THREE-DIMENSIONAL THEORY OF THE CERENKOV FREE-ELECTRON LASER

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Abstract

We present an analytical theory for the operation of a Cerenkov free-electron laser which includes diffraction of the optical mode in the direction transverse to the electron beam. Because the width of the optical mode depends on gain, the usual cubic dispersion relation is replaced by a 5/2-power dispersion relation, which allows two roots. These roots both have positive real parts, indicating that they are slow waves. For a narrow electron beam, the optical mode is much wider than the beam, thus reducing the gain by an order of magnitude from that predicted by the two-dimensional theory. In the limit of a wide electron beam, the two-dimensional theory is recovered.

INTRODUCTION

Compact narrow-band far-infrared, or terahertz (THz), sources have potential applications in a large number of fields including biology, chemistry, and materials science[1, 2]. The current THz sources in existence either produce short-pulsed broadband radiation, or require very large facilities. The exceptions to these are CO_2 pumped FIR lasers and backward-wave oscillators (BWOs). FIR lasers only have discreet lines, making them impractical for spectroscopy, and BWOs do not reach short enough wavelengths. Free-Electron Lasers (FELs) based on either the Smith-Purcell effect or Cerenkov radiation offer the possibility of a source capable of producing narrow-band THz radiation.

In a Cerenkov FEL the electron beam interacts with the evanescent wave of a single-sided dielectric waveguide. Since the wave has a forward group velocity, the instability is convective and the device works as an amplifier. Oscillation is achieved by feedback from reflections at the ends of the structure. The most powerful CFELs use ampere beams in cylindrical waveguides [3] but smaller devices can be constructed using milliampere beams in planar geometries [4]. The theory of cylindrical devices has been developed in detail, and the agreement with experimental results is good [3]. The theory for planar geometries has been worked out in two dimensions [5]. We extend this theory to three dimensions by including diffraction of the optical beam in the direction transverse to the electron beam, parallel to the surface of the dielectric [6]. We show that the gain is reduced by an order of magnitude and the fundamental nature of the conventional three-wave interaction is altered.

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THEORY

To model the CFEL in three-dimensions we let the electron beam pass above a dielectric slab, bounded below by a conductor, as shown in Figure 1. The electron beam travels in $\hat{z}$, while $\hat{x}$ extends above the slab and $\hat{y}$ is the transverse direction. We assume a beam of width W, and a dielectric slab of depth H.

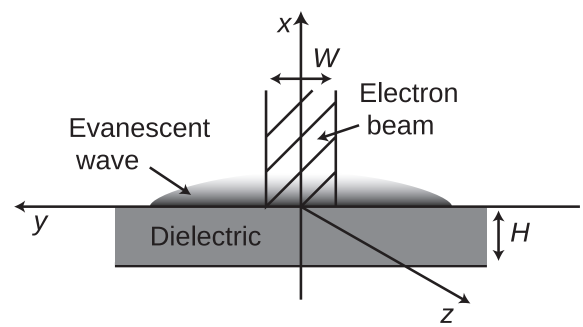

Figure 1: Geometry used for the three-dimensional model. The electron beam has a width W and is allowed to extend to infinity in x. The dielectric had a depth H and is wide compared to both the evanescent scale height and mode width.

To make the problem tractable, we start by making three assumptions. First, we assume the electron beam is uniform and extends infinitely far in $\hat{x}$. This introduces negligible errors to the theory because the electron beam height is actually on the order of the evanescent scale height. The assumption is corrected at the end of the calculation by using a filling factor. Next, we assume the operating frequency of the device is much larger than the plasma frequency or $\omega \gg \omega_e$. This is justified by considering the operating parameters of the most recent CFEL experiments at Dartmouth [4], shown in Table 1. For these parameters $\omega \approx 10^{12}$ Hz and $\omega_e \approx 10^9$ Hz. Finally, we assume that the mode width, Δy, is much greater than the evanescent scale height, Δx. For the Dartmouth parameters $\Delta x = \beta\gamma\lambda/4\pi \approx 80\mu$m, where βc is the electron beam velocity, $\gamma = 1/\sqrt{1-\beta^2}$ and λ is the operating wavelength. Diffraction arguments suggest that the evanescent mode width is on the order of $\Delta y = \sqrt{\beta\lambda Z_g/2\pi} \approx$mm where Z_g is the gain length.

