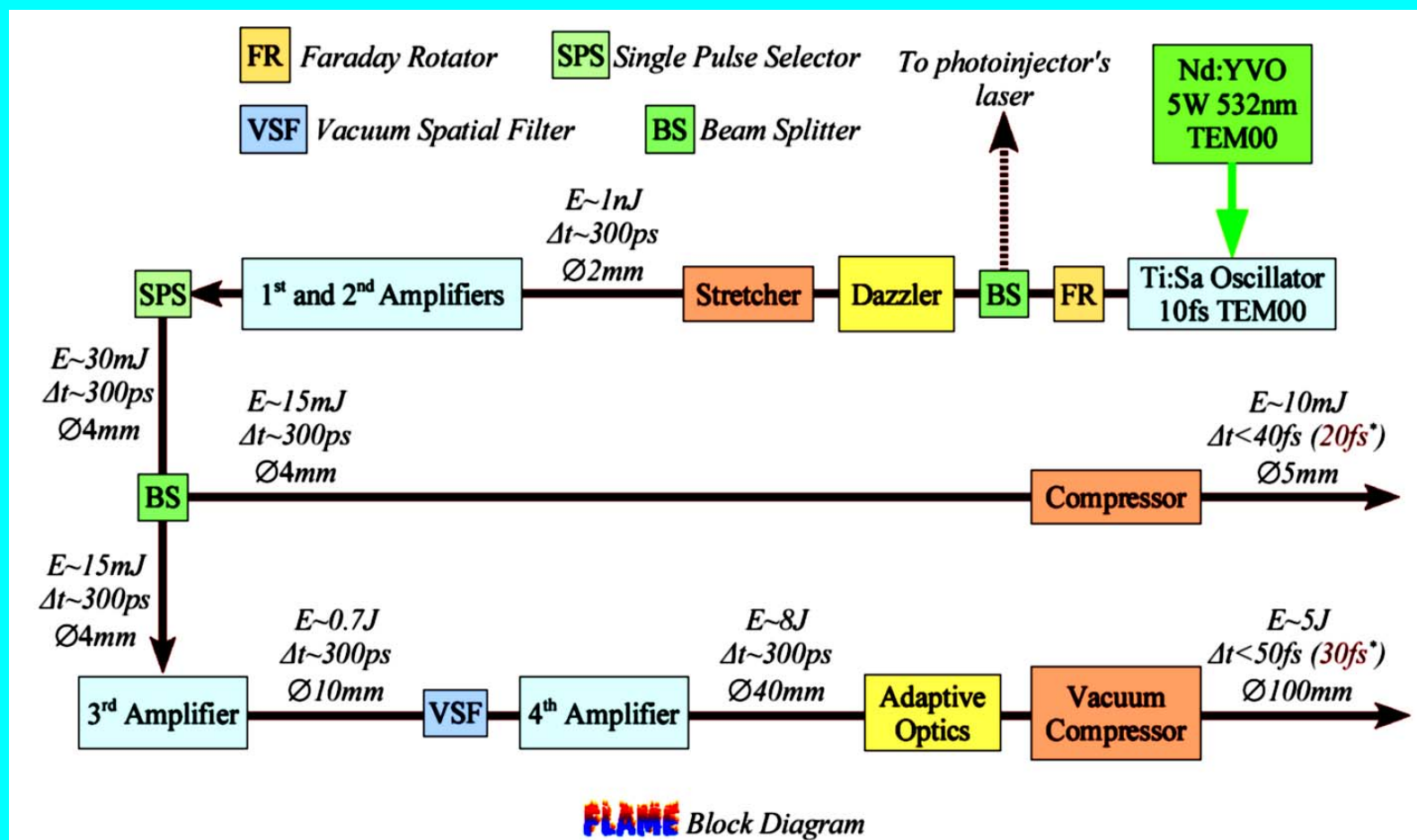


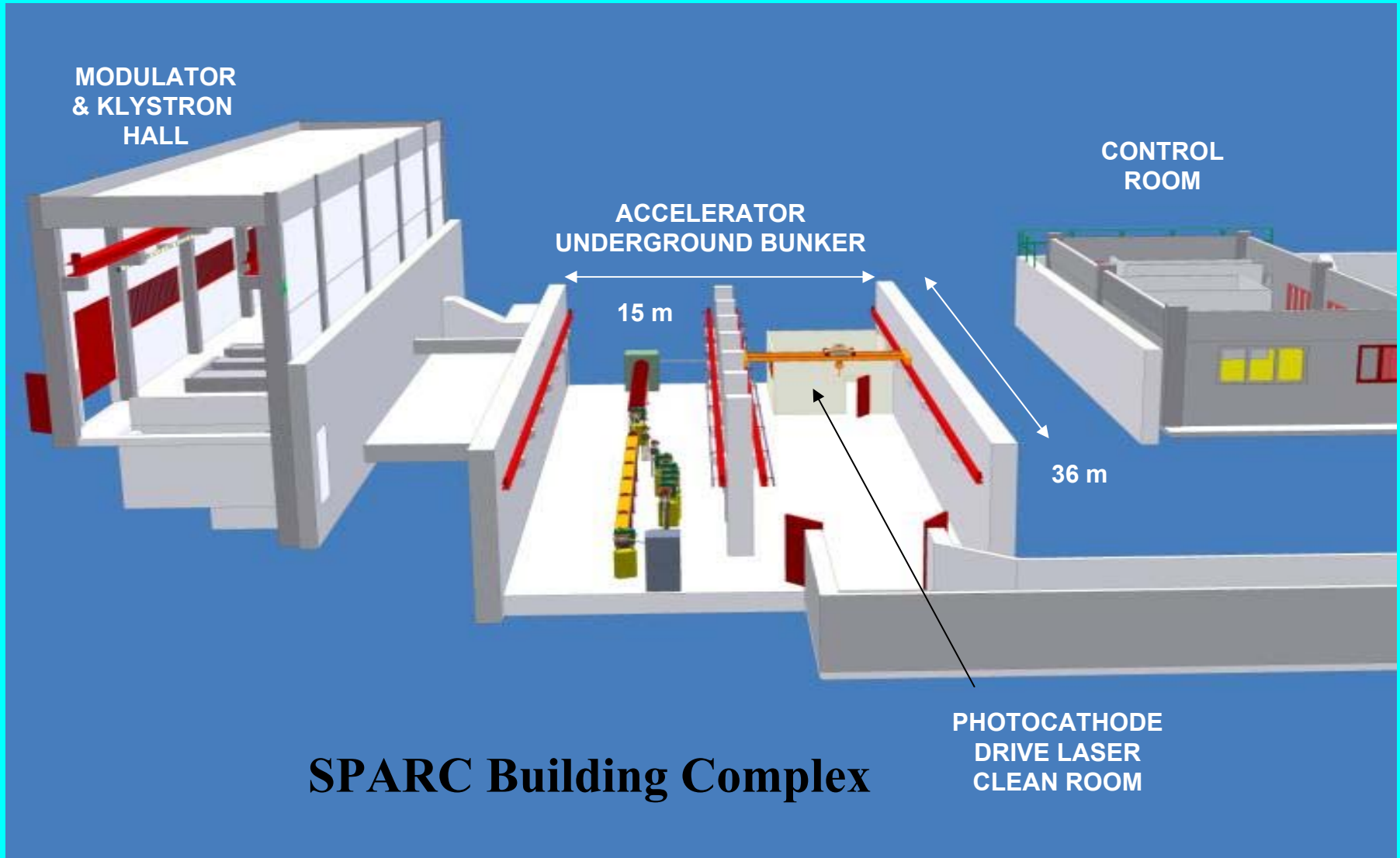
La Sapienza
Università degli Studi di Roma



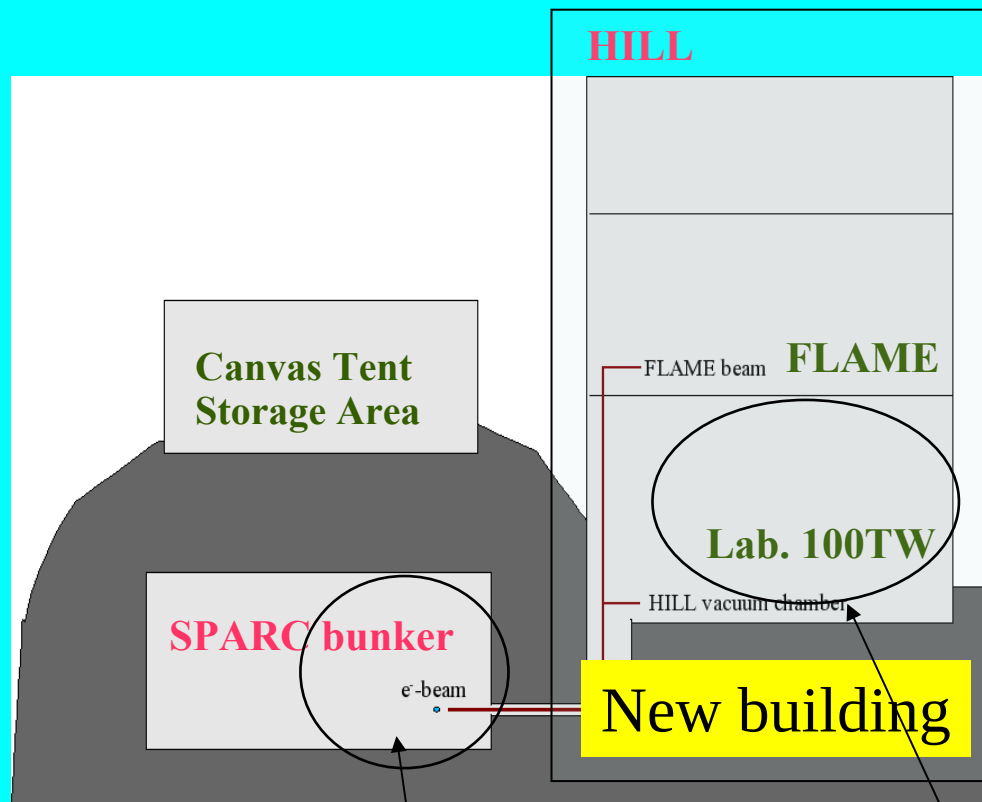
Frascati Laser for Acceleration and Multidisciplinary Experiments (FLAME)

SECOND INGREDIENT

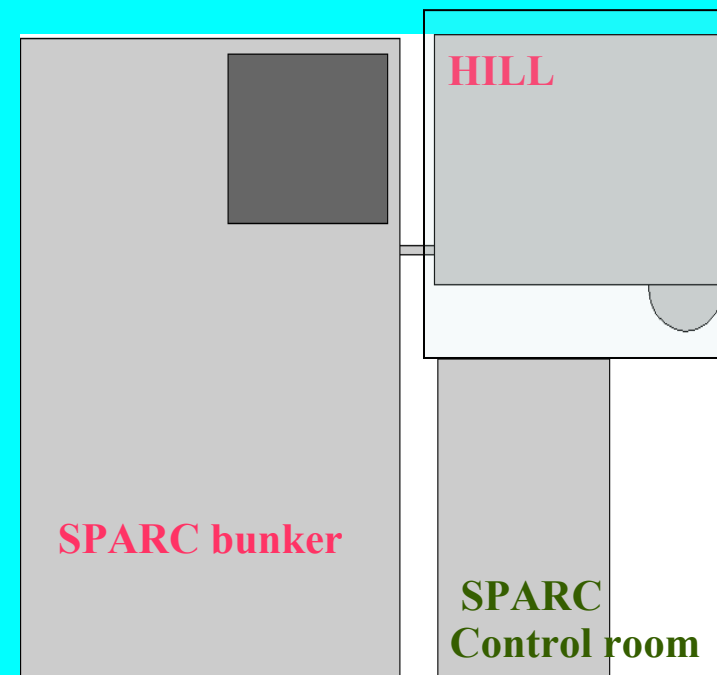




Section view

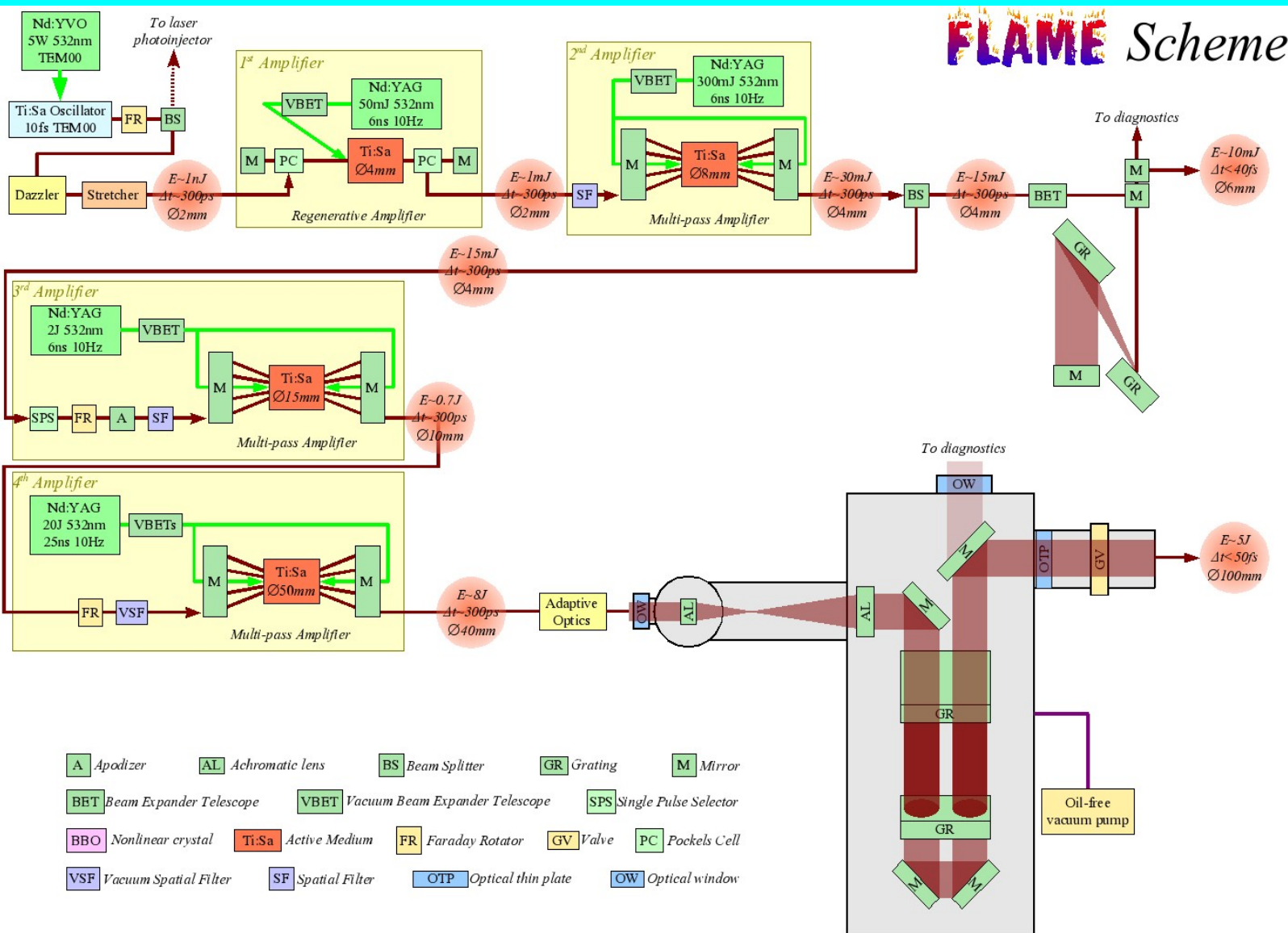


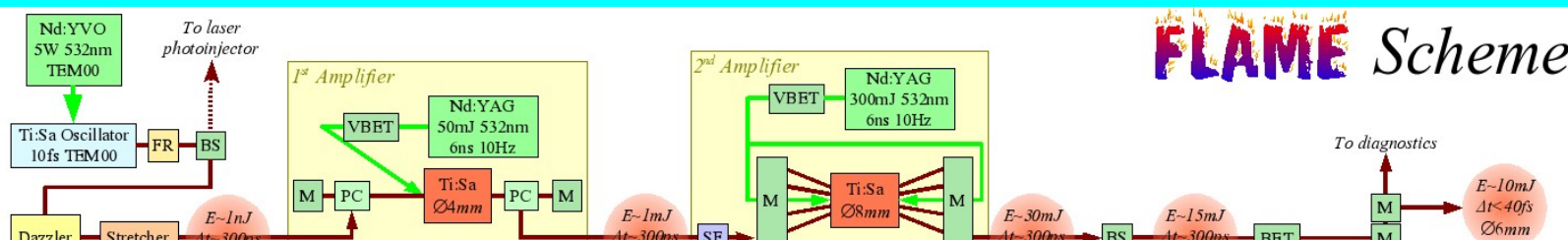
Top view



LWFA with **self-injection** + Thomson scattering

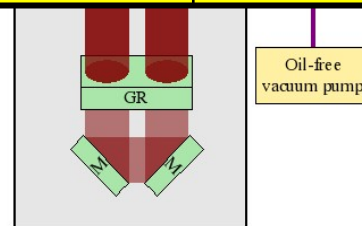
LWFA with **external injection** + Thomson scattering



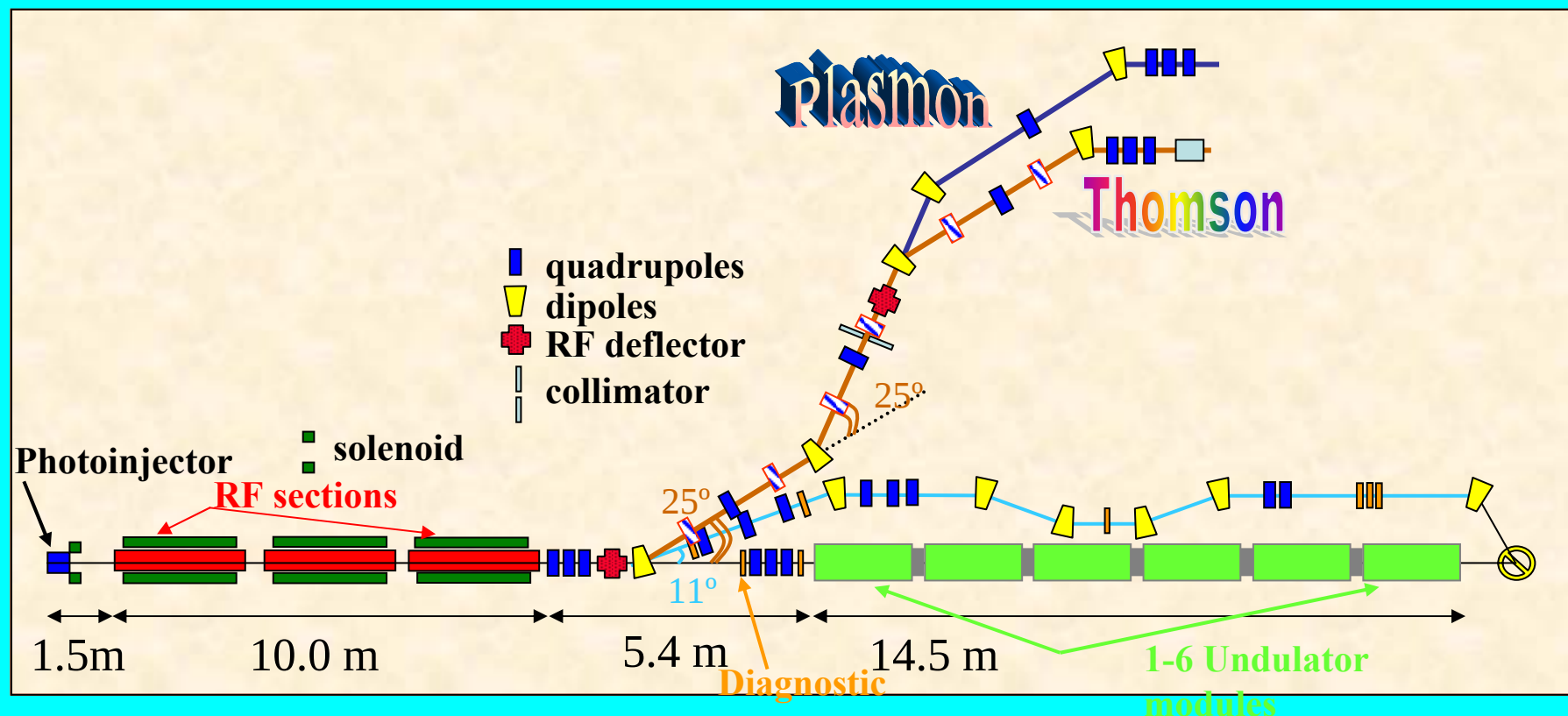


Type	Wavelength	Delivered energy	Duration	Contrast
Phase 1 CPA Ti:Sa	0.8 micron	5J	50fs	$>10^6$
Phase 2 CPA Ti:Sa with <i>OPCPA</i>	0.8 micron	5J	30fs	$>10^8$

