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ANALYTICAL AND NUMERICAL COMPARISON OF DIFFERENT APPROACHES TO THE DESCRIPTION OF SASE IN HIGH GAIN FELs

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Abstract

Correlation function theory which has been developed recently gives rigorous statistical description of the SASE FEL operation. It directly deals with the values averaged over many shots. There are two other approaches which are based either on Vlasov equation or on direct solution of particle motion equations. Both of them perform calculations for some particular initial conditions. After that one can either consider the result as a "typical" sample, or repeat calculations for other initial conditions and then average the results. To check the validity of these three approaches it might be interesting to compare them with each other. In this paper we present the results of such comparison obtained for the 1-D FEL model. We show that two-particle correlation function approximation is equivalent to the quasilinear approximation for the Vlasov equation approach. These two approximations are in a good agreement with the results of direct solution of particle motion equations at linear and early saturation stages. To obtain this agreement at strong saturation, high order harmonics in Vlasov equation have to be taken into account, which corresponds to taking into account of three and more particle correlations in the correlation function approach.

INTRODUCTION

SASE FELs are widely used now as bright sources of coherent radiation in X-ray region [1]. Radiation of such FELs is a result of initial inhomogeneity of particle density, and therefore its parameters fluctuate significantly from shot to shot and within one pulse. To determine these parameters in a single shot one has to solve particle motion equations together with Maxwell equations. For a real shot it is not possible not only because of large number of particles but also because of unknown initial conditions. However, parameters averaged over many shots can be found by standard methods of statistical mechanics. Recently the correlation function theory, which deals with such parameters, was developed based on BBGKY chain of equations. Detailed description of this theory for general case is given elsewhere [2]. In this paper we shell consider simple 1-D case where the theory can be verified by comparing with other approaches based on direct solution of motion equations and solution of Vlasov equation for random smoothed density distribution in one-particle phase space.

BASIC EQUATIONS AND NUMERICAL SOLUTION ALGORITHM

The 1-D approximation is widely used in the FEL theory [3]. Despite the fact that it does not have rigorous foundation, it gives a good qualitative description of the physical processes taking place in single pass FEL.

Equations of Motion

Interaction of two particles in FEL occurs mainly through radiation. Knowing the particle trajectory one can calculate its radiation field and formally substitute it to the motion equations. But, unlike the case of the Coulomb interaction, the resulting set of equations will contain retardation. Fortunately some approximations relevant to FEL radiation field allow to eliminate retardation by special choice of independent variable and to obtain a set of ordinary differential equations. In 1-D case the resulting set of equations can be written in the following simple form:

$$\frac{dz^{(k)}}{d\theta}=1+2\Delta^{(k)};\quad \frac{d\Delta^{(k)}}{d\theta}=\frac{1}{N_\lambda}\sum_{z^{(l)}<z^{(k)}}\Phi\left(z^{(k)},z^{(l)}\right). \quad (1)$$

where $z^{(k)}$ and $\Delta^{(k)}$ - longitudinal coordinate and relative energy deviation of the *k*-th particle, N_λ - number of particles per radiation wavelength, $\theta=2\gamma_\parallel^2\left(t-z\right)$ - special "time" variable (see [2] for details) . Particular form of interaction "force" $\Phi\left(z^{(k)},z^{(l)}\right)$ depends on the used approximation. Further we shell restrict our consideration to the model of charged sheets. In this model $\Phi\left(z^{(k)},z^{(l)}\right)=-8\rho^3\cos\left(z^{(k)}-z^{(l)}\right)$, ρ - Pierce parameter. For the numerical solution of (1) it is also convenient to make substitution $z^{(k)}=\theta+\tilde{z}^{(k)}$ .The resulting set of equations is given bellow:

$$\frac{d\tilde{z}^{(k)}}{d\theta}=2\Delta^{(k)};\quad \frac{d\Delta^{(k)}}{d\theta}=-\frac{1}{N_\lambda}8\rho^3\,\mathrm{Re}\left(e^{i\tilde{z}^{(k)}}S^{(k)}\right). \quad (2)$$

where $S^{(k)}=\sum_{z^{(l)}<z^{(k)}}e^{-i\tilde{z}^{(l)}}$. The set of equations (2) can be easily solved numerically using any simple difference scheme. The only difficulty is the large number of particles. One also needs to remember that to obtain averaged values the system has to be solved many times with "random" initial conditions. Calculation of the value $S^{(k)}$ can be simplified substantially if one sorts particles at each integration step so, that $\tilde{z}^{(l)}<\tilde{z}^{(k)}$ if $i<k$. In this case $S^{(k)}=S^{(k-1)}+e^{-i\tilde{z}^{(k-1)}}$. The order of particle arrangement

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after each integration step is not violated very much and almost any sorting algorithm is efficient in this case.

