Numerical solution of the FEL correlation function equation

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Outline

1. Introduction
2. Basic definitions and derivation of kinetic equations
3. Simplification of the equations for the case of narrow beam and stationary current
4. Algorithm of numerical solution
5. Simulation results
6. Conclusion
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Derivation of kinetic equations

Step 1: Equations of particle longitudinal motion

Assumptions:

1. Transverse motion is not effected by FEL operation. Electron trajectories are given (and known) functions of longitudinal coordinate in undulator and electron initial transverse coordinates in 4-D phase space.

2. Interaction between two electrons in longitudinal direction is carried out through radiation field. All other collective forces are neglected.

3. Radiation field of electrons has narrow bandwidth and obeys paraxial wave equation. Therefore interaction force can be averaged over several undulator periods.
\[
\frac{dz_i}{dt} = 1 - \frac{1}{2\gamma^2} + \frac{\Delta_i}{\gamma^2} - \Delta\beta(z_i, X_i)
\]

\[
\frac{d\Delta_i}{dt} = \sum_{l \neq i} \Phi[z_i, X_i, z_l(t'_l), X_l]
\]

\[
t - z_i = t'_l - z_l(t'_l)
\]

energy deviation \(\delta\gamma/\gamma_0\) of the \(i\)-th particle

velocity shift due to betatron oscillations

longitudinal interaction "force" between two particles

retardation

transverse trajectory

\[
\Phi(1,2) = 2\frac{r_0 k_0 K(z_1)K(z_2)}{\gamma_0 (1 + K^2(z_2))}\sin\left(k_w(z_1 - z_2) + k_0 \left(\frac{\vec{R}_1(X_1, z_1) - \vec{R}_2(X_2, z_2)}{2(z_1 - z_2)} + \phi_1 - \phi_2\right)\right)\theta(z_1 - z_2)\theta(z_2)
\]
Motion equations with retardation

\[
\frac{dz_i}{dt} = 1 - \frac{1}{2\gamma^2} + \frac{\Delta_i}{\gamma^2} - \Delta \beta(z_i, X_i)
\]
\[
\frac{d\Delta_i}{dt} = \sum_{l \neq i} \Phi[z_i, X_i, z_l(t'_i), X_l]
\]
\[t - z_i = t'_i - z_i(t'_i)\]

System of ordinary differential equations

\[
\frac{dz_i}{d\xi} \approx 2\gamma^2 \left[1 + 2\Delta_i - 2\gamma^2 \Delta \beta(z_i, X_i)\right]
\]
\[
\frac{d\Delta_i}{d\xi} \approx 2\gamma^2 \sum_{l \neq i} \Phi(z_i, X_i, z_l, X_l)
\]

\[
\theta = 2\gamma^2 \xi = 2\gamma^2 (t - z) \quad (c = 1)
\]

New “time” variable

\[\xi = t - z\]
\[ \Delta t = \Delta \xi + \Delta z \]

\[ \Delta \xi = \Delta t (1 - V_z) \]

\[ \frac{df}{d\xi} = \frac{df}{dt} \frac{1}{1 - V_z} \]

New independent variable is just a new parameterization of the electron world lines in the space-time continuum.
Step 2: Continuity equation for the $N$-particles distribution function

\[
L(i) = \left(1 + 2\Delta_i - 2\gamma \Delta \beta(z_i, X_i)\right) \frac{\partial}{\partial z_i}
\]

\[
V(i, j) = -N\Phi(z_i, X_i, z_j, X_j) \frac{\partial}{\partial \Delta_i}
\]

\[
\left[ \frac{\partial}{\partial \theta} + \sum_{i=1}^{N} L(i) - \frac{1}{N} \sum_{i \neq j}^{N} V(i, j) \right] f_N(1, \ldots, N; \theta) = 0
\]

distribution function in $6 \times N$ - dimensional phase space
Probability to find the system of $N$ particles in the $6 \times N$ dimensional phase space volume $dX_1 \ldots dX_N$ at the “time” moment $\theta$

\[ f_N(1, \ldots, N; \theta) d\{1\} \ldots d\{N\} = f_N(X_1, \ldots, X_N; \theta) dX_1 \ldots dX_N \]

**Probability density**

**6D vector of one particle coordinates and momenta**

**m-particle distribution function**

\[ f_m = \int f_N dX_{m+1} \ldots dX_N \]

**Probability to find one particle in the 6D phase space volume $dX_1$**

\[ f_1(X_1; \theta) dX_1 \]