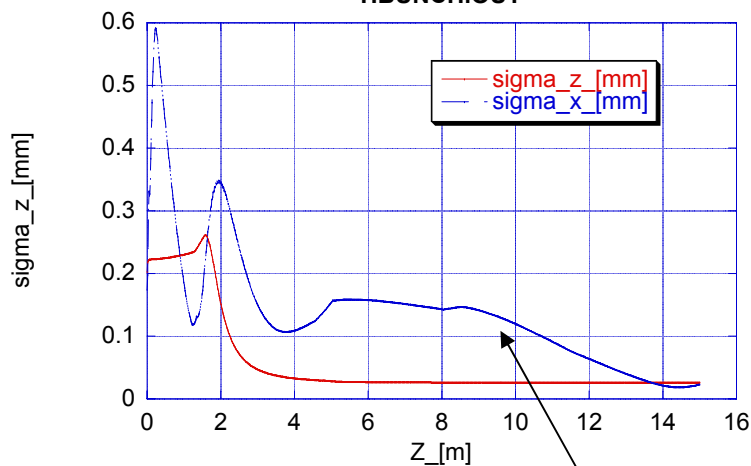
A Apodizer **AL** Achromatic lens **BS** Beam Splitter **GR** Grating **M** Mirror
BET Beam Expander Telescope **VBET** Vacuum Beam Expander Telescope **SPS** Single Pulse Selector
BBO Nonlinear crystal **Ti:Sa** Active Medium **FR** Faraday Rotator **GV** Valve **PC** Pockels Cell
VSF Vacuum Spatial Filter **SF** Spatial Filter **OTP** Optical thin plate **OW** Optical window



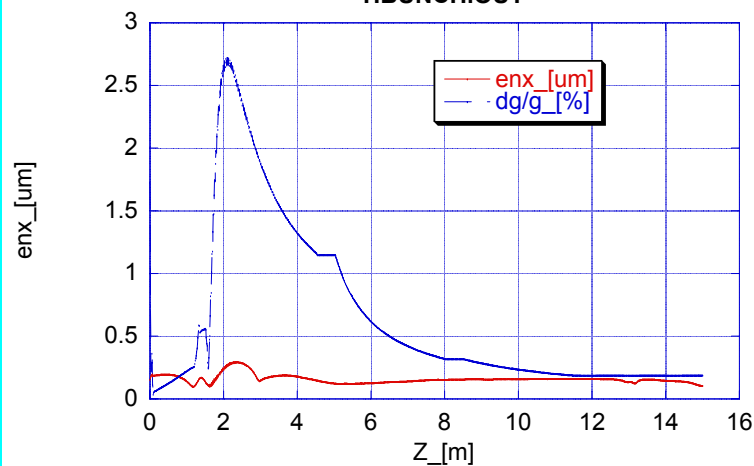
Schematic layout



HBUNCH.OUT



HBUNCH.OUT

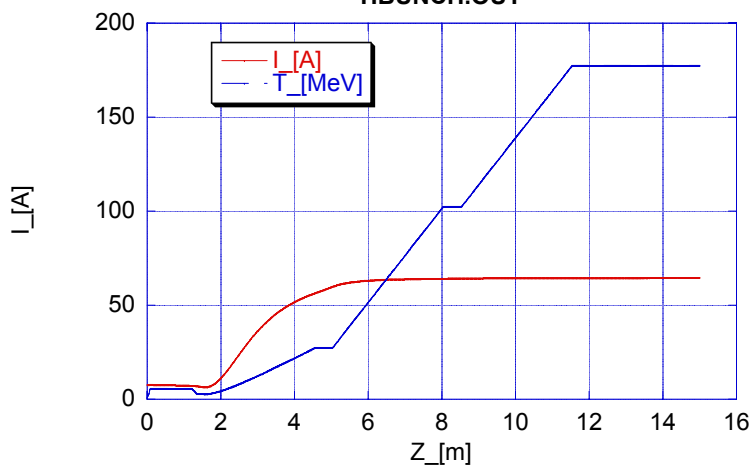


$Q = 20$ pC

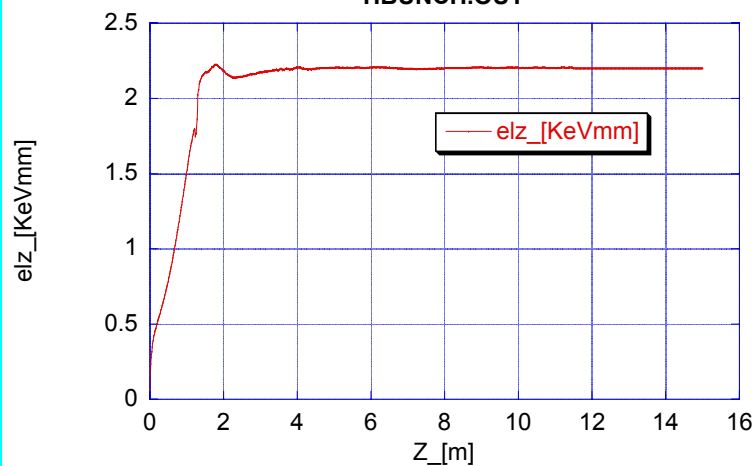
slight focusing

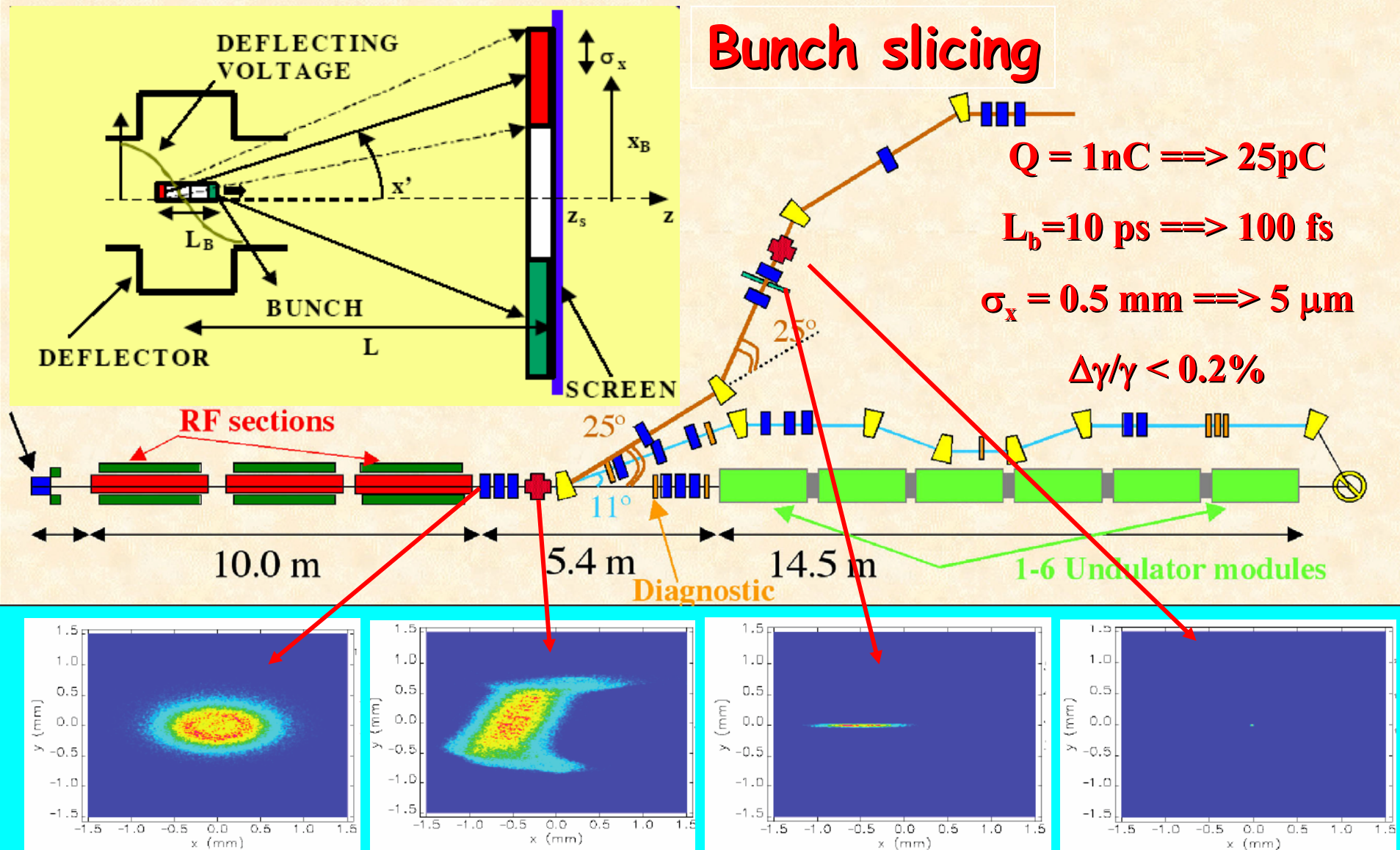
after compression $\sigma_z = 25$ μm, $\Delta\gamma/\gamma = 0.2\%$, $\epsilon_{nx} < 0.3$ μm

HBUNCH.OUT

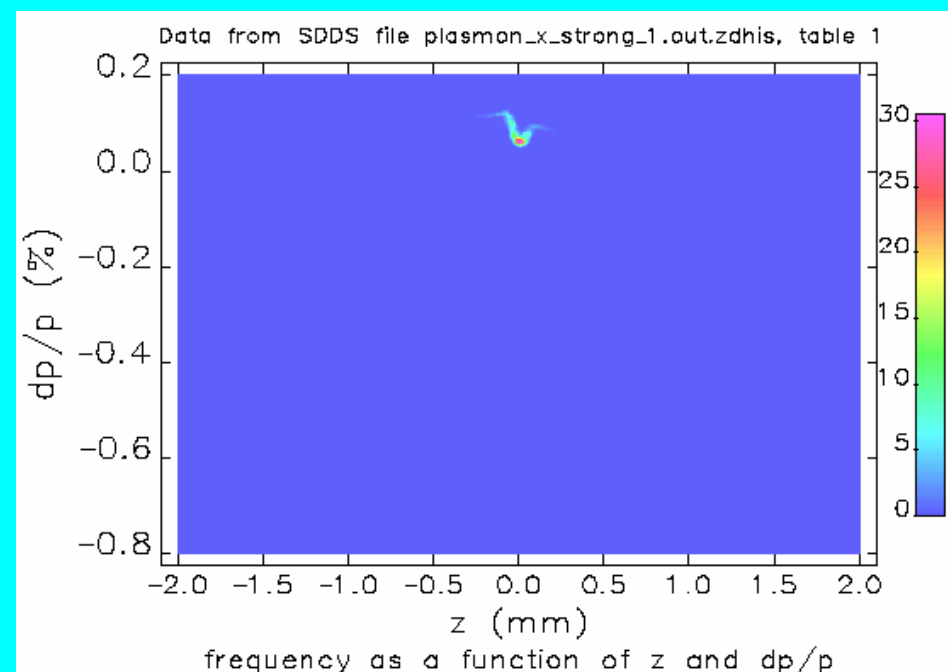
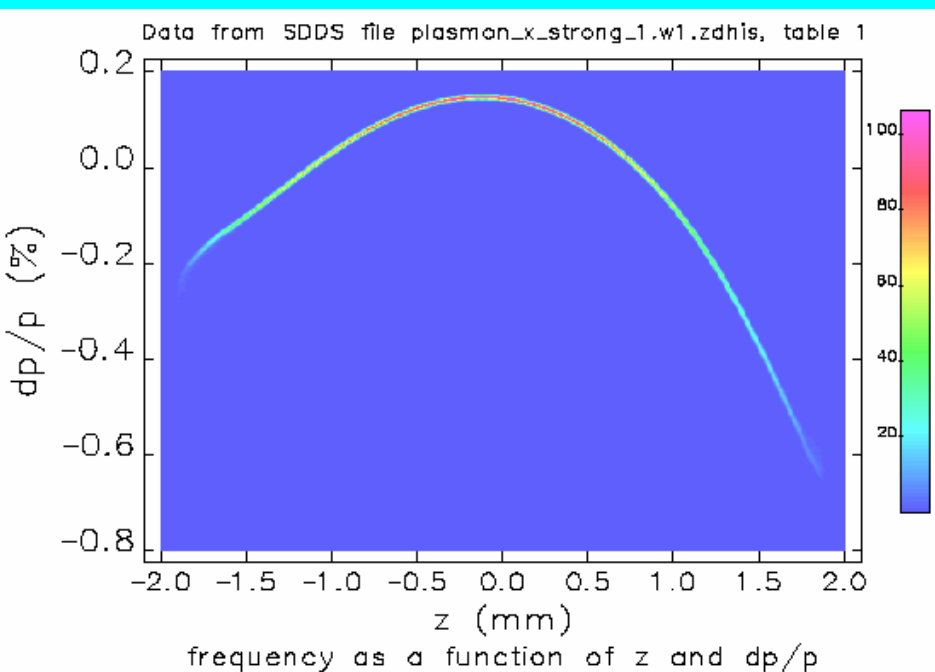


HBUNCH.OUT





beam energy distribution at start/end of the beam line

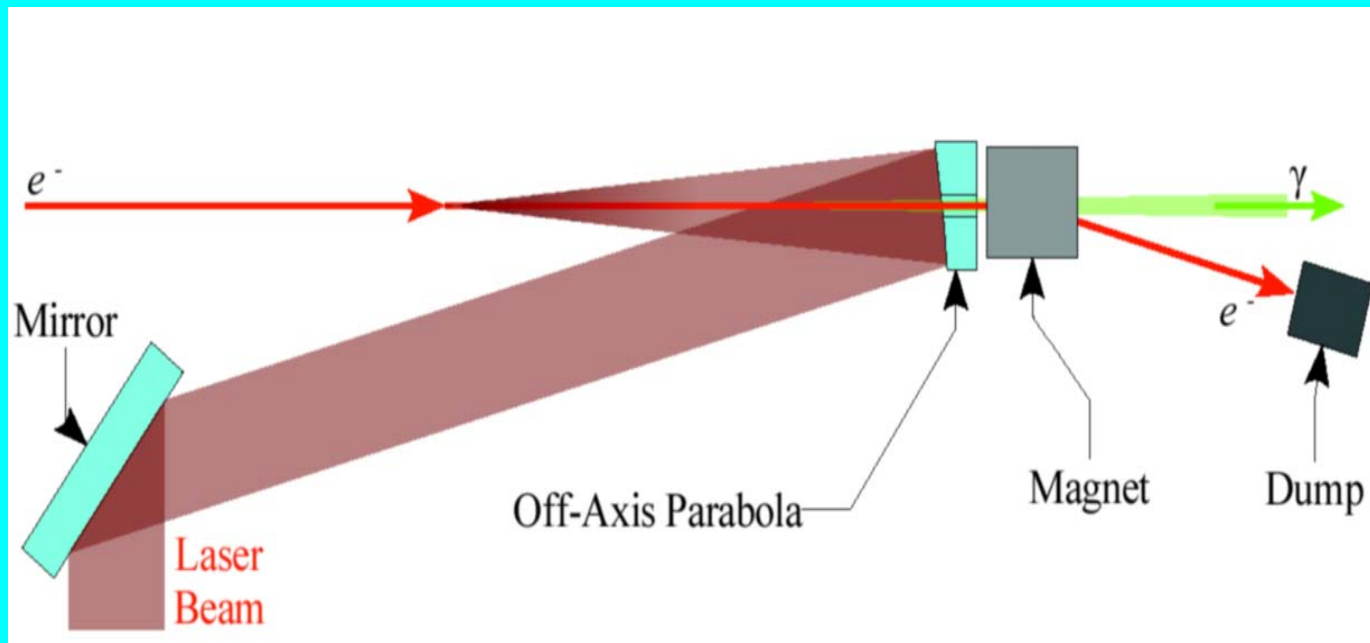


start

end

60 fs bunch (initially uncompressed)
-> 10 fs with RF compression, i.e. by
combining velocity bunching and bunch-slicing

Experimental set-up for the generation of tunable X-ray radiation via Thomson scattering of optical photons by relativistic electron bunches



THOMSON BACK-SCATTERING

$$\nu_T = \nu_0 \frac{1 - \beta \cos \alpha_L}{1 - \beta \cos \theta} \approx \nu_0 \frac{4\gamma^2}{1 + \theta^2 \gamma^2} \approx 4\gamma^2 \nu_0$$

for $\alpha_L = \pi$ and $\theta \ll 1$ or $\theta = 0$

e^- (1 GeV); $\lambda_0 = 1\mu\text{m}$, $E_0 = 1.24 \text{ eV}$ $\longrightarrow$
 $E_T = 20 \text{ MeV}$