Vlasov Equation and Quasilinear Approximation

The Vlasov equation for the considered model can be easily obtained directly from the motion equations (1):

$$\left(\frac{\partial}{\partial\theta}+(1+2\Delta)\frac{\partial}{\partial z}\right)f(z,\Delta,\theta)= \\ =-\frac{\partial}{\partial\Delta}f(z,\Delta,\theta)\int_0^z\int\Phi(z-z')f(z',\Delta',\theta)\,d\Delta'dz' \quad (3)$$

It is convenient to introduce the slow varying amplitudes of distribution function $\tilde{f}^{(n)}(z,\Delta,\theta)$ which we shell call harmonics. For numerical solution it also convenient to consider amplitudes $\tilde{f}^{(n)}(z,\Delta,\theta)$ be periodic in θ with some period T:

$$f(z,\Delta,\theta)=\tilde{f}^{(0)}(z,\Delta,\theta)+2\,\mathrm{Re}\left(\sum_n\tilde{f}^{(n)}(z,\Delta,\theta)e^{in(z-\theta)}\right) \quad (4)$$
$$\tilde{f}^{(n)}(z,\Delta,\theta)=\sum_\nu f_\nu^{(n)}(z,\Delta)e^{-i\nu\theta} \qquad \nu_m=(2\pi/T)\cdot m$$

For the costing beam in this model one has to use the following initial condition: $f_\nu^{(0)}(0,\Delta)=0$ at all $\nu\neq 0$. Value of T is determined from the condition that the bunch length corresponding to one period has to be larger than two slippage lengths in the whole undulator.

Substituting (4) and explicit expression of $\Phi(z-z')$ into (3) we obtain the following set of equations:

$$\left((1+2\Delta)\frac{\partial}{\partial z}-i\nu\right)f_\nu^{(0)}(z,\Delta)= \\ =\frac{\partial}{\partial\Delta}\sum_{\nu'}\left(A_{\nu'}(z)f_{\nu'-\nu}^{(1)*}(z,\Delta)+A_{\nu'}^*(z)f_{\nu'+\nu}^{(1)}(z,\Delta)\right)$$

$$\left((1+2\Delta)\frac{\partial}{\partial z}+i(2n\Delta-\nu)\right)f_\nu^{(n)}(z,\Delta)= \\ =\frac{\partial}{\partial\Delta}\sum_{\nu'}\left(A_{\nu'}(z)f_{\nu-\nu'}^{(n-1)}(z,\Delta)+A_{\nu'}^*(z)f_{\nu+\nu'}^{(n+1)}(z,\Delta)\right) \quad (5)$$

$$A_\nu(z)=4\rho^3\int_0^z\int f_\nu^{(1)}(z',\Delta')\,d\Delta'dz'$$

This system can be solved numerically using explicate difference scheme. It should be noted here that $f_\nu^{(n)}(z,\Delta)$ are random functions and to get averaged values this system has to be solved multiple times with different initial conditions.

One also can simplify (6) by using so called quasilinear approximation [4]. In this approximation we assume that saturation takes place due to the growth of the energy spread only. We neglect all high order harmonics $f_\nu^{(n)}(z,\Delta)$ at $n > 1$ as well as $f_\nu^{(0)}(z,\Delta)$ at $\nu\neq 0$.