**Probability to find one particle in the volume $dX_1$ and another particle in the volume $dX_2$**

\[ f_2(X_1, X_2; \theta) dX_1 dX_2 \]
Step 3: **BBGKY chain of equations**

\[
\begin{align*}
\frac{\partial}{\partial \theta} + L(1) & f_1(1; \theta) = \int V(1,2) f_2(1,2; \theta) d\{2\}, \\
\left[ \frac{\partial}{\partial \theta} + L(1) + L(2) - \frac{1}{N} [V(1,2) + V(2,1)] \right] f_2(1,2; \theta) = \\
& = \int [V(1,3) + V(2,3)] f_3(1,2,3; \theta) d\{3\}, \\
\end{align*}
\]

\[f_{n-1} = \int \{ f_m \} d\{m\}\]

Step 4: **Truncation of the BBGKY chain**

**Correlation functions decomposition**

\[
\begin{align*}
    f_1(1, \theta) & = F(1, \theta) \\
    f_2(1,2, \theta) & = F(1, \theta)F(2, \theta) + G(1,2, \theta) \\
    f_3(1,2,3, \theta) & = F(1, \theta)F(2, \theta)F(3, \theta) + F(1, \theta)G(2,3, \theta) + \\
    & + F(2, \theta)G(1,3, \theta) + F(3, \theta)G(1,2, \theta) + H(1,2,3, \theta)
\end{align*}
\]

\[G(1,2,0) = H(1,2,3,0) = ... = 0\]

**Natural initial condition for the non-interacting particles**
We assume that \( H(1,2,3,\theta) \ll G(1,2,\theta) \)

This assumption seems to be reasonable because the number of interacting particles (the number of particles on cooperation length) is large.

The similar assumption is valid in plasma physics where one has large number of particles in the Debye sphere.
**Final system of kinetic equations**

\[
\frac{\partial}{\partial \theta} F(1, \theta) + v_1 \frac{\partial}{\partial z_1} F(1, \theta) + N \frac{\partial F(1, \theta)}{\partial \Delta_1} \int \Phi(1, 2) F(2, \theta) d2 = -N \int \Phi(1, 2) \frac{\partial G(1, 2, \theta)}{\partial \Delta_1} d2
\]

\[
\frac{\partial}{\partial \theta} G(1, 2, \theta) + v_1 \frac{\partial}{\partial z_1} G(1, 2, \theta) + v_2 \frac{\partial}{\partial z_2} G(1, 2, \theta) + N \frac{\partial F(1, \theta)}{\partial \Delta_1} \int \Phi(1, 3) G(2, 3, \theta) d3 +
\]

\[
+ N \frac{\partial F(2, \theta)}{\partial \Delta_2} \int \Phi(2, 3) G(1, 3, \theta) d3 = -\Phi(1, 2) F(2) \frac{\partial F(1)}{\partial \Delta_1} - \Phi(2, 1) F(1) \frac{\partial F(2)}{\partial \Delta_2}
\]

**Shot noise induced source term**

\[v(i) = 1 + 2\Delta_i - 2\gamma_i^2 \left( \frac{\dot{x}_i^2(z_i, X_i)}{2} + \frac{\dot{y}_i^2(z_i, X_i)}{2} + \frac{x_i^2(z_i, X_i)}{2\beta_u^2(z_i)} + \frac{y_i^2(z_i, X_i)}{2\beta_u^2(z_i)} \right)\]

**Initial conditions**

\[F(1, 0) = F_0(z_1, \Delta_1), \quad G(1, 2, 0) = 0\]

**Longitudinal “velocity”**

\[dz/d\theta\]
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Stationary current and narrow beam case

1. All time derivatives are equal to zero

\[
\frac{\partial}{\partial \theta} = 0
\]

2. Transverse beam size is constant and small

\[
2\pi \frac{\sigma^2}{\lambda_0 L_g} \ll 1
\]