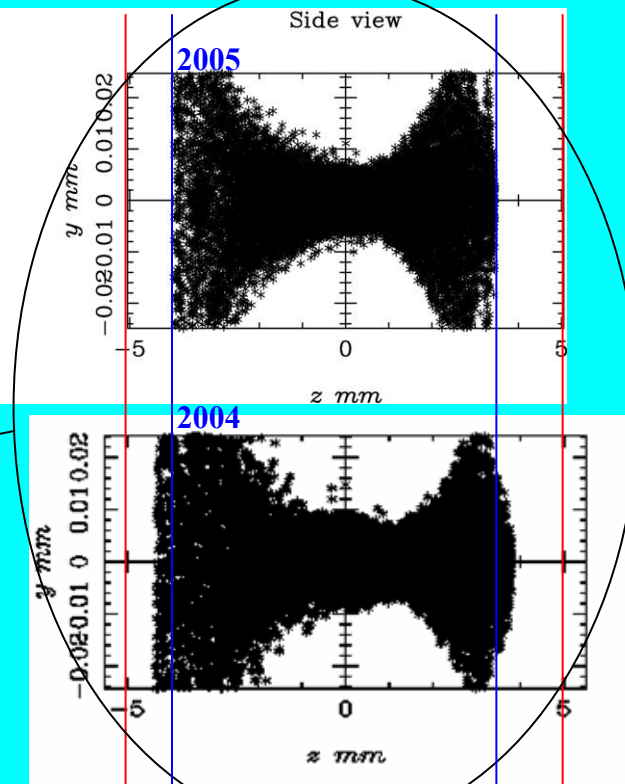
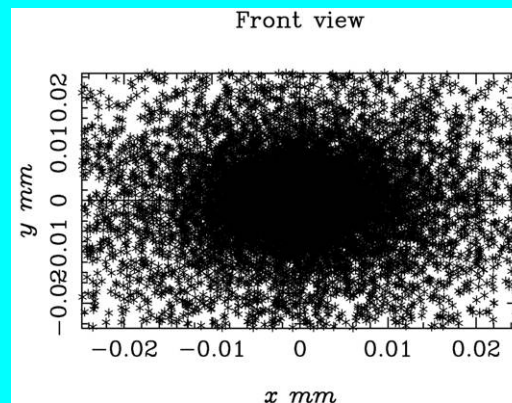
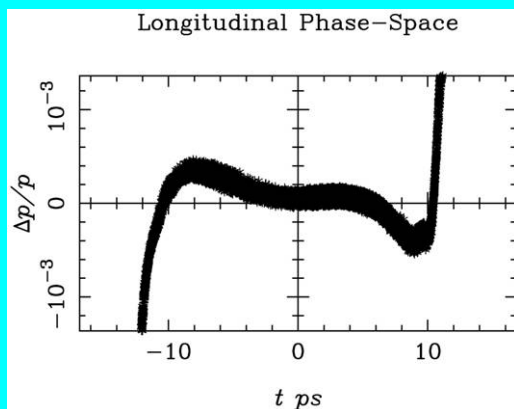
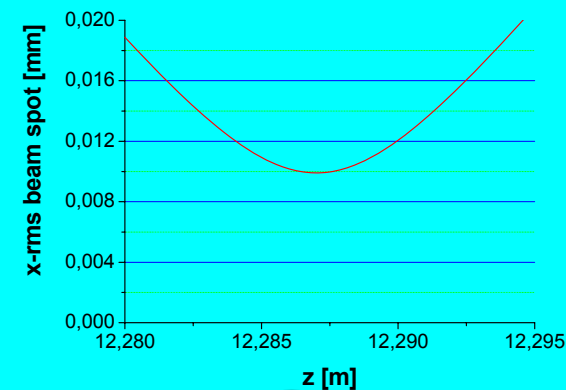
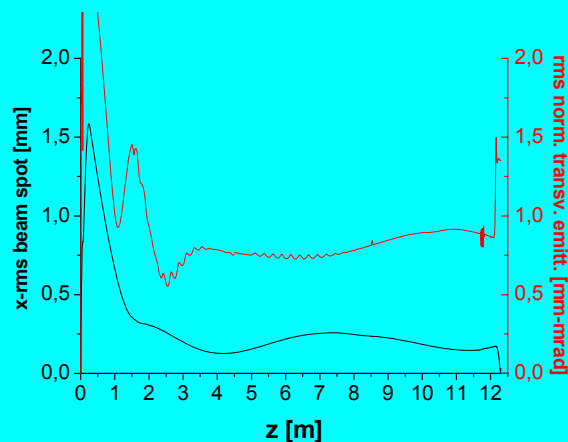
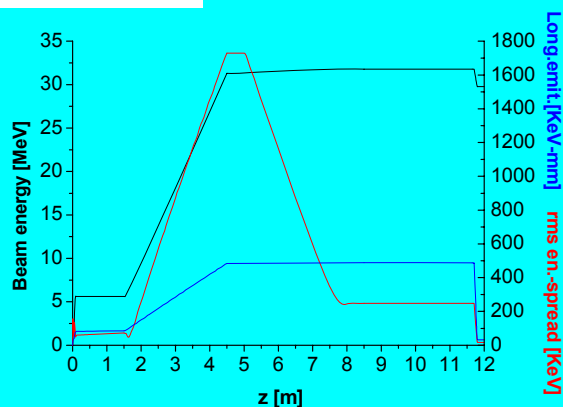
$\lambda_T = 6 \times 10^{-8} \mu\text{m}$,

e^- (200 MeV); $\lambda_0 = 1\mu\text{m}$, $E_0 = 1.24 \text{ eV}$ $\longrightarrow$
 $E_T = 800 \text{ KeV}$

$\lambda_T = 1.56 \times 10^{-6} \mu\text{m}$,

e^- (29 MeV); $\lambda_0 = 0.8\mu\text{m}$, $E_0 = 1.5 \text{ eV}$
 $E_T = 20 \text{ KeV}$

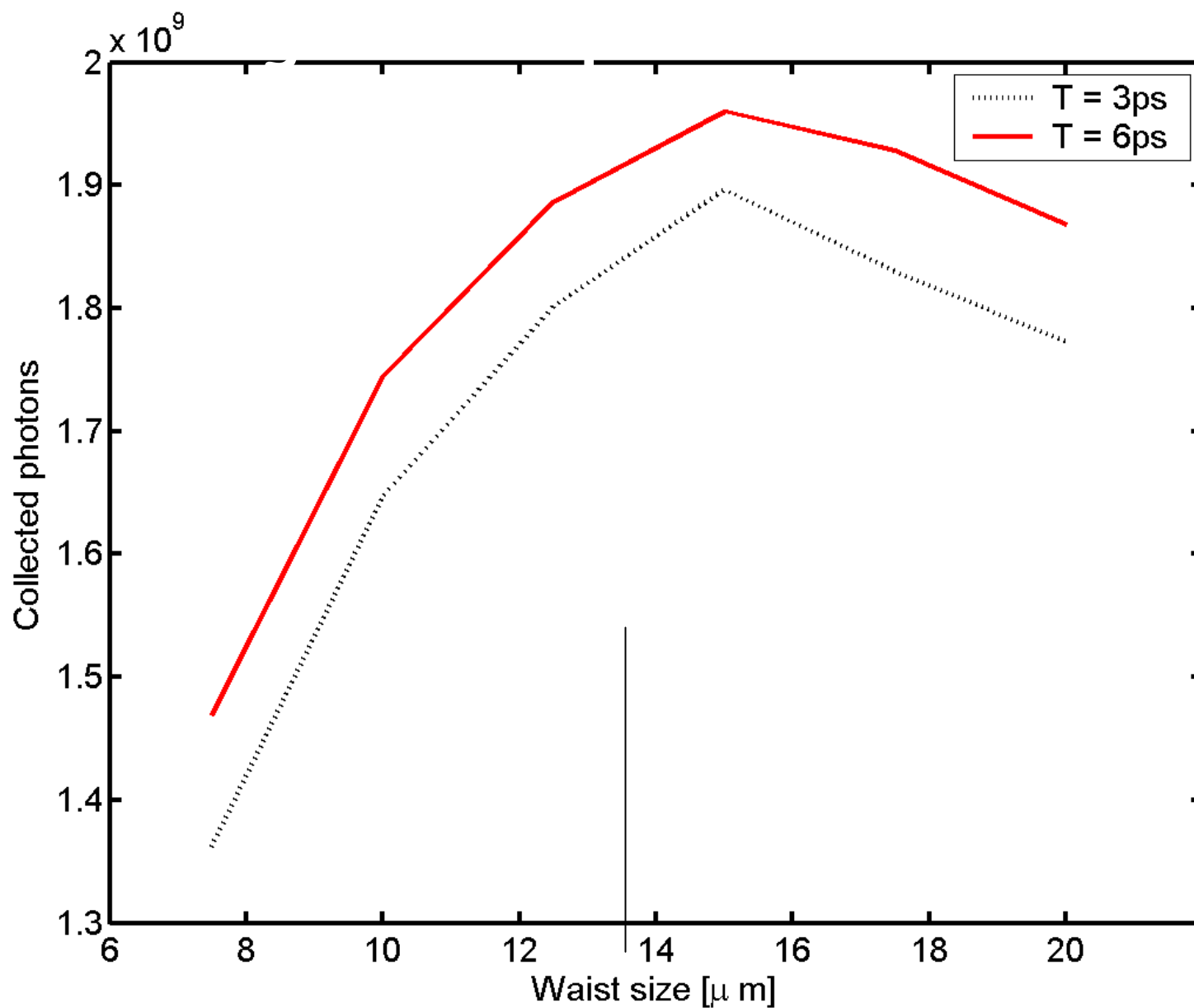
$\lambda_T = 0.5 \times 10^{-4} \mu\text{m}$,

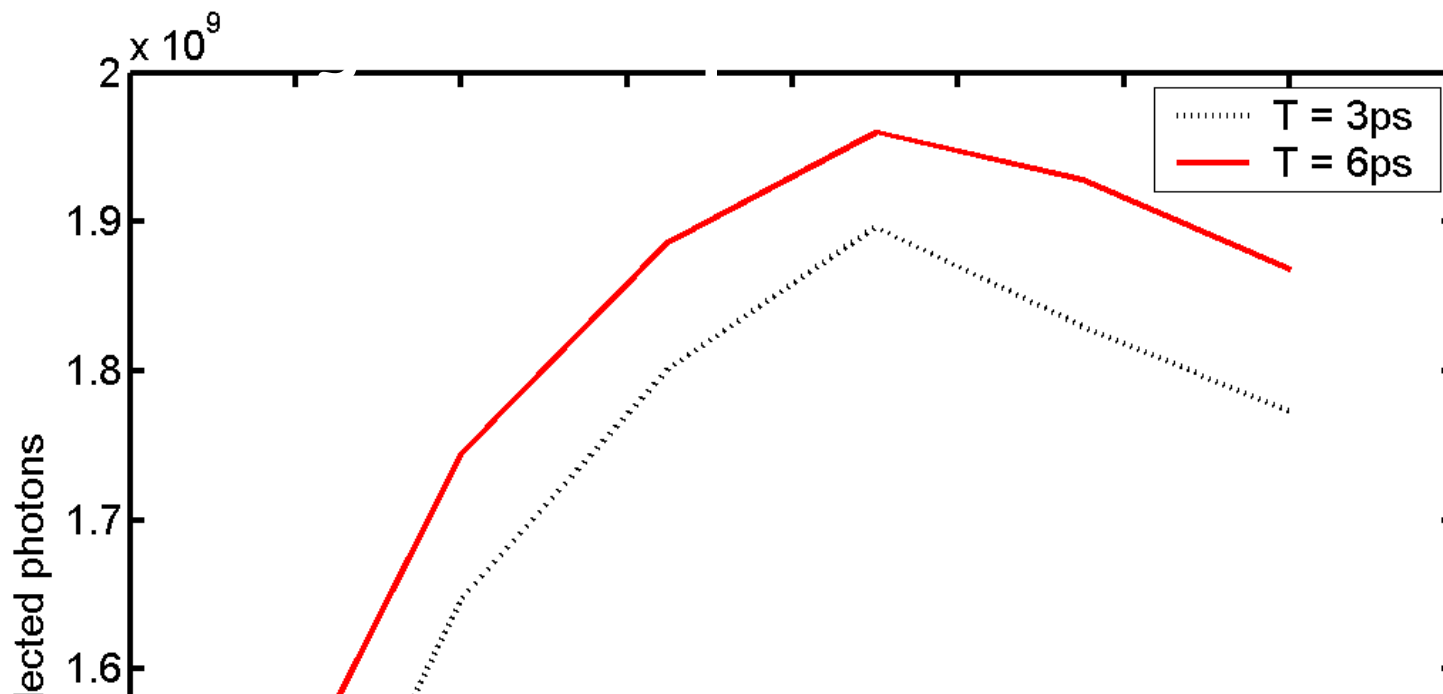


In the 2005 simulation (autumn), playing especially with the injection phase of the II TW structure (involved in the linear correlation correction) have been possible obtain a bunch of about 3ps (~ 1 mm) shorter, with a denser core

Angular and spectral distribution of the TS radiation in the case of an unguided 3 ps laser pulse (12.5 μm beam waist)

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.





Bunch

Current best working point

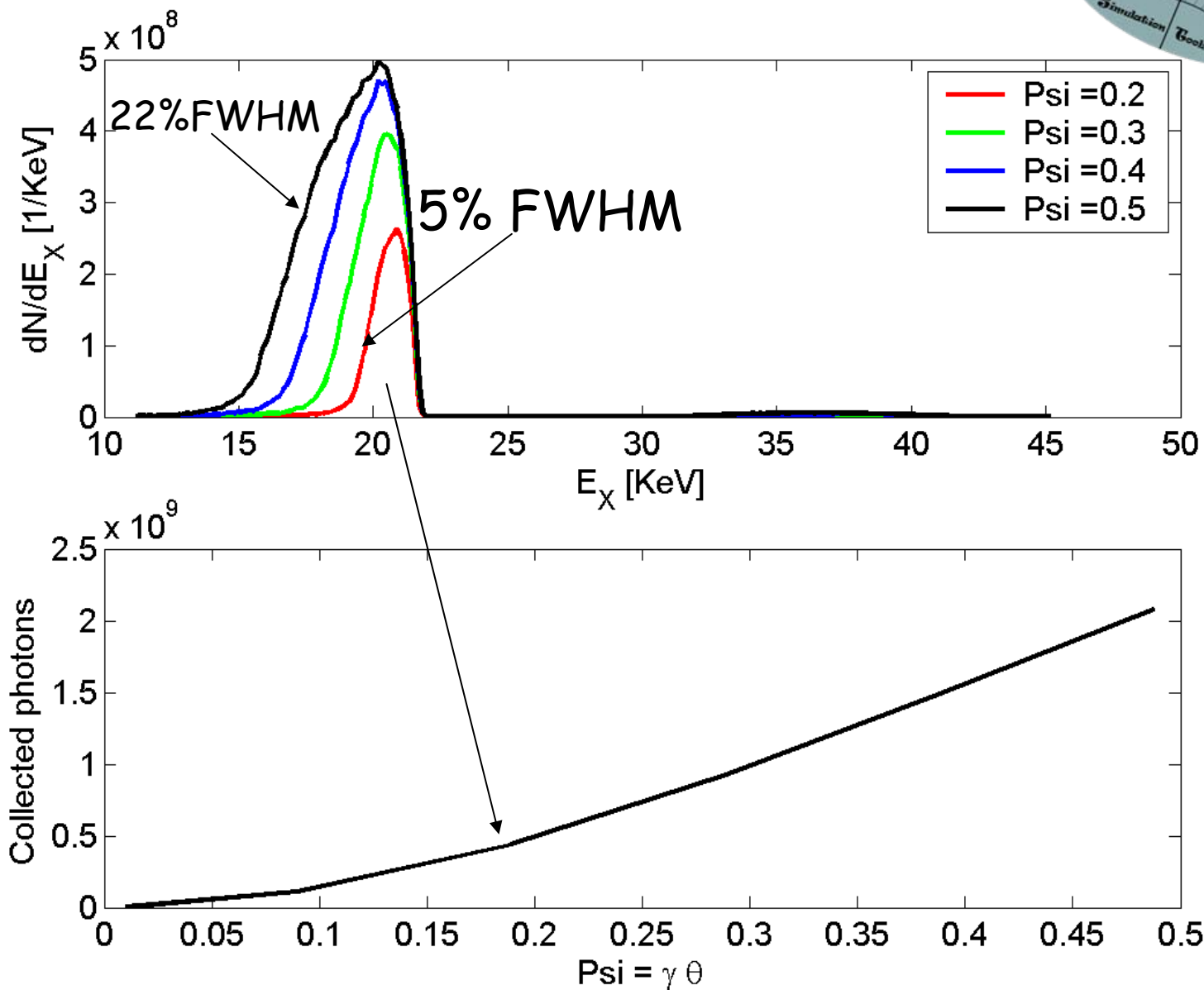
Pulse

- 2.5nC
- 8ps long (full size)
- $13\text{ }\mu\text{m}$ rms tr. Size
- 1.5 mm mrad norm emittance
- 0.1% energy spread



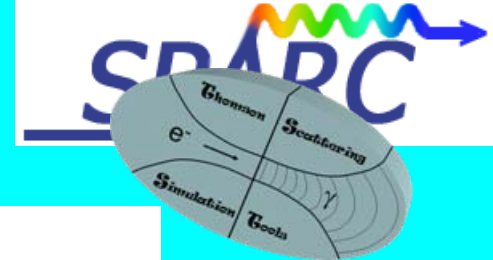
- TEM00
- 5J in 6ps
- $w_0 = 15\text{ }\mu\text{m}$

Waist size [μm]



QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

PLASMON X

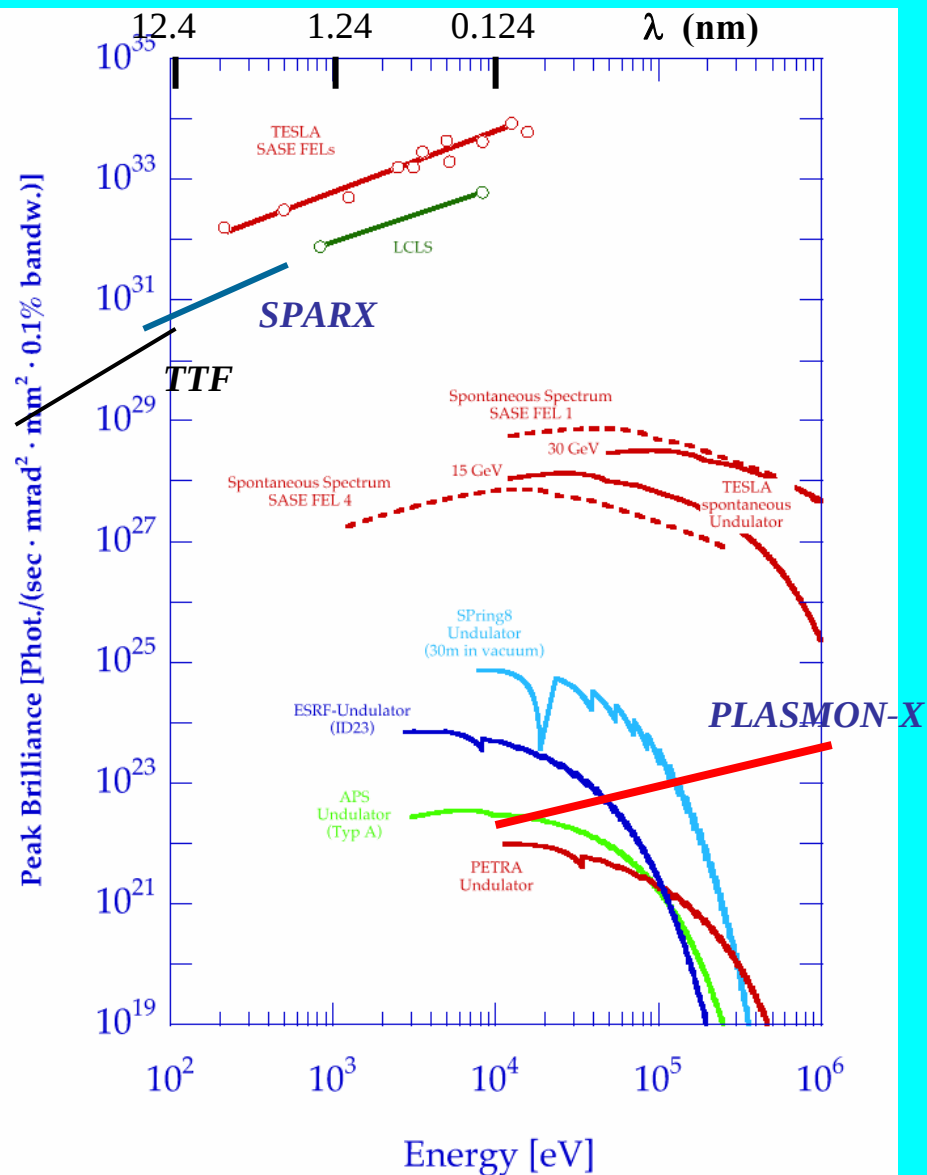


QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

SASE-FELs will allow an
unprecedented upgrade in
Source Brilliance

Covering from the VUV to
the 1 Å X-ray spectral range:
new Research Frontiers

Compact Thomson Sources
extend SR to hard X-ray range
allowing
Advanced Radiological
Imaging **inside** Hospitals



FEL resonance condition

$$\lambda = \lambda_w \frac{(1 + a_w^2)}{2\gamma^2} \quad (\text{magnetostatic wiggler})$$