Correlation Function Equation

The set of equations for the two-particle correlation function for the case of coasting beam is given bellow:

$$(1+2\Delta_1)\frac{\partial}{\partial z_1}F(1)=-\int d\{2\}\Phi(1,2)\frac{\partial}{\partial\Delta_1}G(1,2)$$

$$\left((1+2\Delta_1)\frac{\partial}{\partial z_1}+(1+2\Delta_2)\frac{\partial}{\partial z_2}\right)G(1,2)+ \\ +\int\left(\frac{\partial F(1)}{\partial\Delta_1}\Phi(1,3)G(2,3)+\frac{\partial F(2)}{\partial\Delta_2}\Phi(2,3)G(1,3)\right)d\{3\}= \\ =-\frac{1}{N_{\lambda_w}}\left(\Phi(1,2)\frac{\partial}{\partial\Delta_1}+\Phi(2,1)\frac{\partial}{\partial\Delta_2}\right)F(1)F(2) \quad (6)$$

To calculated current and radiation spectral distributions one needs to know two-time correlation function which obeys the following equation:

$$\left(\frac{\partial}{\partial\theta_1}+(1+2\Delta_1)\frac{\partial}{\partial z_1}\right)G_2(1,2;\theta_1-\theta_2)= \\ =-\frac{\partial F(1,\theta_1)}{\partial\Delta_1}\int\Phi(1,3)G_2(3,2;\theta_1-\theta_2)d\{3\}$$

It has to be solved with the initial condition $G_2(1,2,0)=G(1,2)$. It also makes sense to introduce slow varying amplitudes or harmonics of the correlation function the following way:

$$G(z_1,\Delta_1;z_2,\Delta_2)=G^{(0)}(z_1,\Delta_1;z_2,\Delta_2)+ \\ +\sum_{n>0}\left(\tilde{G}^{(n)}(z_1,\Delta_1;z_2,\Delta_2)e^{in(z_1-z_2)}+\kappa.c.\right) \quad (7)$$

By substituting (7) into (6) one can show that high order harmonics are not exited if only two-particle correlation function is taken into account.

For the high order harmonics to appear one needs to take into account many particles correlations. They may play important role at saturation if its mechanism is not quasilinear. In this case one has to consider three and more particles correlations.

RESULTS OF SIMULATIONS

In simulations we used the following set of parameters: $\rho=\sigma_e=9\cdot 10^{-4}$ - Pearce parameter and energy spread, $N_\lambda=500$ - number of particles per radiation wavelength, $N_w=2000$ - number of undulator periods. The bunch length in the case of direct solution of particle motion equations was 4000 of radiation wavelengths. Therefore, the total number of particles was $2\cdot 10^6$. Averaging was

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done over 2000 sets of initial conditions. In quasilinear simulations averaging was done over $2\cdot 10^4$ sets. When harmonics were taken into account (0, 2nd and 3d) we used 1300 sets as simulation was rather slow.

Corresponding Quantities for Comparison

To compare simulation results obtained from three approaches we need to find corresponding quantities which can be matched. If $\tilde{z}^{(k)}(\theta)$ and $\Delta^{(k)}(\theta)$ are solutions of particle motion equations (2) then the bunching factor can be found from the expression:

$$B(z,\theta)=\left\langle e^{i\tilde{z}^{(k)}(\theta)}\right\rangle\Big|_{\left|z-\tilde{z}^{(k)}(\theta)\right|<\pi}$$

Analog of the two-time correlation function, integrated by energies in this case will be $\left\langle B(z_1,\theta_1)B^*(z_2,\theta_2)\right\rangle \approx C(z_1,z_2,\theta_1-\theta_2)$ (valid for long bunch). Here averaging is done over initial conditions,

We shell use the following quantities for comparison - square of the bunching factor

$$\left|b(z)\right|^2=\int G(z,\Delta_1,z,\Delta_2)d\Delta_1 d\Delta_2 \approx$$
$$\approx C(z,z,0)\approx \int\sum_{\nu}\left\langle f_{\nu}^{(1)}(z,\Delta_1)f_{\nu}^{(1)*}(z,\Delta_2)\right\rangle d\Delta_1 d\Delta_2$$

and current spectral density

$$J_{\nu}(z)=\int G_2(z,\Delta_1,z,\Delta_2,\tau)e^{i(1+\nu)\tau}d\Delta_1 d\Delta_2 d\tau \approx$$
$$\approx\int C(z,z,\tau)e^{i\nu\tau}d\tau \approx T\int\left\langle f_{\nu}^{(1)}(z,\Delta_1)f_{\nu}^{(1)*}(z,\Delta_2)\right\rangle d\Delta_1 d\Delta_2$$

Comparison of Different Approaches

The results of comparison are presented in Fig. 1 and Fig. 2.