We find the fields within and above the dielectric by solving Maxwell's equations and requiring the solutions to meet the appropriate boundary conditions. Only TM modes

Table 1: Parameters used in the Dartmouth experiments and all theoretical predictions [4]

Dielectric thickness	350 μm
Index of refraction	3.6
Beam energy	30 keV
Beam thickness	70 μm
Beam current	1 mA

of the form

$$E_z = E_0(x,y)\,e^{i(kz-\omega t)} \qquad (1)$$
$$\mathbf{H}_t = \mathbf{H}_0(x,y)\,e^{i(kz-\omega t)}$$

are considered, where k is the wave vector in the $\hat{z}$ direction. We find the fields above the dielectric by starting in the electron rest frame, where we can treat the electron beam as an isotropic dielectric. We then transform the fields into the laboratory frame. By setting the fields above and within the dielectric equal at the boundary, we find the dispersion relation. When there is no beam present, we find the two-dimensional dispersion relation,

$$D(\omega,k,k_y) = \frac{n^2\left(\omega^2-k^2c^2\right)}{n^2\omega^2-k^2c^2}\kappa(k_y)\cot\left[\kappa(k_y)H\right]+\alpha(k_y) \qquad (2)$$

where n is the index of refraction of the dielectric slab, k_y is the $\hat{y}$ component of the wave vector, $\kappa(k_y)$ is the $\hat{x}$ component of the wave vector within the dielectric, and $\alpha(k_y)$ is the $\hat{x}$ component of the wave vector above the dielectric. The dispersion relation is plotted in Figure 2.

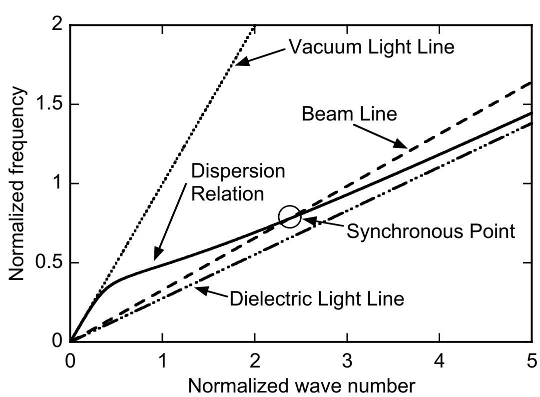

Figure 2: The two-dimensional dispersion curve for the parameters used in experiments at Dartmouth (see Table 1). Only the first order branch is shown.

When the electron beam is present, we anticipate that the highest gain will be found in the vicinity of the synchronous point, shown in Fig. 2, where the electron-beam line intersects the dispersion curve or, $\omega_0 = \beta c k_0$. We also assume that most waves are traveling parallel to the direction of beam travel so $k_y \approx 0$. We can then expand the dispersion relation about this point and find,

$$(\delta\omega-\beta c\delta k)\left[D_\omega(\delta\omega-\beta_g c\delta k)+D_y k_y^2\right]\bar{E}(k_y) = \frac{W}{2\pi}\frac{\omega_e^2\alpha_0}{\gamma^3}\int_{-\infty}^{\infty}dk_y'\bar{E}'\left(k_y'\right)\,\mathrm{sinc}\left[\frac{W}{2}\left(k_y-k_y'\right)\right] \qquad (3)$$

where

$$D_\omega = \left.\frac{\partial D(\omega,k,k_y)}{\partial\omega}\right|_{(\omega_0,k_0,0)} \qquad (4)$$
$$D_y = \left.\frac{\partial D(\omega,k,k_y)}{\partial k_y^2}\right|_{(\omega_0,k_0,0)}$$

and $\delta\omega$ and δk are the shifts in frequency and wave vector from the synchronous point, β_g is the group velocity, $\alpha_0 = \alpha(\omega_0,k_0,k_y=0)$ and k_y' is a dummy variable. Computations show that $D_\omega > 0$, $D_y < 0$ and $\beta_g > 0$. In general this equation must be solved numerically, but for either a wide or narrow electron beam, analytical solutions to the dispersion relation can be found.