Example : for $\lambda=1\text{\AA}$, $\lambda_w=1\text{cm}$, $E=3.5\text{GeV}$

$$\lambda = \lambda_{\text{pump}} \frac{(1 + a_w^2)}{4\gamma^2} \quad (\text{electromagnetic wiggler})$$

Example : for $\lambda=1\text{\AA}$, $\lambda_{\text{pump}}=1\mu\text{m}$, $E=25\text{MeV}$

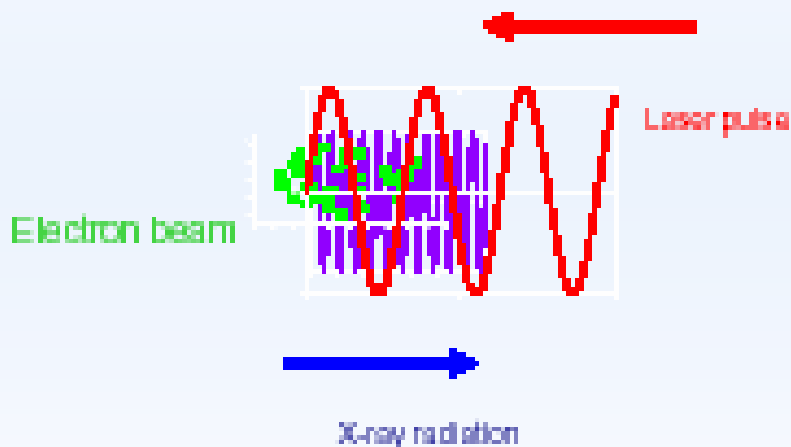
Toward Coherent X-rays: exploring coherent emission mechanisms (FEL-like) in Thomson Sources

COLLECTIVE EFFECTS IN THE THOMSON BACK-SCATTERING BETWEEN A LASER PULSE AND A RELATIVISTIC ELECTRON BEAM

A. Bacci, L. Serafini *INFN-Sezione di Milano, Via Celoria, 16, 20133 Milano (Italy)* C. Maroli, V. Petrillo *Dipartimento di Fisica dell'Università di Milano e INFN-Sezione di Milano, Via Celoria, 16, 20133 Milano (Italy)* M. Ferrario *INFN-LNF, Via Fermi 40, 00044 Frascati (RM), Italy*

FEL-Conf. 2005

The physical system



Laser pulse characteristics:

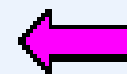
wavelength $\lambda=0,8 \mu\text{m}$, power 1TW, time duration $T=5 \text{ ps}$
Circular polarization, focal spot diameter $w_0 > 50 \text{ micron}$

Electron beam characteristics:

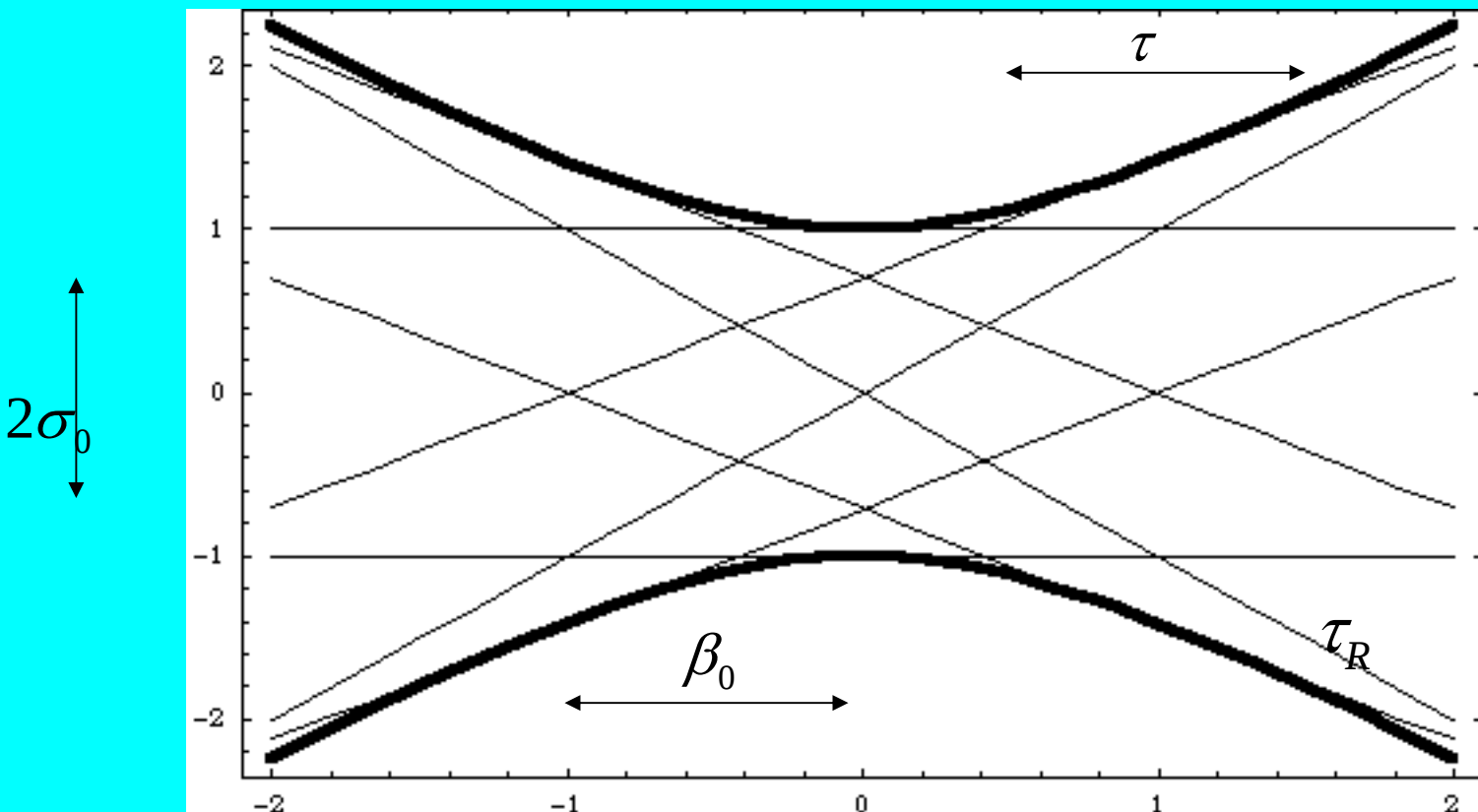
Counterpropagating respect the laser pulse
Energy 15 MeV ($\gamma=30$), spot size $\sigma_0=10 \mu\text{m}$, length $L_b=100\text{-}200\mu\text{m}$, charge $Q=1 \text{ nC}$

Radiation characteristics:

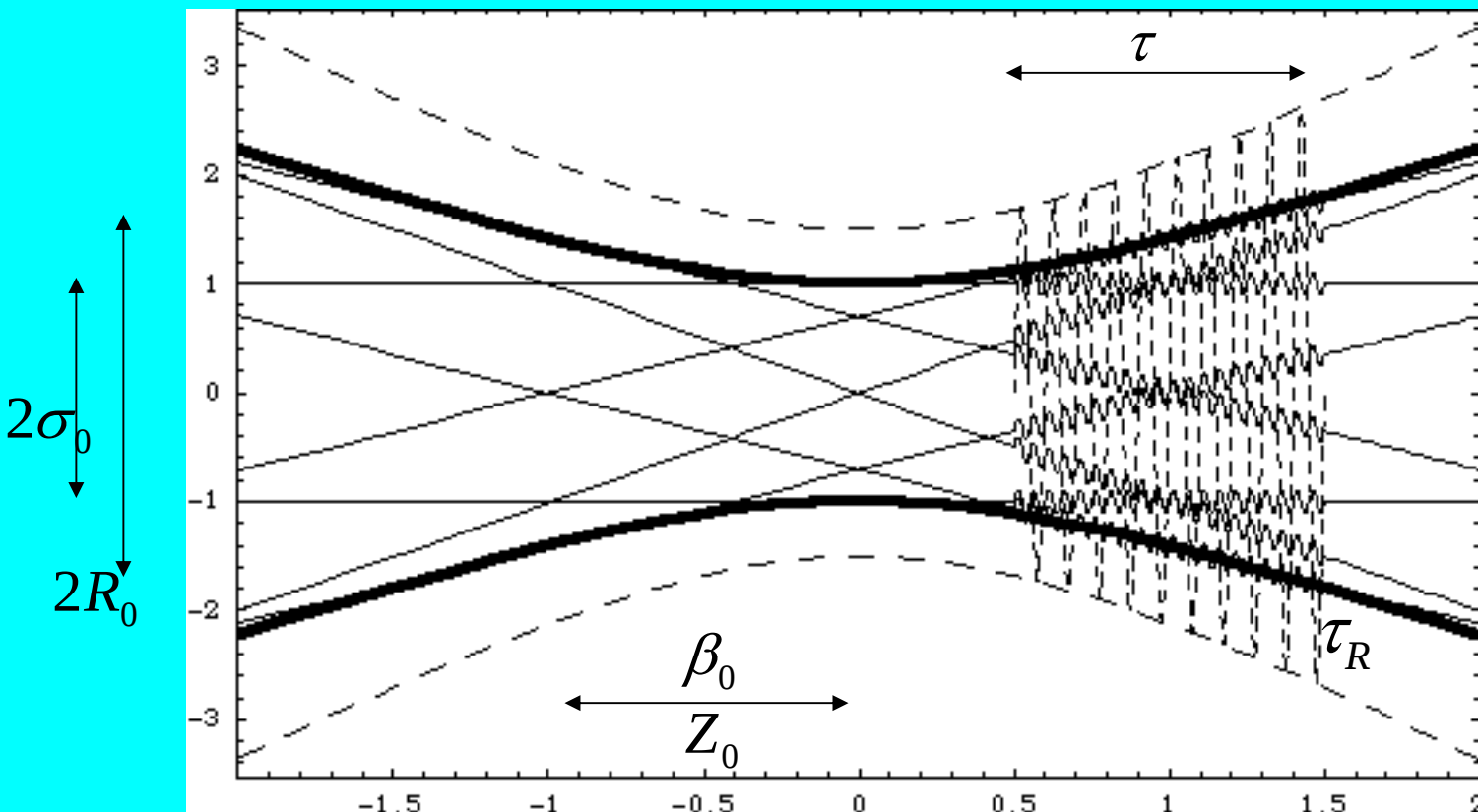
Wavelength $\lambda=2.2 \text{ Ang}$



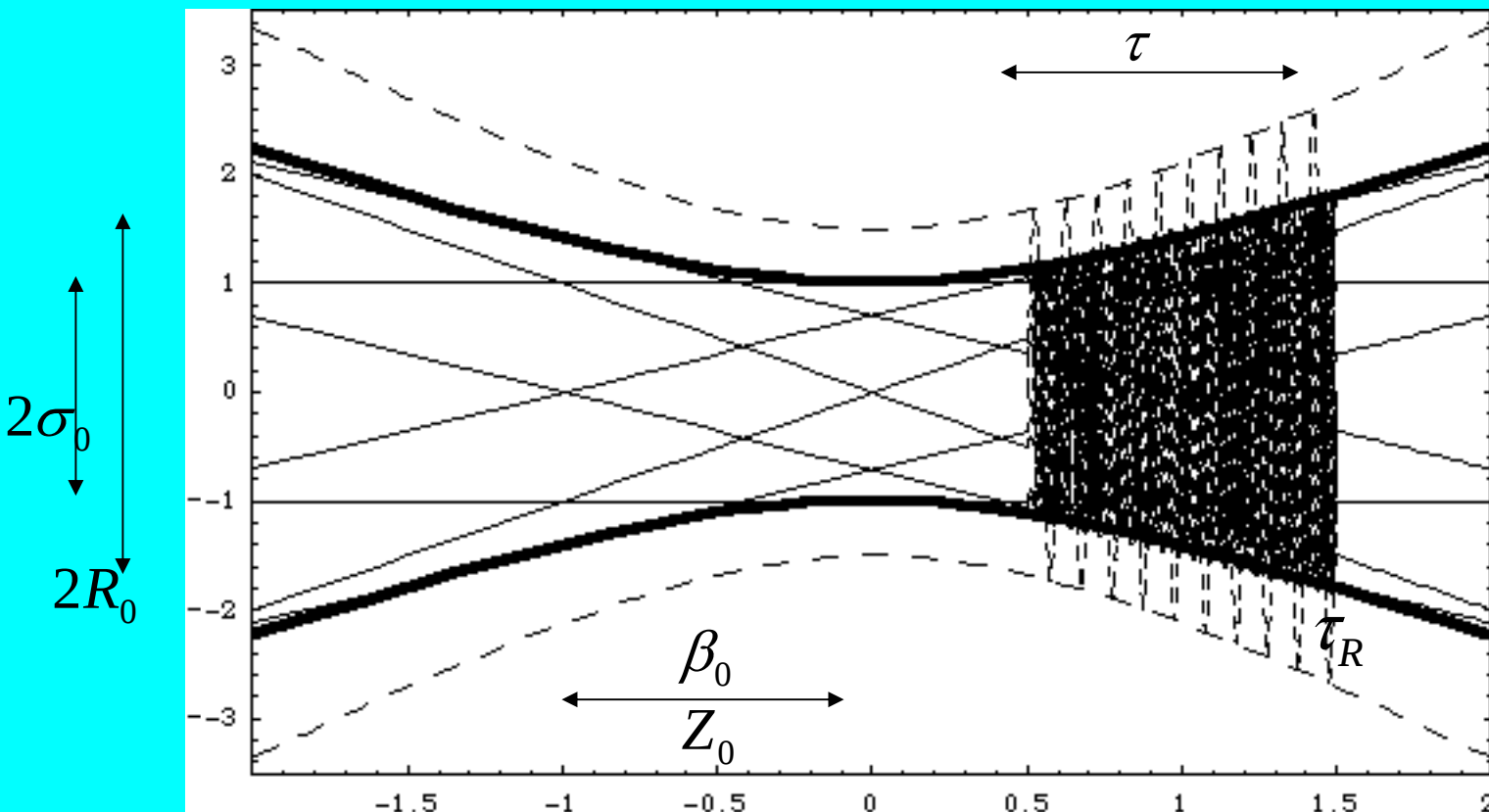
Conditions to operate a Thomson Source in FEL mode $\lambda_R = \lambda \frac{(1 + a_0^2)}{4\gamma^2}$



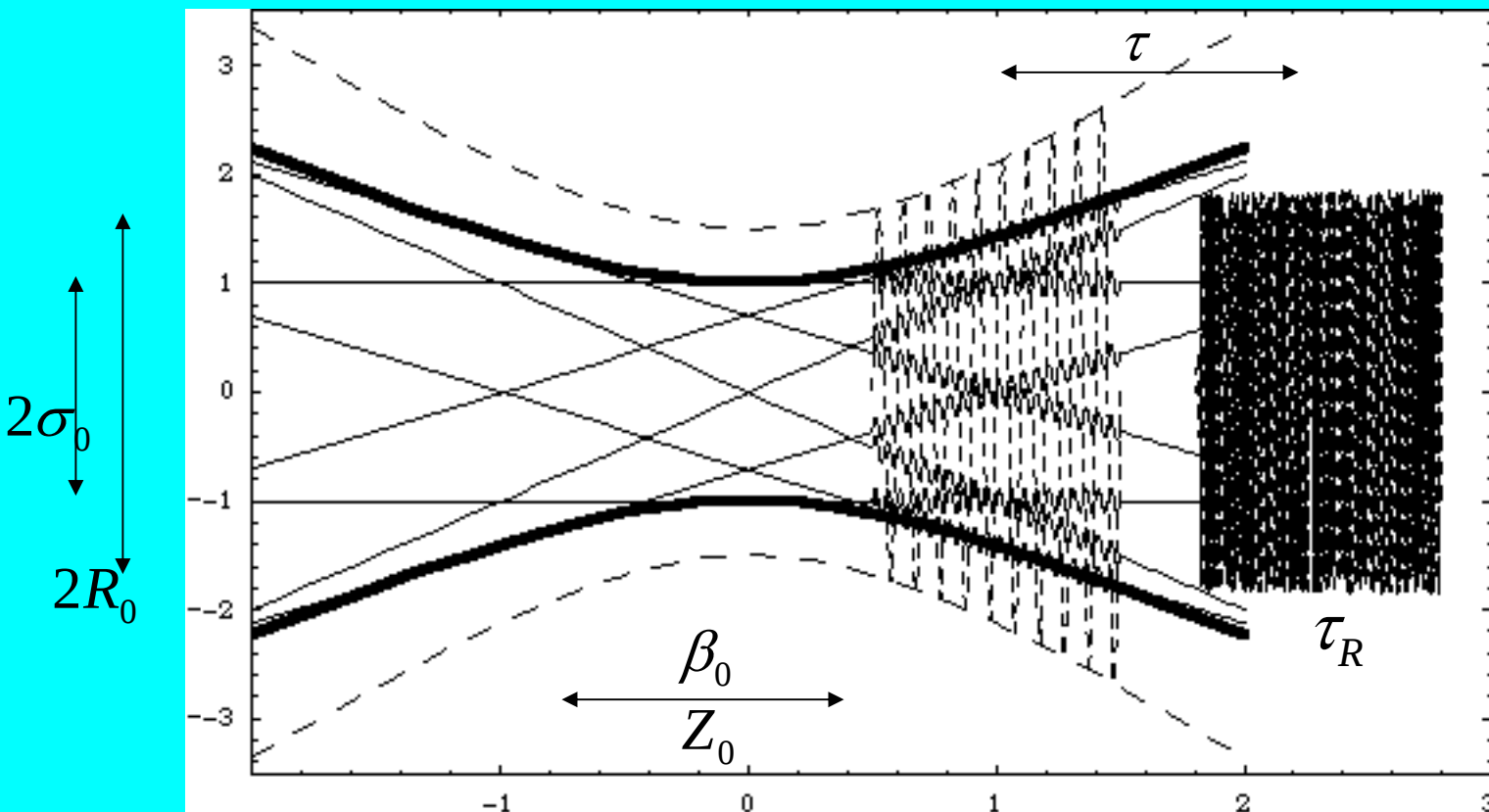
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Conditions to operate a Thomson Source in FEL mode $\lambda_R = \lambda \frac{(1 + a_0^2)}{4\gamma^2}$



Laser

$$a_0 = 8.5 \cdot 10^{-6} \frac{\lambda \sqrt{P}}{R_0}$$

$$Z_0 = \frac{4\pi R_0^2}{\lambda}$$

e-beam

$$\sigma_0 = \sqrt{\frac{\varepsilon_n \beta_0}{\gamma}} ; I ; \Delta\gamma/\gamma$$

FEL

$$\lambda_R = \lambda \frac{(1 + a_0^2)}{4\gamma^2}$$

$$\rho = \frac{10^{-2}}{\gamma} \sqrt[3]{I \lambda^4 P / \sigma_0^4}$$

$$L_g = \frac{\lambda}{4\pi\rho}$$

$$\frac{\Delta\lambda_R}{\lambda_R} = \rho$$

Conditions to operate a Thomson Source in FEL mode

Laser length

$$c\tau = 10L_g$$

*Beam-laser overlap
(laser uniform in x, y, z)*

$$R_0 = 2\sigma_0 \quad c\tau \leq 2Z_0$$

Laser ripples

$$\Delta \equiv \frac{\Delta a_0}{a_0} \quad \frac{\Delta\lambda_R}{\lambda_R} = \frac{2a_0^2}{1 + a_0^2} \Delta \quad \longrightarrow \quad \Delta \leq \rho \frac{1 + a_0^2}{2a_0^2}$$

Laser bandwidth

$$\frac{\Delta\lambda_R}{\lambda_R} = \frac{\Delta\lambda}{\lambda} = \frac{c\tau}{\lambda} \quad \longrightarrow \quad c\tau \geq \frac{\lambda}{\rho}$$