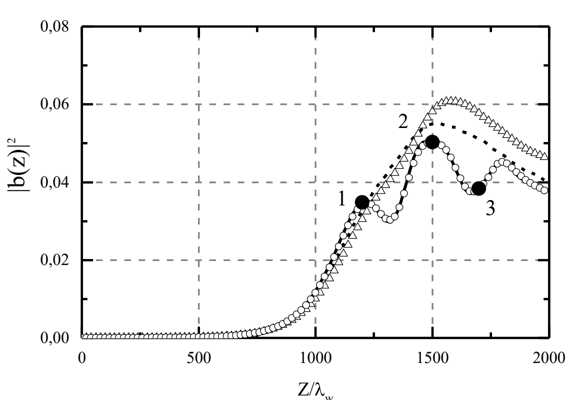

Figure 1: Dependence of bunching factor on beam position in undulator. Solid line - correlation function theory, circles – Vlasov equation in quasilinear approximation, dashed line - direct solution of motion equations, triangles - Vlasov equation with high order harmonics

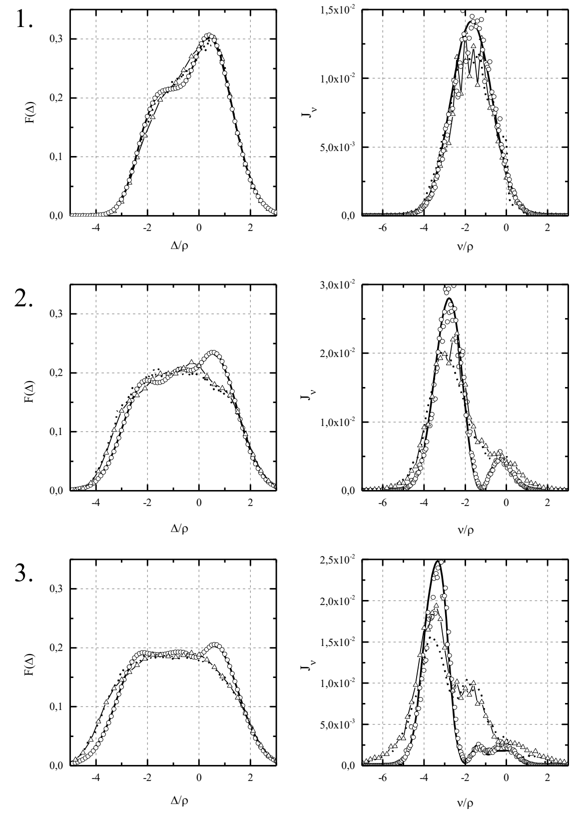

Figure 2: Energy distribution and current spectrum at different points in undulator shown in Fig. 1. Legend is the same as in Fig.1.

It worth noting that perfect agreement of the correlation function theory with quasilinear approximation is not surprising. Value $\tilde{g}(1;2)=\sum_{\nu}f_{\nu}^{(1)}(z_1,\Delta_1)f_{\nu}^{(1)*}(z_2,\Delta_2)$ corresponds to $\tilde{G}^{(1)}(1;2)$. It obeys exactly the same equation without the RHS. term responsible for shot noise. And for the three-particle correlation function $H(1,2,3)=\tilde{H}_0(1,2,3)e^{i(z_3-z_2)}+\tilde{H}_2(1,2,3)e^{i(2z_1-z_2-z_3)}+\ldots$ one can find the following corresponding values composed of zero and high order harmonics:

$$\tilde{h}_0(1,2,3)=\sum_{\nu}\sum_{\nu'\neq\nu}\left(f_{\nu-\nu'}^{(0)}(z_1,\Delta_1)f_{\nu}^{(1)*}(z_2,\Delta_2)f_{\nu'}^{(1)}(z_3,\Delta_3)\right)$$
$$\tilde{h}_2(1,2,3)=\sum_{\nu}\sum_{\nu'}\left(f_{\nu+\nu'}^{(2)}(z_1,\Delta_1)f_{\nu}^{(1)*}(z_2,\Delta_2)f_{\nu'}^{(1)*}(z_3,\Delta_3)\right)$$

which means that appearance of high order harmonics is related to many-particle correlations.

CONCLUSION

Direct solution of motion equations showed that at deep saturation stage high order harmonics in Vlasov equation have to be taken into account. In correlation function theory these harmonics are related to many-particle correlations. Neglecting of these correlations in BBGKY chain is equivalent to quasilinear approximation.

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