Wide Electron Beam

When the electron beam is wide, $W \gg \Delta y$, the sinc function behaves like a delta function, so we evaluate the integral at $k_y' = k_y$. We recover the two-dimensional cubic dispersion relation,

$$(\delta\omega-\beta c\delta k)^2(\delta\omega-\beta_g c\delta k) = \frac{\omega_e^2}{\gamma^3}\frac{\alpha_0}{D_\omega}, \qquad (5)$$

the roots of which are shown in Figure 3. As in the two-dimensional case, one root corresponds to a fast wave, one to a slow wave which gains energy from the electron beam, and one to the structure wave, which decays. For steady-state amplifier operation we take $\delta\omega = 0$ and find that the gain in three dimensions with a wide beam,

$$\mu_{3D-wide} = -Im(\delta k) = \frac{\sqrt{3}}{2}\left(\frac{\omega_e^2}{\gamma^3\beta^2\beta_g c^3}\frac{\alpha_0}{D_\omega}\right)^{1/3} \qquad (6)$$

is identical to the two-dimensional gain.

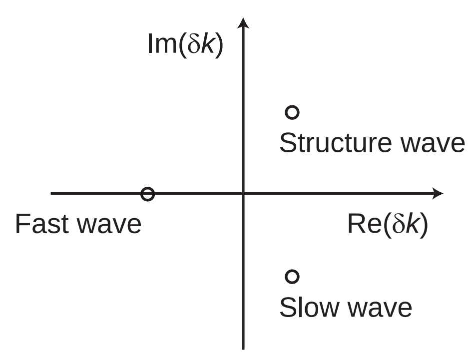

Figure 3: Roots of the dispersion relation for a wide electron beam. These are the same as for the two-dimensional result.

Narrow Electron Beam

When the electron beam is narrow, $W \ll \Delta y$, we approximate sinc$(x) \approx 1$. The integral becomes trivial, and we are able to invert the Fourier transform of the field using contour integration. The dispersion relation in this limit is

$$E_0(0,y) = \frac{W}{2}\frac{\omega_e^2}{\gamma^3}\frac{\alpha_0}{D_y}\frac{E_0(0,0)}{(\delta\omega - \beta c\delta k)^2}\frac{e^{y\Delta k_y}}{\Delta k_y} \quad (7)$$

where

$$\Delta k_y^2 = \frac{D_\omega}{D_y}(\delta\omega - \beta_g c\delta k) \quad (8)$$

and the sign of the root is chosen such that for positive y $Re(\Delta k_y) < 0$ and for negative y $Re(\Delta k_y) > 0$. Solving this dispersion relation on axis for the steady-state amplifier case, $y = 0$ and $\delta\omega = 0$, we find the roots are

$$\delta k_n = K^{2/5} e^{i(4/5)n\pi} \quad (9)$$

where

$$K = \frac{W}{2}\frac{\omega_e^2}{\gamma^3\beta^2\beta_g^{1/2}c^{5/2}}\frac{\alpha_0}{\sqrt{D_\omega |D_y|}}. \quad (10)$$

Figure 4 shows these five roots. Only two, $n = 2$ and $n = 3$ are physically admissible solutions; for the others, the fields diverge at $y = \pm\infty$. Of these the $n = 2$ root has positive gain and the $n = 3$ root has negative gain where the gain is given by

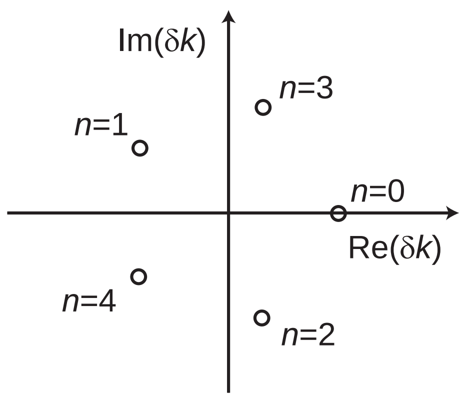

Figure 4: Roots of the dispersion relation for a narrow electron beam. Because of physical constraints, only $n = 2$ and $n = 3$ roots are true solutions. The $n = 2$ root has positive gain, while $n = 3$ has negative gain. Both modes have the same width and are slow waves $(Re(\delta k) > 0)$.

$$\mu_{3D-narrow} = -Im(\delta k_2) = K^{2/5}\sin\left(\frac{2}{5}\pi\right). \quad (11)$$