Conditions to operate a Thomson Source in FEL mode

FEL bandwidth broadening due to beam emittance

$$\lambda_R = \lambda \frac{(1 + a_0^2 + \gamma^2 \theta^2)}{4\gamma^2}$$

$$\frac{\Delta\lambda_R}{\lambda_R} \approx \frac{\gamma^2 \theta^2}{1 + a_0^2} \approx \frac{4\varepsilon_n^2}{\sigma_0^2} \begin{cases} \frac{\Delta\lambda_R}{\lambda_R} \leq \alpha\rho \\ \alpha \geq 1 \end{cases} \quad \longrightarrow \quad \varepsilon_n \leq \frac{1}{2} \sqrt{\alpha\rho} \sigma_0$$

$$L_g = \frac{\lambda}{4\pi\rho} \quad Z_R = \frac{4\pi\sigma_0^2}{\lambda_R} \quad \longrightarrow \quad \rho = \frac{Z_R}{L_G} \frac{\lambda_R \lambda}{8\pi^2 \sigma_0^2}$$

Generalized Pellegrini criterion

$$\varepsilon_n \leq \sqrt{\alpha} \sqrt{\frac{Z_R}{L_G}} \frac{\lambda_R \gamma}{2\sqrt{2}\pi}$$

Generalized Pellegrini criterion

$$\varepsilon_n \leq \sqrt{\alpha} \sqrt{\frac{Z_R}{L_G}} \frac{\lambda_R \gamma}{2\sqrt{2}\pi}$$

LCLS

$\lambda_R \cong 1$ Angstrom

FEL-Thomson

$$\gamma = 3 \cdot 10^4$$

$$\lambda_w = 2.5 \text{ cm}$$

$$L_g \cong Z_R \cong 10 \text{ m}$$

$$\varepsilon_n < \frac{3 \cdot 10^4 10^{-10}}{4\pi} \cong 0.25 \text{ } \mu\text{m}$$

$$\gamma = 30$$

$$\lambda = 1 \text{ } \mu\text{m}$$

$$L_g \cong 100 \text{ } \mu\text{m} ; Z_R \cong 10 \text{ m}$$

$$\varepsilon_n < \sqrt{10^5} \frac{30 \cdot 10^{-10}}{2\sqrt{2}\pi} \cong 0.11 \text{ } \mu\text{m}$$

Satisfying all conditions above implies:

$$\varepsilon_n = \frac{\lambda}{5.62}$$

$$U = 47 \lambda / \Delta^2$$

$$P = 3.1 \cdot 10^9 / \Delta$$

$$a_0 = 1.12$$

$$\gamma = 0.068 \sqrt[3]{\frac{I}{\Delta^2}}$$

$$\lambda_R = 120 \lambda \sqrt[3]{\frac{\Delta^4}{I^2}}$$

$$\tau = 1.5 \cdot 10^{-8} \lambda / \Delta$$

$$\rho = 0.18 \Delta$$

$$\sigma_0 = 0.21 \lambda / \sqrt{\Delta}$$

$$L_G = 0.44 \lambda / \Delta$$

$$\bar{\rho} = 6 \cdot 10^{11} \lambda \sqrt[3]{\frac{\Delta^5}{I}}$$

$$Z_0 = 2.2 \lambda / \Delta$$

$$\beta_0 = 0.018 \lambda \sqrt[3]{\frac{I}{\Delta^5}}$$

For the case of a Ti:Sa laser we derive:

$$\varepsilon_n = 0.14 \text{ } \mu m$$

$$\gamma = 1.47 \sqrt[3]{\frac{I}{\Delta^2}}$$

$$L_G = 35 / \Delta \text{ } [\mu m]$$

$$U = 0.37 / \Delta^2 \text{ } [J]$$

$$\lambda_R = 2063 \Delta \sqrt[3]{\frac{\Delta}{I^2}} \text{ } [Angstrom]$$

$$\bar{\rho} = 222 \Delta \sqrt[3]{\frac{\Delta^2}{I}}$$

$$P = 0.31 / \Delta \text{ } [TW]$$

$$\tau = 1.2 / \Delta \text{ } [ps]$$

$$Z_0 = 0.18 / \Delta \text{ } [mm]$$

$$a_0 = 1.12$$

$$\rho = 1.8 \cdot 10^{-3} \Delta$$

$$\beta_0 = \frac{0.03}{\Delta} \sqrt[3]{\frac{I}{\Delta^2}} \text{ } [mm]$$

$$\sigma_0 = 1.7 / \sqrt{\Delta} \text{ } [\mu m]$$

$$\Delta \text{ } [\%] ; I \text{ } [A]$$

Setting $\Delta = 0.5 \%$ and $I = 1700 \text{ A}$

Classical SASE

$$\varepsilon_n = 0.14 \text{ } \mu\text{m}$$

$$\gamma = 28$$

$$L_G = 70 \text{ } \mu\text{m}$$

$$U = 1.5 \text{ J}$$

$$\lambda_R = 5.7 \text{ Angstrom}$$

$$\bar{\rho} = 6$$

$$P = 3.1 \text{ TW}$$

$$\tau = 2.4 \text{ ps}$$

$$Z_0 = 0.36 \text{ mm}$$

$$a_0 = 1.12$$

$$\rho = 9 \cdot 10^{-4}$$

$$\beta_0 = 1.14 \text{ mm}$$

$$\sigma_0 = 2.4 \text{ } \mu\text{m}$$

Setting $\Delta = 0.1 \%$ and $I = 2500 \text{ A}$
Quantum FEL

$$\varepsilon_n = 0.14 \text{ } \mu\text{m}$$

$$\gamma = 93$$

$$L_G = 350 \text{ } \mu\text{m}$$

$$U = 37 \text{ J}$$

$$\lambda_R = 0.52 \text{ Angstrom}$$

$$\bar{\rho} = 0.35$$

$$P = 0.6 \text{ TW}$$

$$\tau = 12 \text{ ps}$$

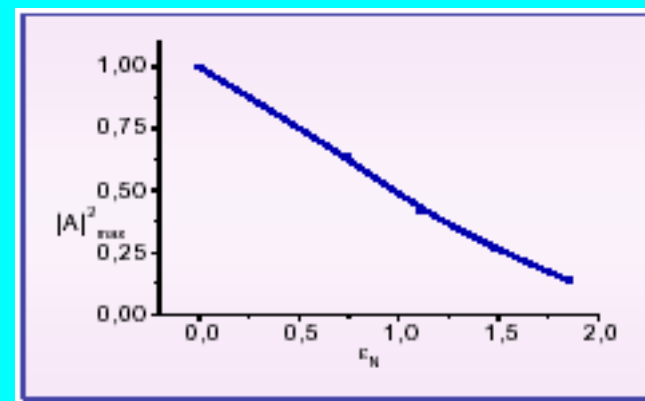
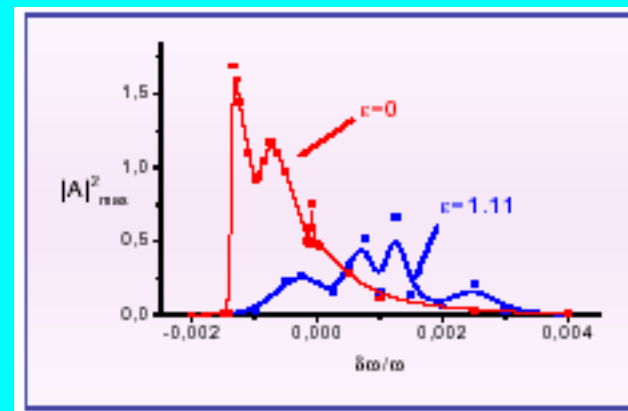
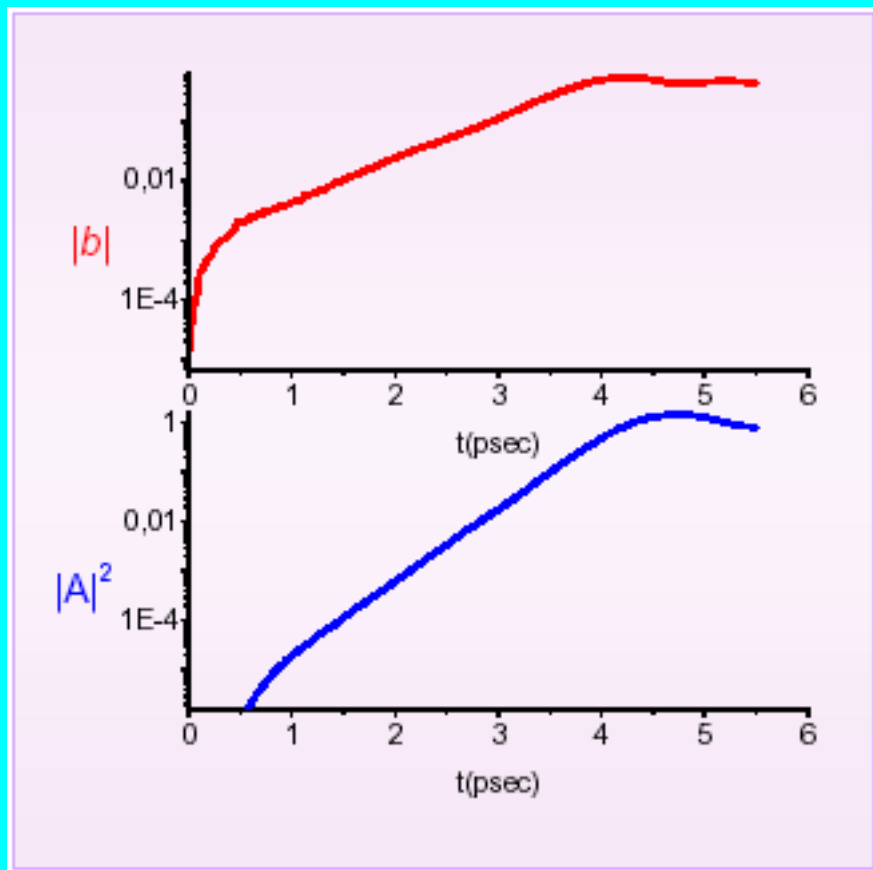
$$Z_0 = 1.8 \text{ mm}$$

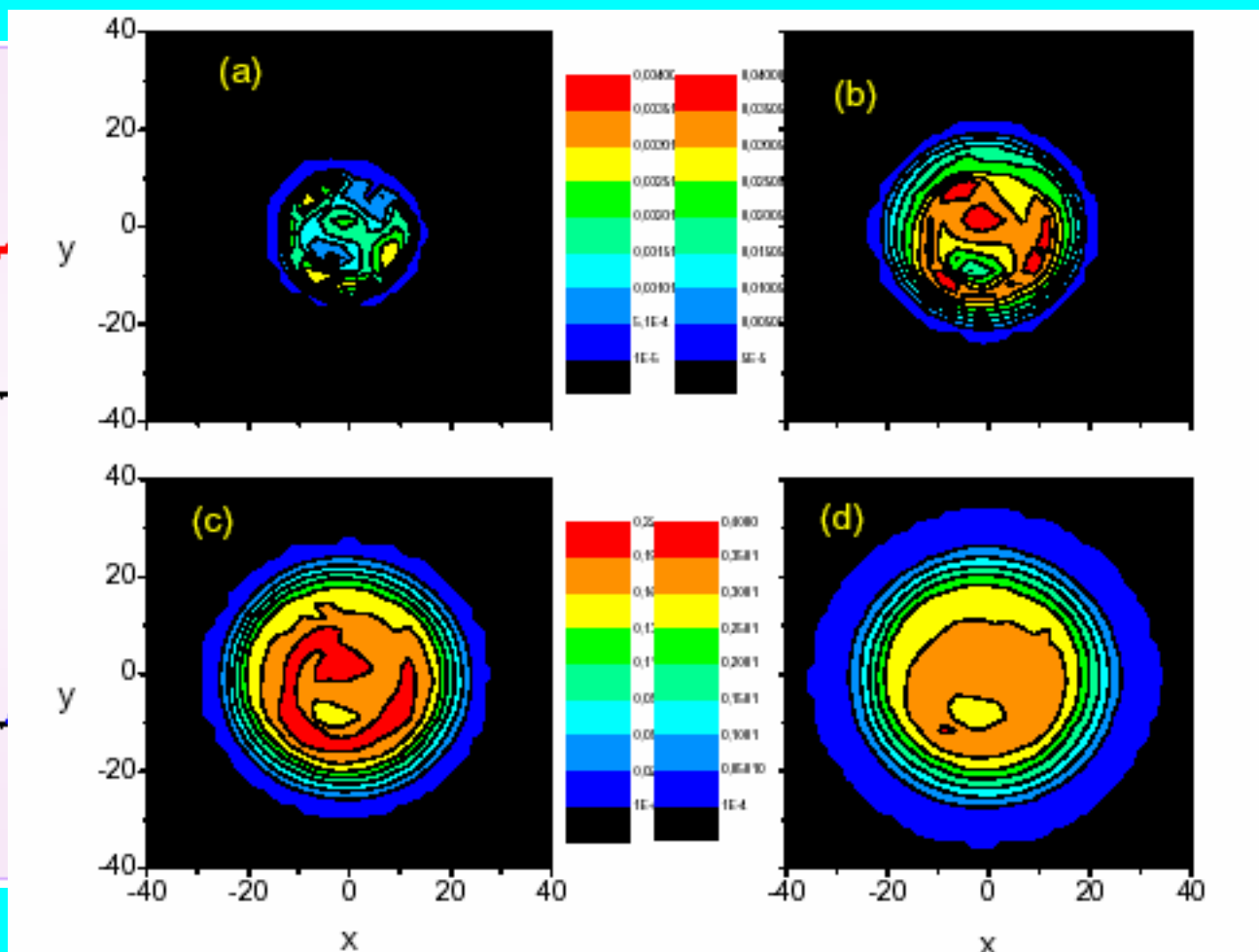
$$a_0 = 1.12$$

$$\rho = 1.8 \cdot 10^{-4}$$

$$\beta_0 = 18.8 \text{ mm}$$

$$\sigma_0 = 5.4 \text{ } \mu\text{m}$$

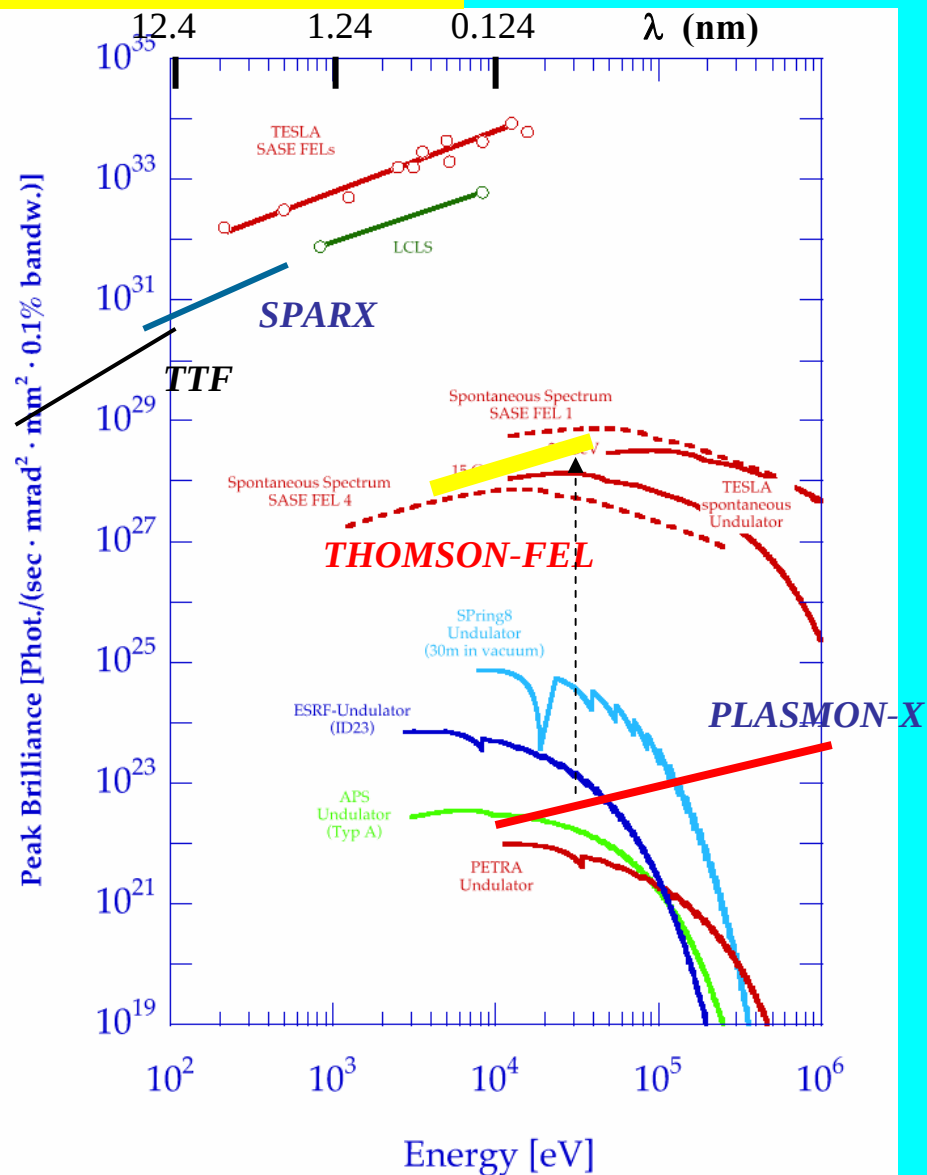




**up to $2 \cdot 10^{10}$ photons per pulse @ 6 keV,
emitted in a coherent diffraction limited radiation beam, $\Delta\theta = 3 \mu\text{rad}$
(cmp. 10^9 ph/pulse in $\Delta\theta = 3 \text{ mrad}$ of incoherent Thomson radiation)**

Compact Thomson Sources
extend SR to hard X-ray range
allowing
Advanced Radiological
Imaging **inside** Hospitals