3. Interaction force can be averaged over transverse distribution

\[
\langle \Phi(1,2) \rangle \perp = -\frac{r_e}{2\sigma^2 k_w \gamma} \frac{K^2}{1 + K^2} \left( \frac{e^{ik_w(z_1 - z_2)}}{1 + i \alpha k_w(z_1 - z_2)} + c.c. \right)
\]
Resulting system of equations for numerical solution

\[\nu_1 \frac{\partial}{\partial z_1} F(1) = -2 \Re \left( \frac{\partial}{\partial \Delta_1} I(z_1, \Delta_1; z_1) \right)\]

\[\frac{1}{2} \left[ (\nu_1 + \nu_2) \left( \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \right) + (\nu_1 - \nu_2) \left( 2i + \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \right) \right] \tilde{G}(1;2) = \]

\[= -\frac{\partial F(1)}{\partial \Delta_1} I^*(z_1; z_2, \Delta_2) - \frac{\partial F(2)}{\partial \Delta_2} I(z_2; z_1, \Delta_1) -
\]

\[-\frac{2\pi}{N_{\lambda_0}} \left( \Phi^*(z_1 - z_2) \frac{\partial}{\partial \Delta_1} + \Phi(z_2 - z_1) \frac{\partial}{\partial \Delta_2} \right) F(1) F(2)\]

\[I(z_1; z_2, \Delta_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(z_1 - z_3) \tilde{G}(2;3) d \{3\}\]

\[G(z_1, \Delta_1; z_2, \Delta_2) = 2 \Re \left( \tilde{G}(z_1, \Delta_1; z_2, \Delta_2) e^{i(z_1 - z_2)} \right)\]
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Algorithm of numerical solution

Explicit difference scheme

\[ G_{p,j+1}^{n,m} = G_{m,j-1}^{n,m} - \frac{\Delta_n - \Delta_m}{1 + \Delta_n + \Delta_m} \left( G_{p,j-1}^{n,m} - G_{m,j+1}^{n,m} + 4iH_z G_{o,j}^{n,m} \right) - \frac{2H_z}{H_\Delta} \frac{1}{1 + \Delta_n + \Delta_m} \left( F_{j,o}^{n} - F_{j,o}^{n-1} \right) l_{1,j}^{m} + \left( F_{j}^{m} - F_{j}^{m-1} \right) l_{2,j}^{n} + \]

\[ - \frac{2\pi}{N_{j,o}} \frac{F_{j}^{m} + F_{j}^{m-1}}{2} \left( F_{j,o}^{n} - F_{j,o}^{n-1} \right) \Phi_{j-o-j}^{*} \]

\[ F_{j,o+1}^{n} = F_{j,o-1}^{n} - \frac{2}{1 + 2\Delta_n} \frac{2H_z}{H_\Delta} \text{Re} \left( l_{2,j}^{n+1} - l_{2,j}^{n} \right) \]

\[ l_{1,j}^{n} = \frac{H_z}{2} \sum_{k=1}^{j-o-1} \left( I G_{o,j,k}^{m} \Phi_{j-k+1}^{o} + I G_{o,j,k+1}^{m} \Phi_{j-k}^{o} \right) \]

\[ l_{2,j}^{n} = \frac{H_z}{2} \sum_{k=1}^{j-1} \left( I G_{o,j,k}^{n} \Phi_{j-k+1}^{o} + I G_{o,j,k+1}^{n} \Phi_{j-k}^{o} \right) \]

\[ I G_{o,j,o}^{n} = H_\Delta \sum_{m=1}^{N_{j,o}} G_{o,j}^{n,m} \]

\[ \tilde{G}(1,2) = \tilde{G}^{*}(2,1) \]
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Results of simulations

Two coordinate distribution of the correlation function amplitude integrated over energy

$L_g$ – the gain length at linear stage
This function characterize spectral properties of the signal in FEL.

\[ J_{\nu}(z) = \int \tilde{G}\left( z - \frac{\tau}{2}, \Delta_1; z + \frac{\tau}{2}, \Delta_2 \right) e^{i\nu \tau} d\Delta_1 d\Delta_2 d\tau \]

This function represents the square of beam microbunching at coordinate \( z \) in undulator.

\[ g(z) = \frac{1}{2\pi} \int J_{\nu}(z) d\nu = \int \tilde{G}(z, \Delta_1, z, \Delta_2) d\Delta_1 d\Delta_2 \]
Growth of microbunching square in undulator

\[ \int G(z, \Delta_1, z, \Delta_2) d\Delta_1 d\Delta_2 \]

\( L_g \) – the gain length at linear stage
Decrease of the r.m.s. spectral bandwidth

$L_g$ – the gain length at linear stage, $N_g$ – number of undulator periods at one gain length
Energy and spectral distributions at different points in undulator
Energy distribution

Normalized spectral distribution

Correlation function amplitude
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Conclusion

✓ We developed the description for saturation in SASE FEL based on rigorous statistical approach.

✓ For the simplest case of narrow electron beam we first obtained non-trivial solution for the correlation and single particle distribution functions nonlinear behaviour.
Thank you for your attention!

The end.