From Equation 7, we find the mode width at the $1/e$ point to be

$$\Delta y = -\frac{2}{Re(\Delta k_y)} = \frac{2}{K^{1/5}}\left|\frac{D_y}{\beta_g c D_\omega}\right|^{1/2}\sec\left(\frac{\pi}{5}\right). \quad (12)$$

DISCUSSION AND COMPARISON WITH EXPERIMENT

Using the parameters in Table 1, we can compare the results of the two-dimensional theory with the three-dimensional theory and experimental observations. The vertical and transverse field profiles for the three-dimensional theory using a narrow electron beam are shown in Figures 5 and 6, respectively. The transverse field profile shows that the mode width, $\Delta y \approx 6$ mm, is large compared to both the width of the electron beam and the scale height of the evanescent wave, as initially assumed.

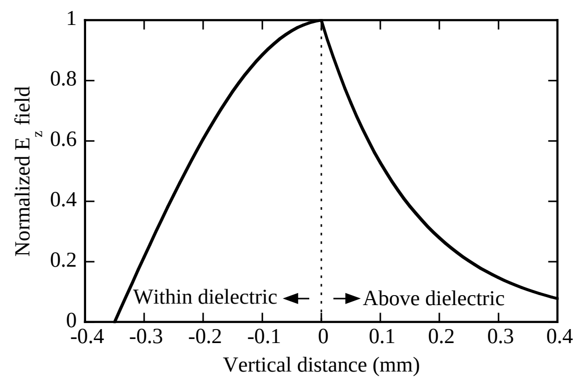

Figure 5: Vertical profile of the electric field. The field height is much smaller than the field width shown in Figure 6.

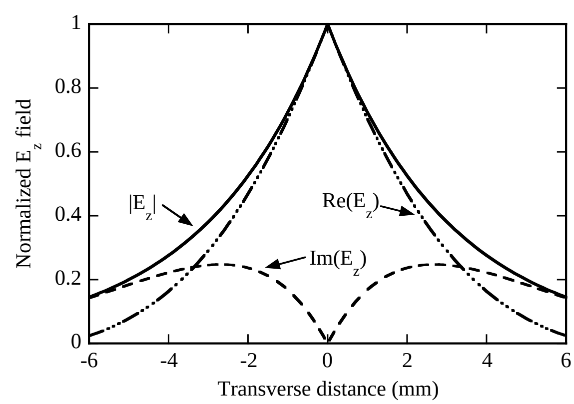

Figure 6: Transverse profile of the electric field. The field is much wider than the electron beam used in experiments. It is also wide compared with the evanescent scale height.

In computing the gain, we recognize that the electron beam only extends above the dielectric a distance comparable to the scale height of the evanescent wave, rather than well beyond the wave. To correct for this we reduce the electron density, or equivalently ω_p^2, by a filling factor [7, 8]

$$F_{fill} = 1 - e^{-W/\Delta x}. \quad (13)$$

We find that the two-dimensional gain is $\mu_{2D} \approx 50/\text{m}$, in good agreement with the theoretical prediction of $60/\text{m}$

by the Dartmouth group [4, 5]. The three-dimensional gain, $\mu_{3D} \approx 10/\mathrm{m}$, is a factor of 5 smaller. This can be understood as diffraction diluting the effective current density by spreading the optical mode over an area much larger than the electron beam. The experimental value for gain reported by the Dartmouth group [4] is $\mu_{exp} \approx 250-450/\mathrm{m}$. This is $5-9$ times larger than the two dimensional theory and $25-45$ times larger than the three-dimensional theory. The origin of this discrepancy is not understood.

Equally interesting are the structure of the dispersion relation in three dimensions and the location of the roots. In place of the cubic dispersion relation characteristic of two-dimensional geometries and axially symmetric structures, we obtain and $5/2$-power dispersion relation. The two roots which satisfy the dispersion relation are labeled $n = 2$ and $n = 3$ in Figure 4. The $n = 2$ root has gain while the $n = 3$ root has loss. Both roots have the same real part, corresponding to the same mode width and indicating that both are slow waves ($Re\,(\delta k) > 0$). This result is surprising because it suggests that transverse diffraction in a three-dimensional structures forbids fast waves. We can understand this result physically by recognizing that when waves are confined to the region near the electron beam the situation is similar to waves guided by a dielectric waveguide. Because its index of refraction exceeds unity, the dielectric waveguide only supports slow waves, so optical modes supported by the CFEL can only be slow waves.

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