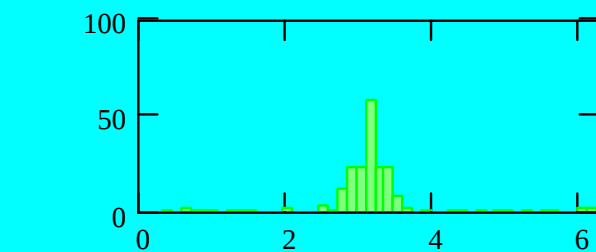
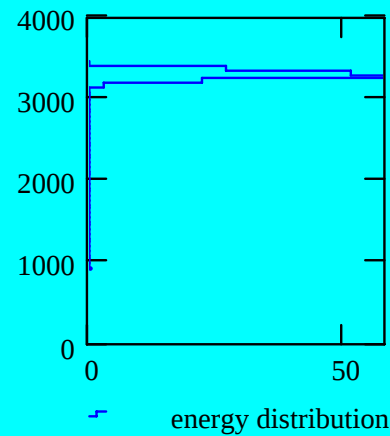
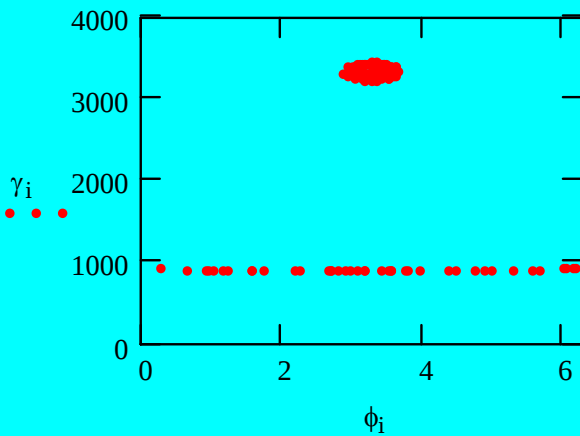
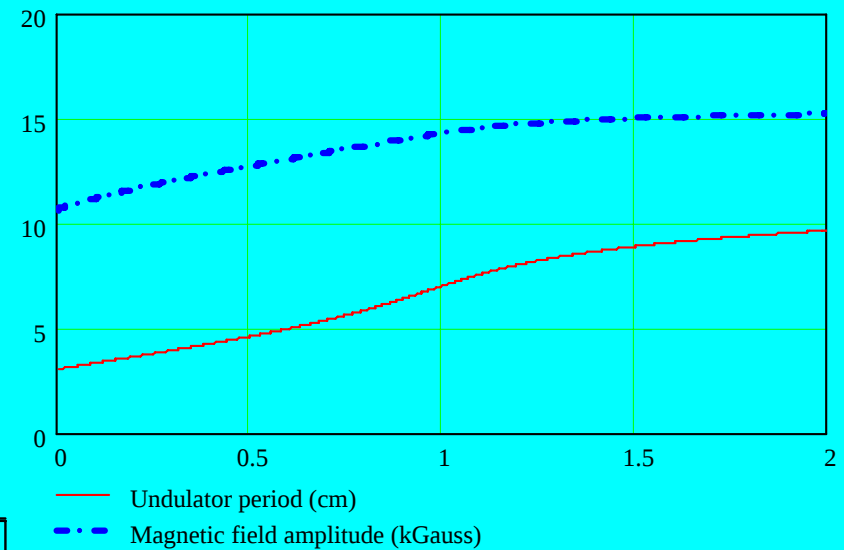
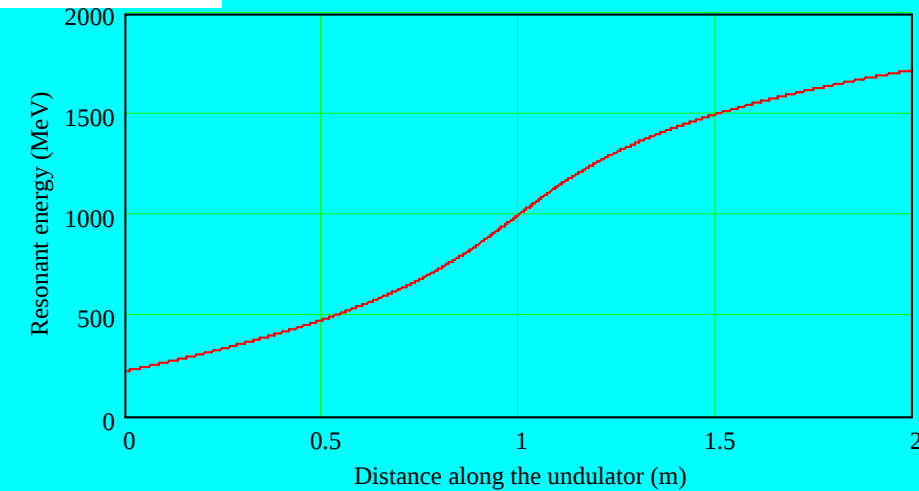
Coherent Thomson Sources
will push the
achievable brilliance
if laser pulses with
 $\Delta I/I < \Delta\omega/\omega = 2 \cdot 10^{-3}$
will be made available



GeV-class IFEL design:

- Application of IFEL scheme as 4th generation light source driver
- Compact-size accelerator
- ESASE benefits intrinsic
 - Exponential gain length reduction
 - Absolute timing synchronization with external laser
 - Control of x-ray radiation pulse envelope
- Advanced Accelerator driven light source
- Design exercise aimed to extend the energy and wavelength reach of planned SPARC linac

Initial e-beam energy (γ value)	210 MeV
Initial e-beam intrinsic energy spread	0.1% (1σ)
Initial e-beam current	1 kA
Laser wavelength	800 nm
Laser peak power	20 TW
Nominal length of wiggler, L_w	200 cm

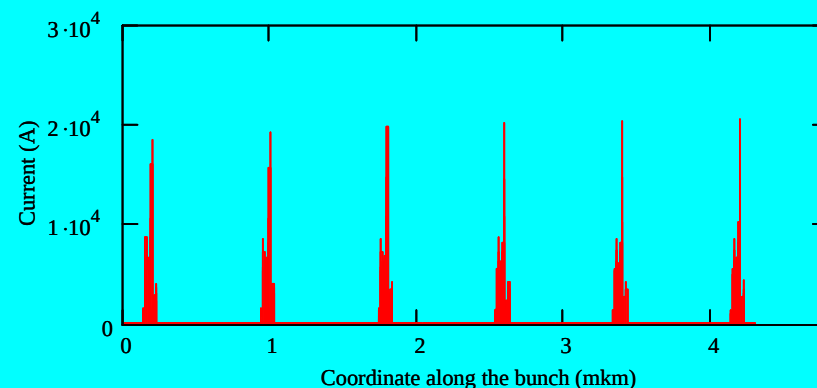
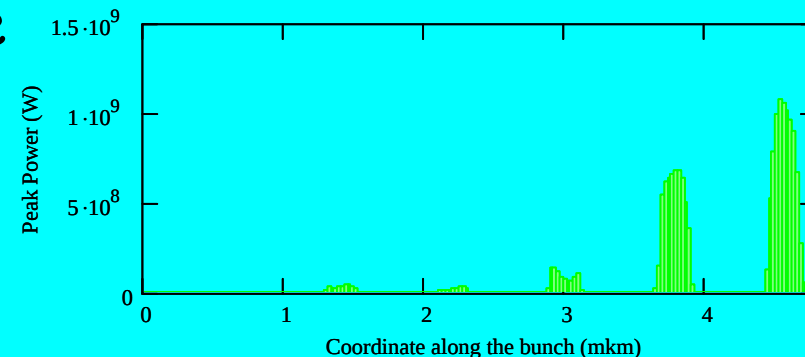


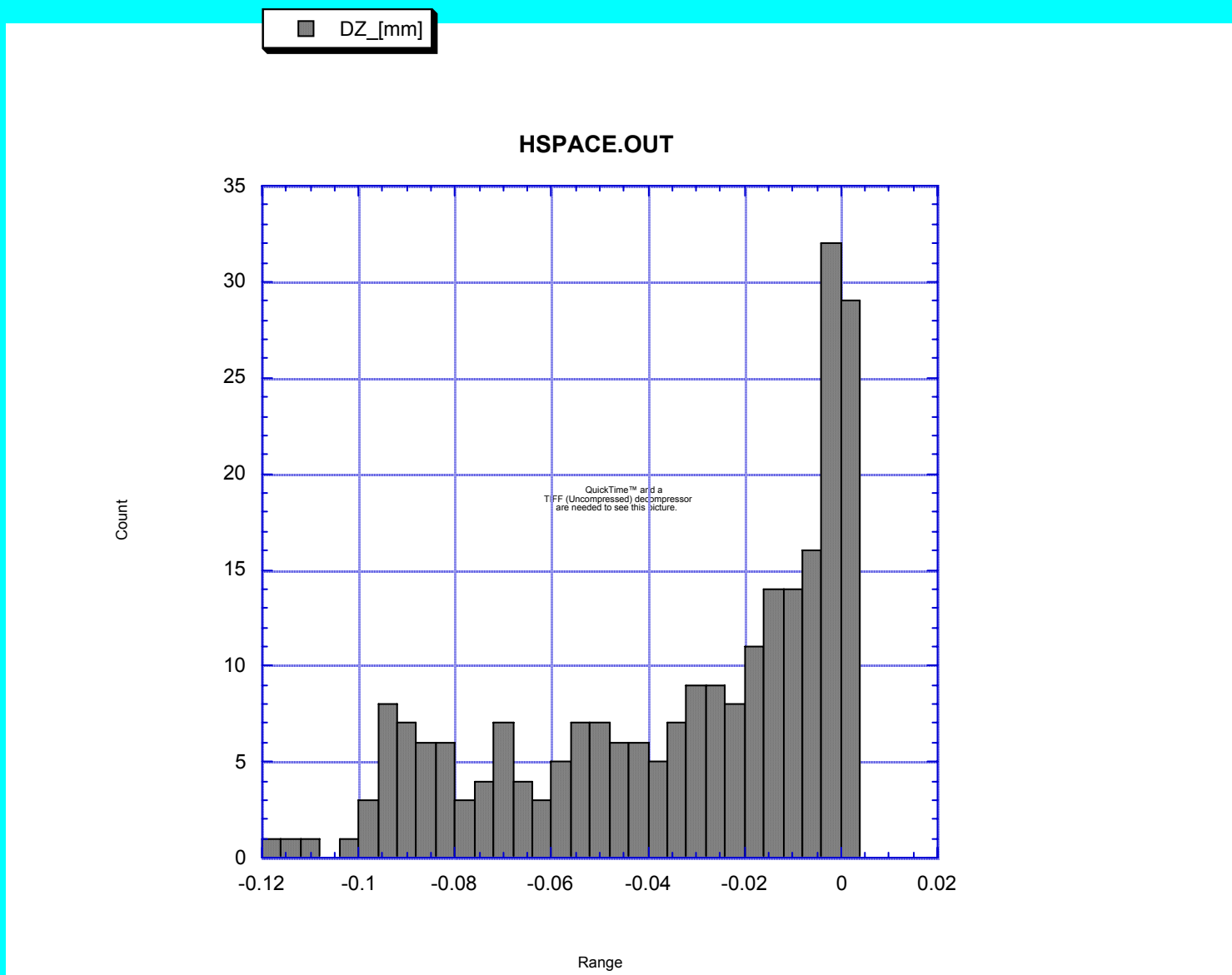
Final output parameters

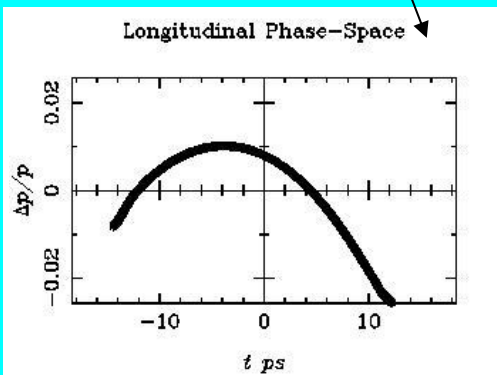
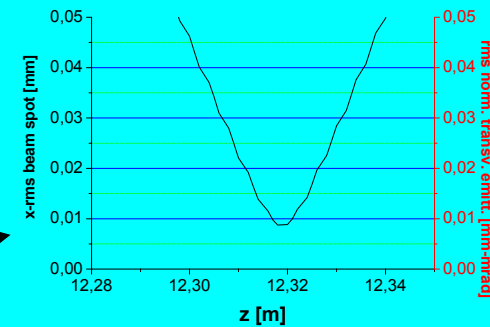
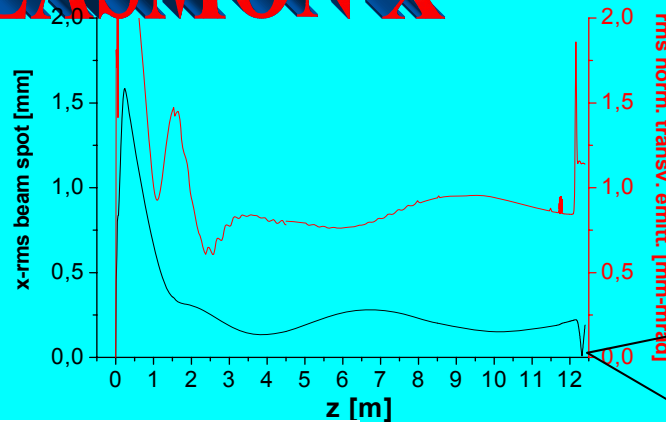
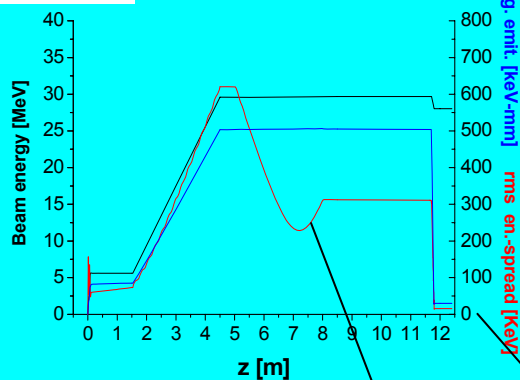
Energy	1.7 GeV
Energy spread	<0.8 %
Microbunch length	250 as
Peak current	> 6 kAmp
Avg. gradient	750 MeV/m

Conclusions

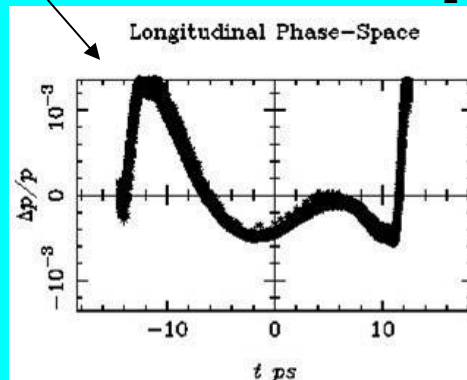
- Sending such a beam into an undulator
 - FEL radiation @ $\lambda = 3$ nm (water window). Peak power 1 GW in 300 attoseconds
- Among laser accelerators, the IFEL offers best control of the longitudinal phase space.
- Preserving ultrashort pulse structure in the radiation output requires some precautions in the design of the FEL amplifier (slippage problems) but can be done.
- Path towards ultrashort probe beams pass through a synergy between laser and accelerator worlds.



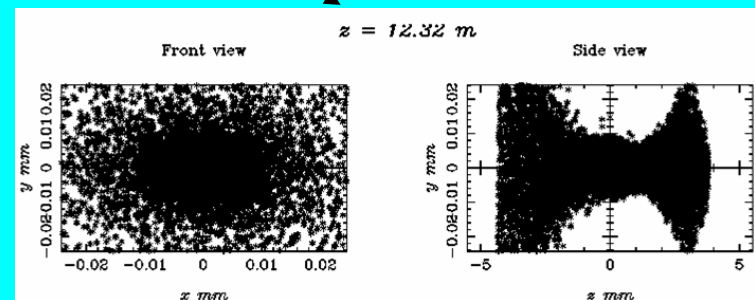




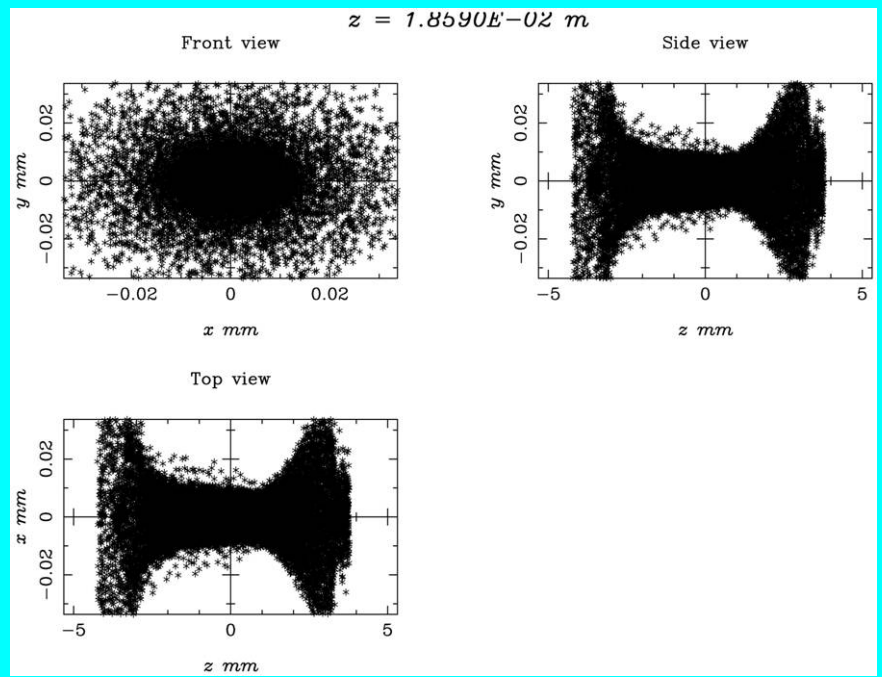
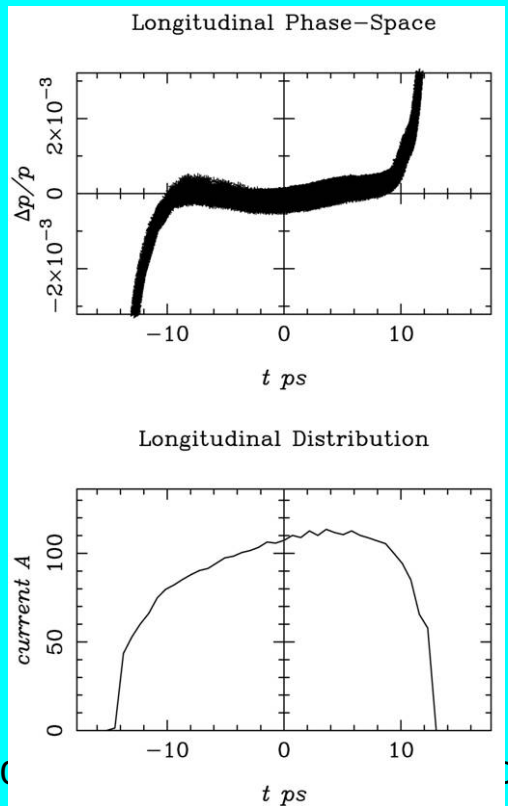
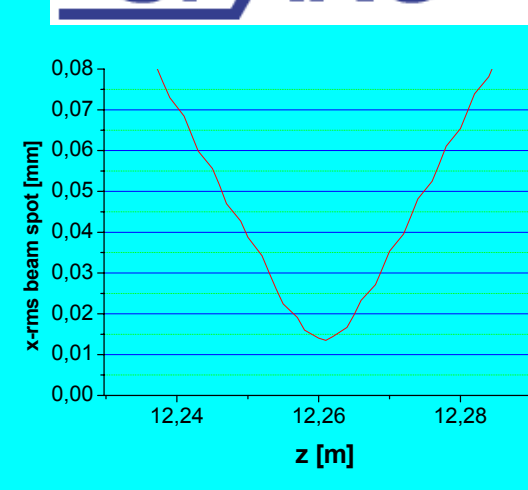
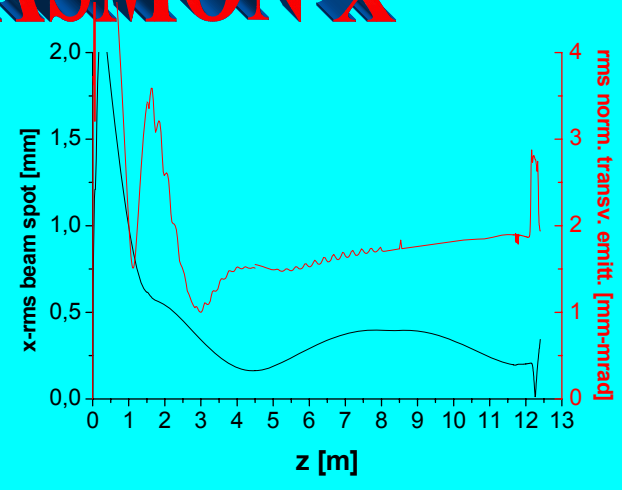
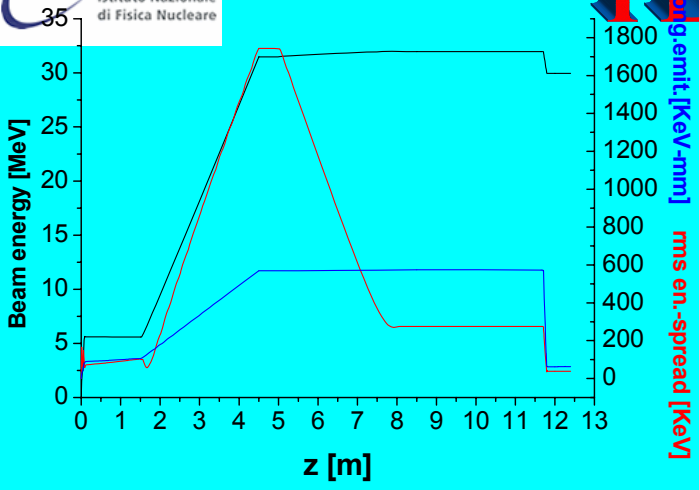
Linear correlation correction, using the second TW acceleration structure: $E_z = 2.70 \text{ MV/m}$, $\Phi_i = 85^\circ$



Quadratic correlation correction, using a dedicated x-band structure : $E_z = 30 \text{ MV/m}$, $\Phi_i = 275^\circ$



The results seem to be good especially for the very low longitudinal energy spread. Keeping in consideration that the beta in the focal spot is about 5 mm, the principal limitation is due to bunch length.

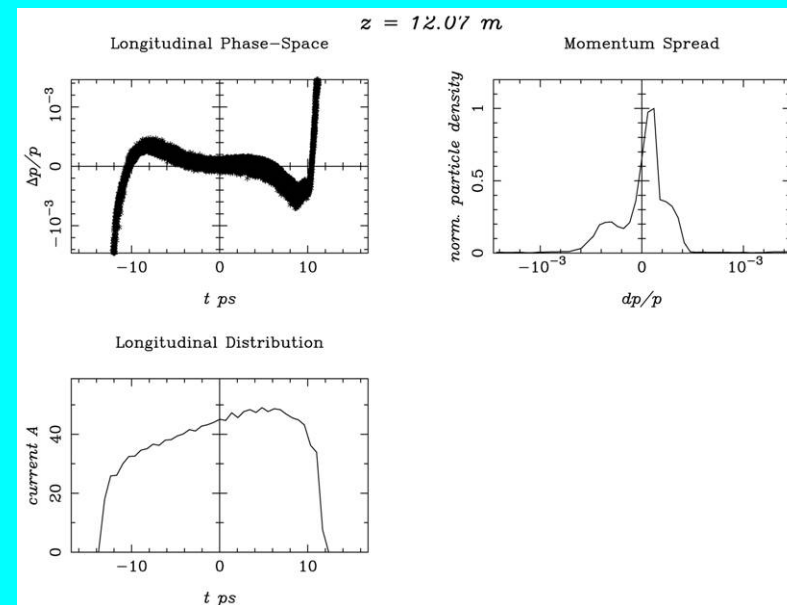
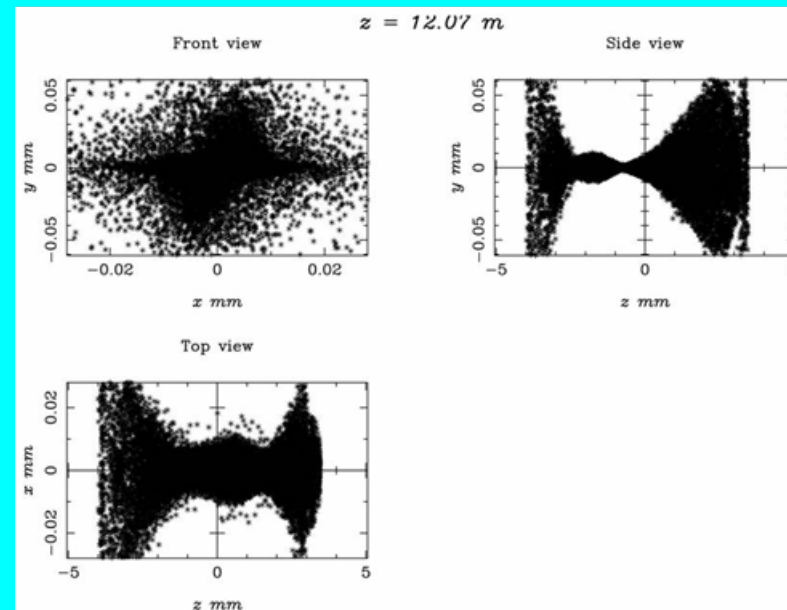
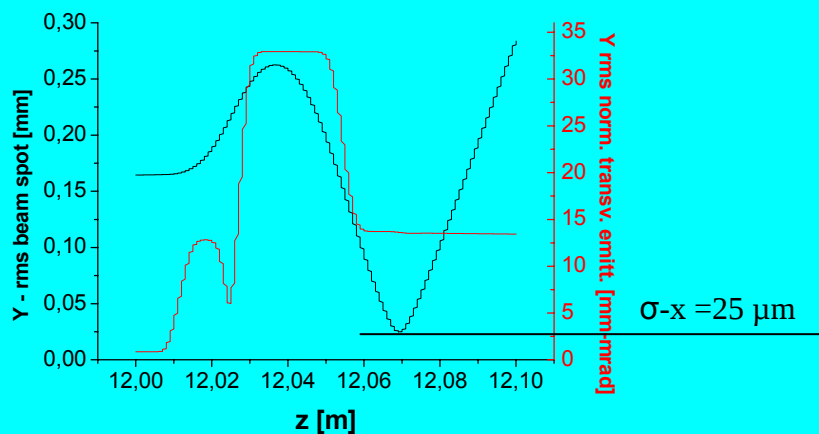
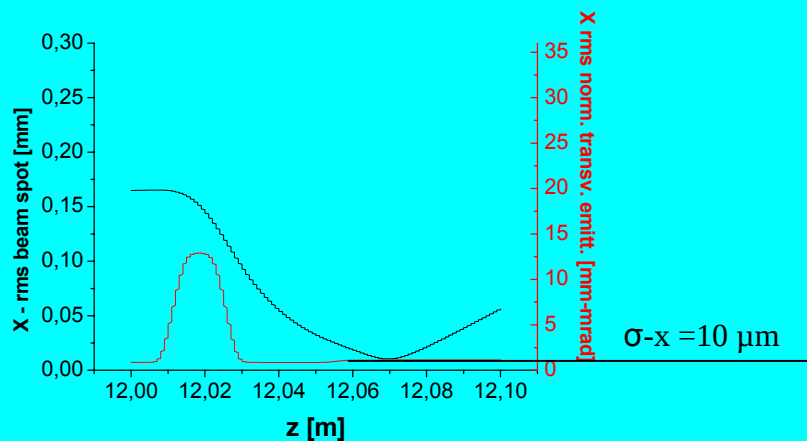


Triplet configuration:

Ideal +1 -2 +1

Q-length = 1.46 cm

Q-Gradient = 300 T/m

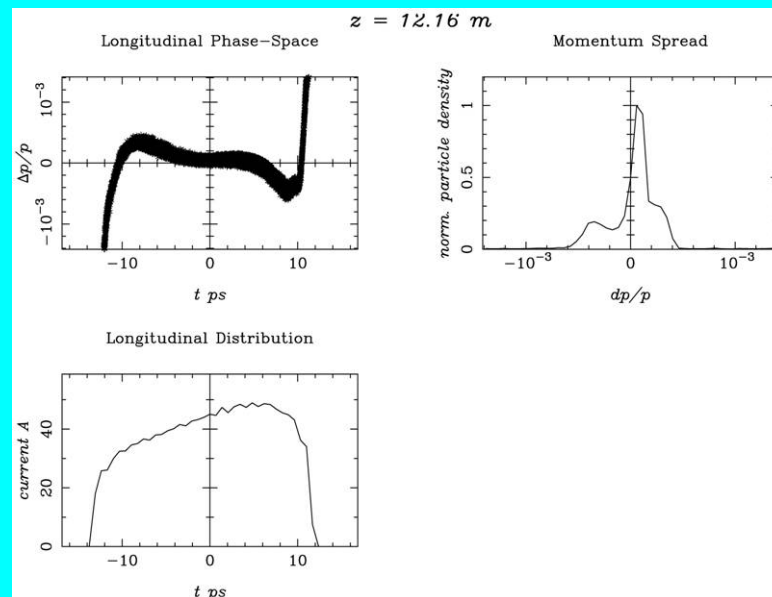
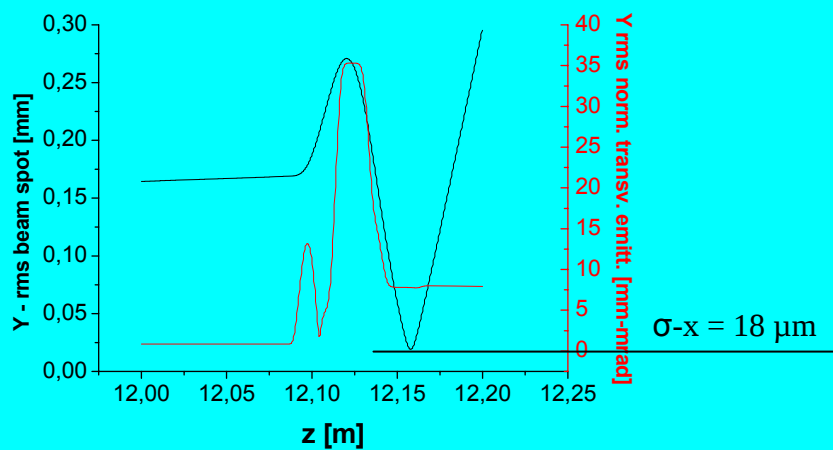
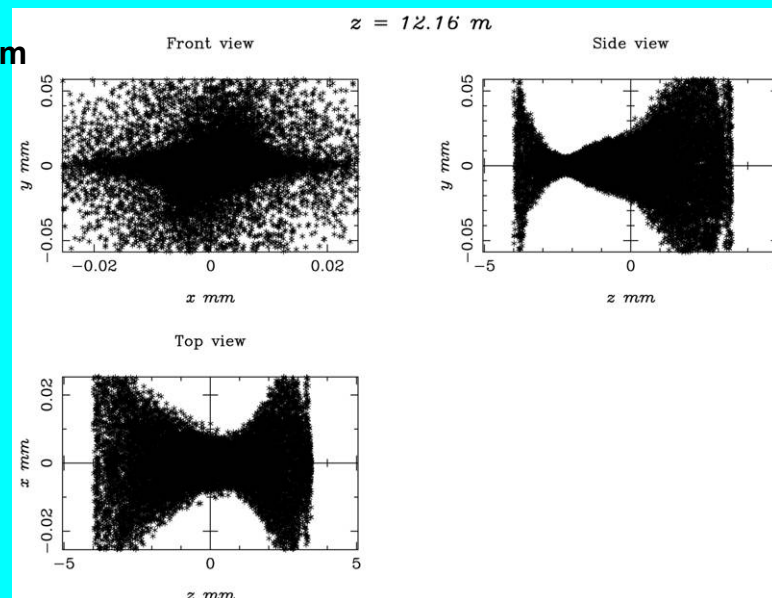
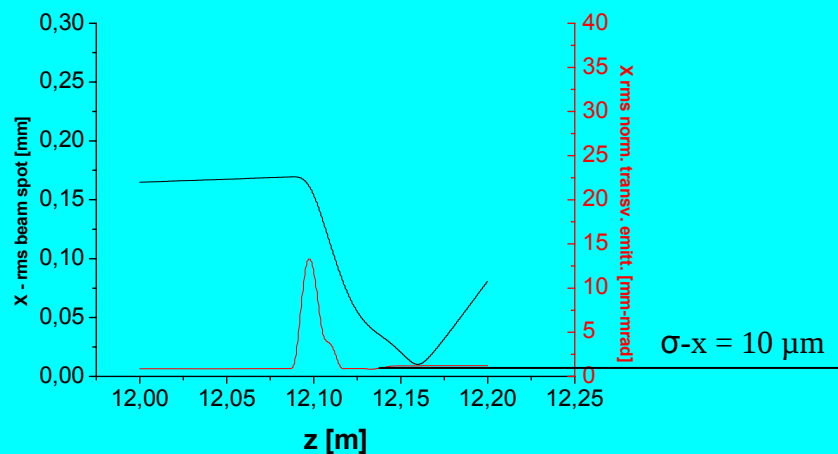


Triplet configuration:

Q1-length = 1.00 cm ; Q2-length = 2.00 cm ; Q3-length = 2.00 cm

Gradient = +300 ; -300 ; +300 (T/m)

Position, Q1 = 12.0968 m; Q2 = 12.12305 m; Q3 = 12.1511 m

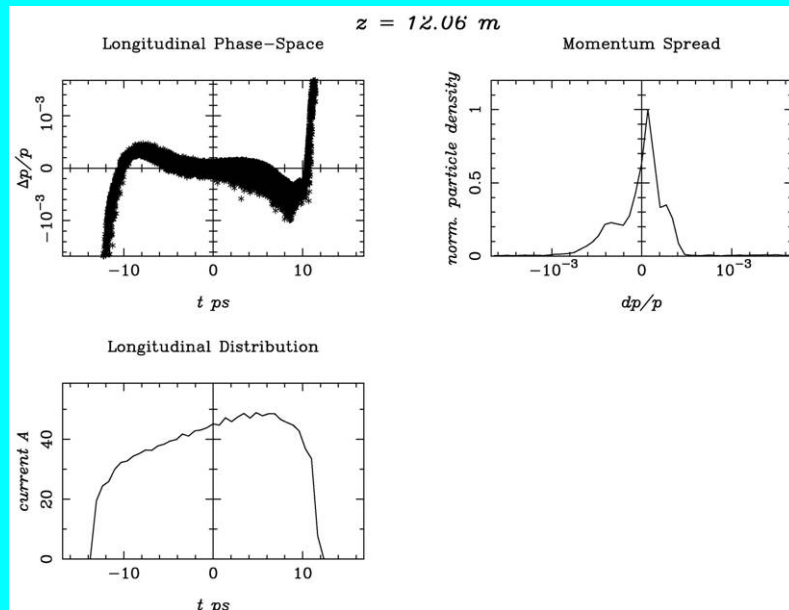
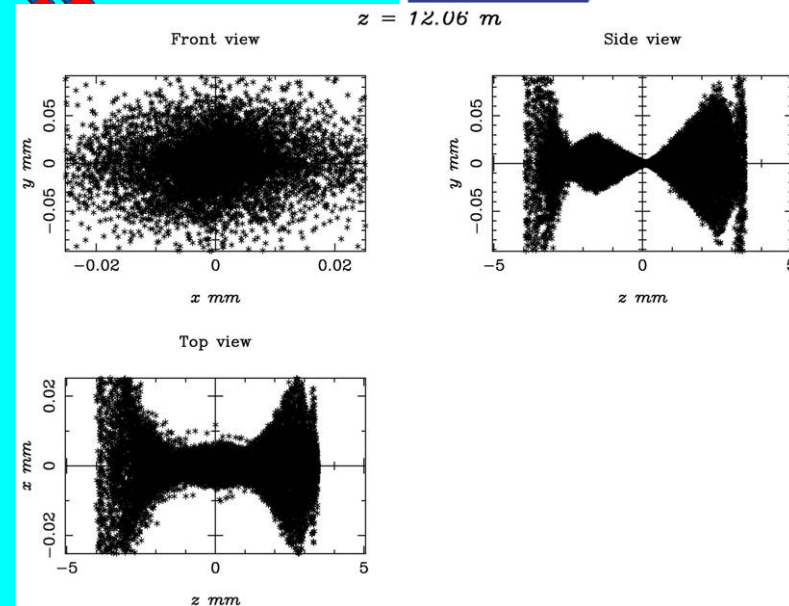
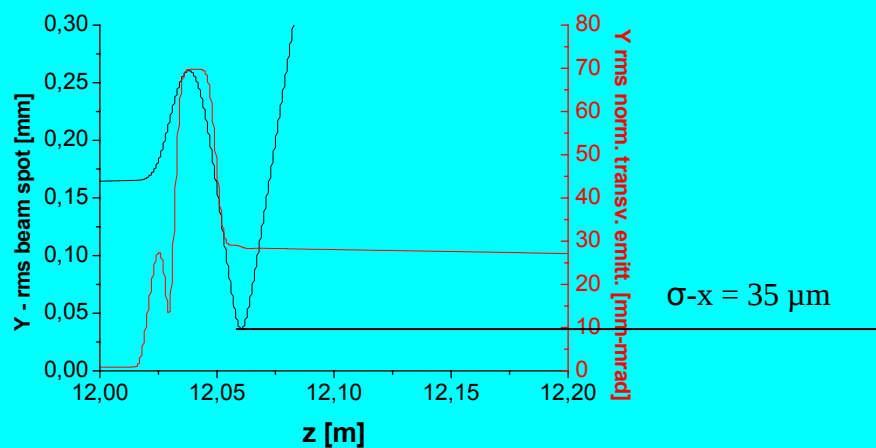
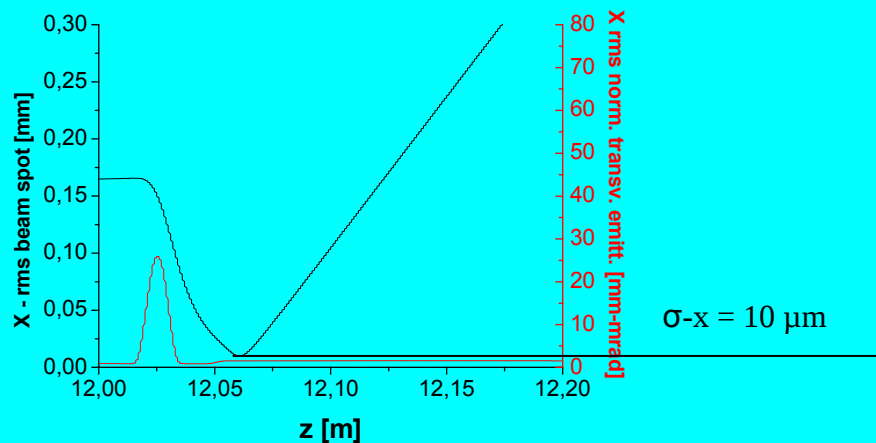


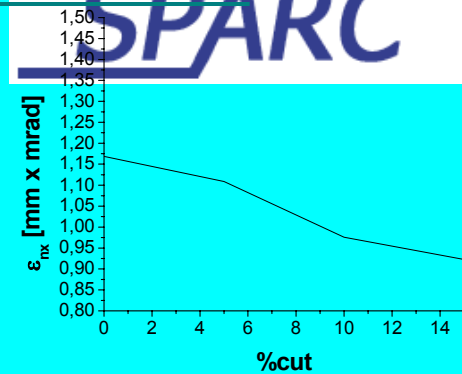
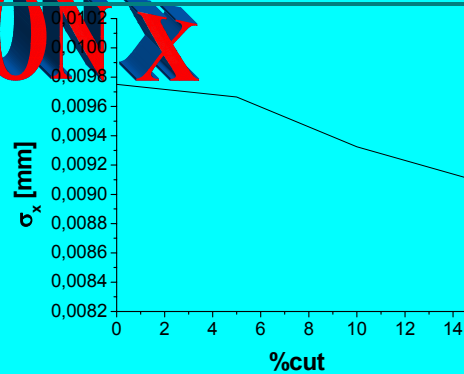
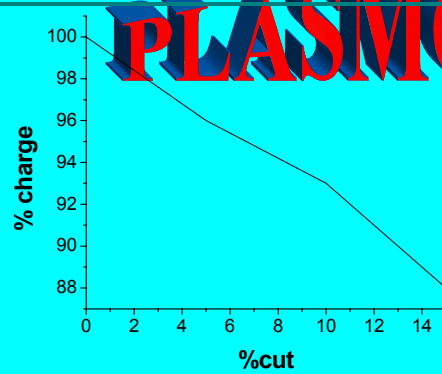
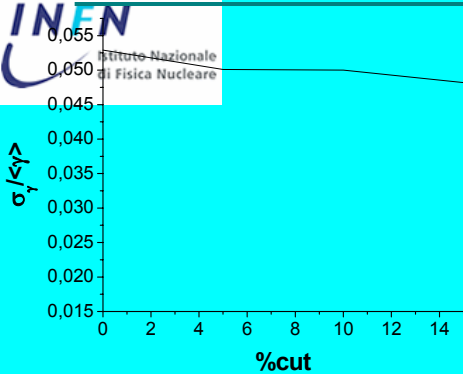
Triplet configuration- thin lens:

Ideal +1 -2 +1

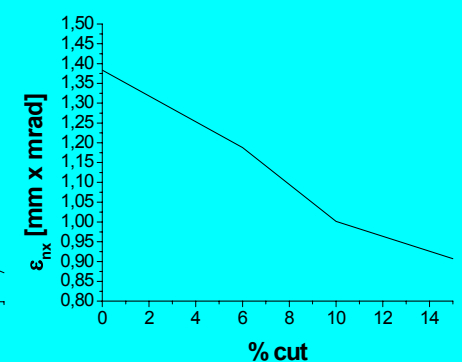
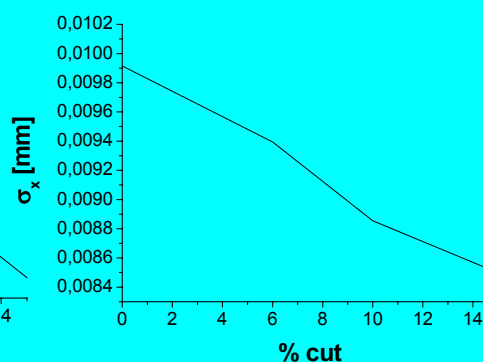
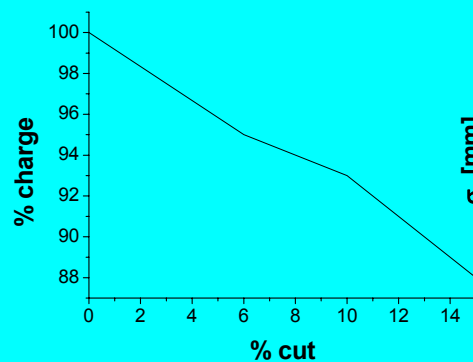
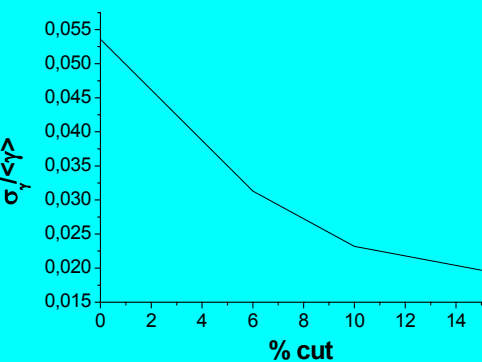
Q-length = 1.00 cm

Q-Gradient = 645 T/m

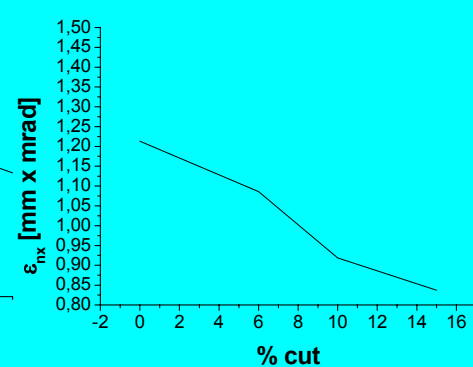
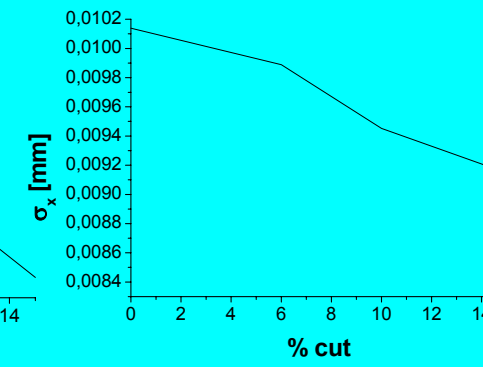
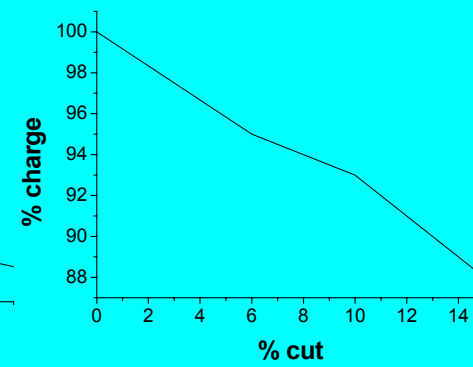
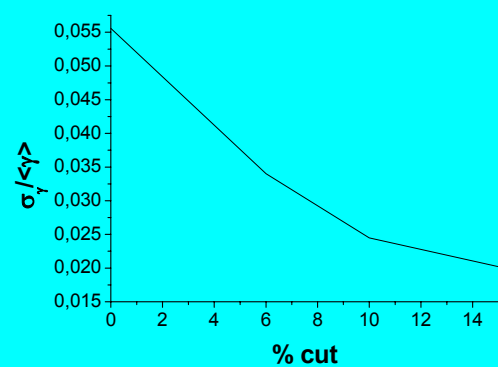




2005 best results (1nC Solenoid lens)



best results with triplet (1nC Solenoid lens) – ONLY X PROJECTION

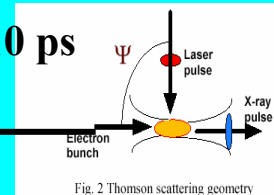


30-50 fs synchr.
Ti:Sa 200-TW
Laser System

$$\varepsilon_n = 2 \mu\text{m}, \sigma = 5 \mu\text{m}, \Delta\gamma/\gamma = 4 \cdot 10^{-4}$$

1 J, 10 ps gaussian

2 nC, 10 ps



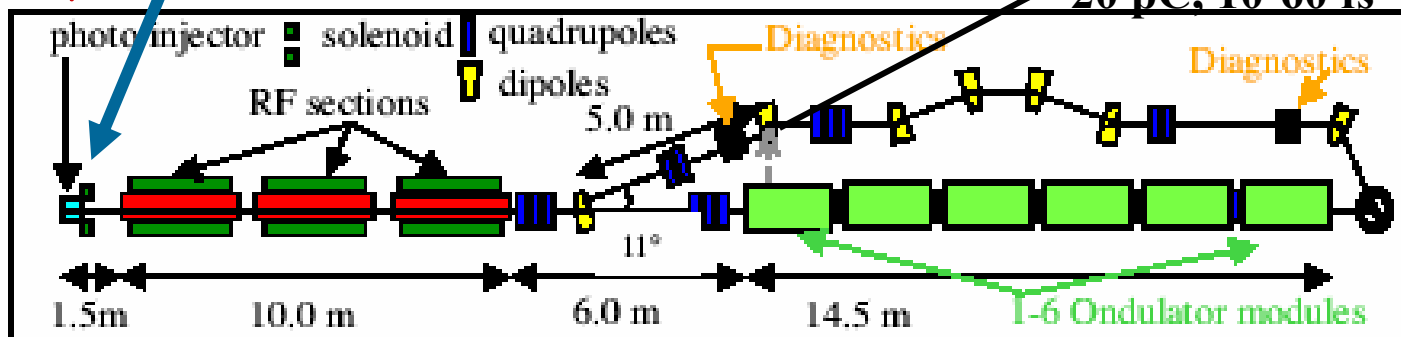
Compr.

1 J, 100 fs gaussian

20 mJ, 10 ps flat top

500 μJ

20 pC, 10-60 fs

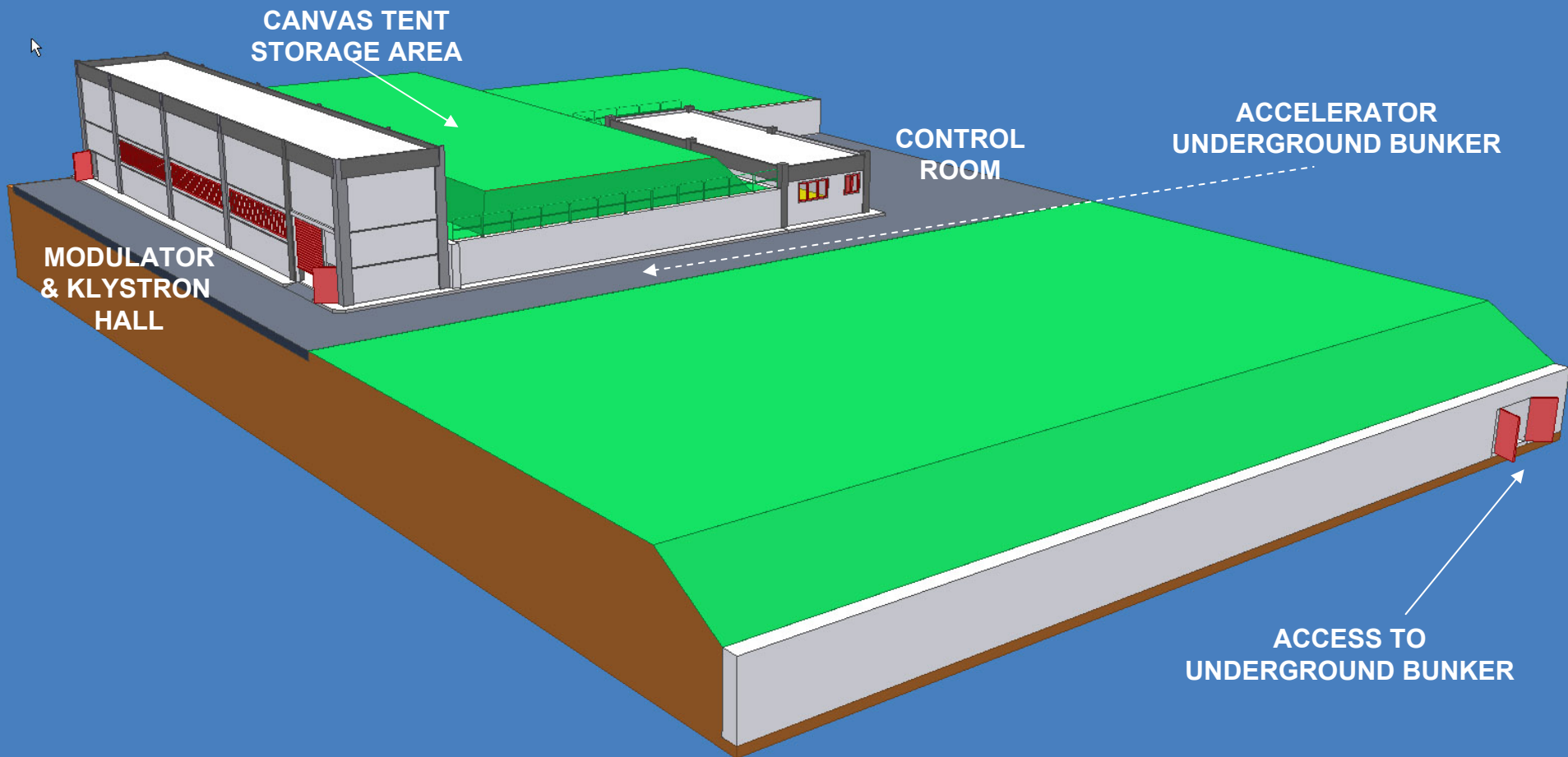


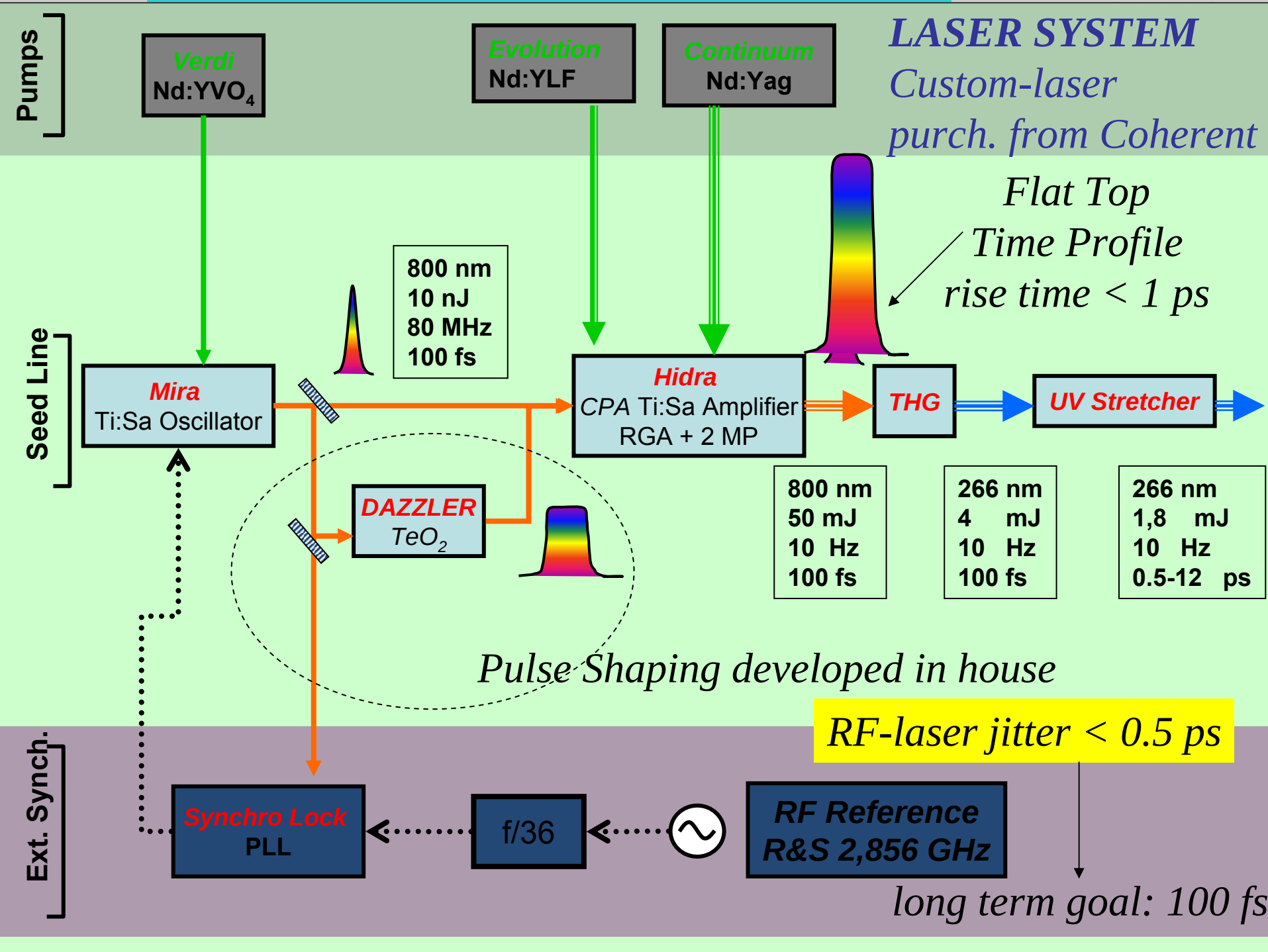
$$\varepsilon_n = 0.2 \mu\text{m}, \sigma = 10 \mu\text{m}$$

1 nC, 10 ps, $\varepsilon_n = 1 \mu\text{m}$

**Two additional beam lines at SPARC for plasma acceleration
and monochromatic X-ray beams**

A 3D Model View of the buildings from outside





Schedule of the planned PLASMONX activity

Activity/Year	2004	2005	2006	2007	2008	2009
Set up cluster for parallel computing						
Test experiments @ CEA - Saclay						
Development of m.w. PIC codes						
Modelling of acceleration schemes						
Modelling of T.S.						
Design of Laser System						
Set up LASER LABORATORY at LNF						
Set up Phase 1: 30fs, 0.1TW			M			
Set up I II amplification stage						
Development of the OPCPA amplifiers						
Installation of laser diagnostics						
Phase 2: 1TW compression test				M		
IV amplification stage						
Phase 3: 100 TW compression test					M	
Self injection LWFA experiments						
External injection LWFA test						M
Set up additional beam-line at LNF						
Set up laser beam transport to LINAC						
Set up laser beam-electron bunch interaction						
Synchronization R&D and setup						
Thomson source test						M
TS beam users (MAMBRO, etc.)						

**2.5 Meuro bid is ongoing
for the FLAME laser**

*QFEL experim. under study
with PLASMONX facility*

With these parameters, $\rho = \frac{1}{\gamma_0} \left(\frac{\omega_b^2 a_{LO}^2}{16\omega_L^2} \right)^{\frac{1}{3}} = 7 \cdot 10^{-4}$.

The gain length $L_g = 110$ micron, and the system is classic

$$\mathbf{A}_L(\underline{r}, t) = \frac{a_{LO}}{\sqrt{2}} (g(\underline{r}, t) e^{-i(k_L z + \omega_L t)} \hat{\mathbf{e}} + cc) + O\left(\frac{\mathcal{A}_L}{w_0}\right)$$

with

$$\hat{\mathbf{e}} = (\mathbf{e}_x + i\mathbf{e}_y) / \sqrt{2}$$

And with envelope

$$g(\underline{r}, t) = \Phi(z + ct) \frac{1 + i \frac{z}{z_0}}{1 + \frac{z^2}{z_0^2}} \exp \left[-4 \frac{x^2 + y^2}{w_0^2 \left(1 + \frac{z^2}{z_0^2}\right)} - 4i \frac{x^2 + y^2}{w_0^2 \left(\frac{z}{z_0} + \frac{z_0}{z}\right)} \right]$$

The 3 dimensional collective equations that describes the collective effects are similar to the FEL equation with the SVEA approximation

$$\begin{aligned}
 \frac{d}{d\bar{t}} \bar{\mathbf{r}}_j(\bar{t}) &= \rho \frac{\mathbf{P}_j(\bar{t})}{\bar{\gamma}_j(\bar{t})} \\
 \frac{d}{d\bar{t}} P_{jz}(\bar{t}) &= -\frac{\bar{a}_{L0}^2}{2\rho\gamma_0^2} \frac{1}{\bar{\gamma}_j} \left[\frac{\partial}{\partial \bar{z}} |g|^2 \right]_{\bar{\mathbf{x}}=\bar{\mathbf{r}}_j} - \\
 &\quad \frac{2}{\bar{\gamma}_j} \text{Re} \left[(g^* \bar{A})_{\bar{\mathbf{x}}=\bar{\mathbf{r}}_j} e^{i\theta_j(\bar{t})} \right] + \dots \\
 \frac{d}{d\bar{t}} \mathbf{P}_{j\perp}(\bar{t}) &= -\frac{\bar{a}_{L0}^2}{2\rho\gamma_0^2} \frac{1}{\bar{\gamma}_j} \left[\nabla_{\perp} |g|^2 \right]_{\bar{\mathbf{x}}=\bar{\mathbf{r}}_j} \\
 &\quad - \frac{4 \frac{k_L}{k} \rho}{1 + \frac{k_L}{k}} \frac{1}{\bar{\gamma}_j} \text{Im} \left[(\nabla_{\perp} (g^* \bar{A}))_{\bar{\mathbf{x}}=\bar{\mathbf{r}}_j} e^{i\theta_j(\bar{t})} \right] + \dots \\
 \left(\frac{\partial}{\partial \bar{t}} + \frac{\partial}{\partial \bar{z}} \right) \bar{A}(\bar{\mathbf{x}}, \bar{t}) - i \frac{k_L}{k} \rho \bar{\nabla}_{\perp}^2 \bar{A} &= b
 \end{aligned}$$