A DESCRIPTION OF GUIDED FEL RADIATION WITH DIELECTRIC WAVEGUIDE EIGENMODES

E. Hemsing*, G. Andonian, J. Rosenzweig, Particle Beam Physics Laboratory, UCLA, USA
A.Gover, Deptartment of Physical Electronics, Tel-Aviv University, Israel

Abstract

We present a description of free-electron laser (FEL) radiation in the high-gain, small-signal regime through an expansion of eigenmodes of a virtual dielectric waveguide. A set of coupled differential equations is derived for the slowly-varying mode expansion coefficients and the electron beam density modulation amplitudes. The equations are decoupled into an algebraic matrix equation for solutions of the self-similar FEL supermodes that propagate with a self-similar profile. For a suitable choice of the form of the virtual dielectric, this virtual dielectric waveguide expansion (VDE) approach has the advantage of describing gain-guided FEL radiation over many Rayleigh lengths in terms of a basis that parallels the standard gaussian modes of free-space paraxial optics.

INTRODUCTION

The optical guiding of light in free-electron lasers (FELs) is a well-known phenomena that results during amplification as the coherent interaction between the electron beam (e-beam) and the electromagnetic (em) field introduces an inward curvature in the phase front of the light, refracting it back towards the lasing core of the e-beam[1, 2]. During the exponential gain process the e-beam can behave like a guiding structure that suppresses diffraction, reducing transverse power losses and enhancing the em field amplification (gain-guiding). The guided em field eventually settles into a propagating, self-similar eigenmode of the FEL system (supermode) with a fixed transverse profile distribution and spot size[3, 4].

Guided modes have been previously explored analytically by direct derivation of the eigenmode equations from the coupled Maxwell-Vlasov equations[3, 5, 6], and through expansions of the FEL signal fields in terms of hollow, conducting-boundary waveguide eigenmodes[4], step-index fiber modes[2], and free-space paraxial waves[7, 8]. Since, in an FEL, the e-beam operates simultaneously as an optical source and as a wave-guiding structure, an em mode description permits investigation of the coupling efficiency and guiding characteristics of individual modes to the e-beam. Of particular interest is the coupling to the well-known Hermite-Gaussian or Laguerre-Gaussian modes that describe free-space waves in the paraxial limit. The propagation and guiding of these modes over many Rayleigh lengths in an FEL interaction can be investigated directly by an expansion of the radiation field in terms of guided eigenmodes that have the form of paraxial modes evaluated at the waist. This connection is useful both in characterizing the free-space propagating radiation fields emitted from the FEL, but also in understanding input radiation coupling, as in the case of seed radiation injection.

* ehemsing@physics.ucla.edu

FIELD EXPANSION AND MODE EXCITATION IN A WAVEGUIDE

In a structure of axial translational symmetry, the radiation fields can be expanded in terms of transverse radiation modes with amplitudes that vary only as a function of the symmetry axis, z. Neglecting backward propagating waves and approximating the fields as dominantly transverse, the radiation field expansion in terms of waveguide modes is,

$$\underline{\tilde{E}}_\perp(\underline{r}) = \sum_q C_q(z)\underline{\tilde{\mathcal{E}}}_{\perp q}(r_\perp)e^{ik_{zq}z}$$

$$\underline{\tilde{H}}_\perp(\underline{r}) = \sum_q C_q(z)\underline{\tilde{\mathcal{H}}}_{\perp q}(r_\perp)e^{ik_{zq}z}. \quad (1)$$

where $\underline{\tilde{\mathcal{H}}}_{\perp q} = (1/Z_q)\hat{e}_z \times \underline{\tilde{\mathcal{E}}}_{\perp q}$, k_{zq} is the q^{th} mode axial wavenumber, and $Z_q = (k/k_{zq})\sqrt{\mu_0/\epsilon_0}$ for TE modes. The modes are orthogonal and normalized to

$$\mathcal{P}_q = \frac{1}{2}\mathrm{Re}\Big[\int\int[\underline{\tilde{\mathcal{E}}}_{\perp \mathrm{q}}(\mathrm{r}_\perp) \times \underline{\tilde{\mathcal{H}}}^*_{\perp \mathrm{q}}(\mathrm{r}_\perp)] \cdot \hat{\mathrm{e}}_\mathrm{z}\mathrm{d}^2\mathbf{r}_\perp\Big]. \quad (2)$$

The mode $\tilde{\mathcal{E}}_{\perp q}$ is an eigenmode of a dielectric medium with transverse variation in the refractive index $n(r_\perp)$. Assuming $\nabla n^2 \ll k$, the eigenmode equation is

$$\nabla^2_\perp\underline{\tilde{\mathcal{E}}}_{\perp q}(r_\perp) + [n(r_\perp)^2k^2 - k^2_{zq}]\underline{\tilde{\mathcal{E}}}_{\perp q}(r_\perp) = 0 \quad (3)$$

where $k = \omega/c$. With equations (1) and (3) and under the paraxial approximation ($|d^2C_q/dz^2| \ll |k^2C_q|$) for the slowly-growing coefficients, the excitation equation for the mode q in the presence of a source current is given by,

$$\frac{d}{dz}C_q(z) = -\frac{1}{4\mathcal{P}_q}e^{-ik_{zq}z}\int\int\underline{\tilde{J}}_\perp(\mathbf{r}) \cdot \underline{\tilde{\mathcal{E}}}^*_{\perp q}(r_\perp)d^2\mathbf{r}_\perp$$
$$-i\sum_{q'}C_{q'}(z)e^{-i\Delta k_{zqq'}z}\kappa^d_{q,q'} \quad (4)$$

where

$$\kappa^d_{q,q'} = \frac{\omega\epsilon_0}{4\mathcal{P}_q}\int\int[n(r_\perp)^2 - 1]\underline{\tilde{\mathcal{E}}}_{\perp q'}(r_\perp) \cdot \underline{\tilde{\mathcal{E}}}^*_{\perp q}(r_\perp)d^2\mathbf{r}_\perp \quad (5)$$

and $\Delta k_{zqq'} = k_{zq} - k_{zq'}$ is the difference between the axial wavenumbers of the modes q and q'. The term $\kappa^d_{q,q'}$ characterizes the mode coupling and represents the virtual polarization currents and charges that must be subtracted when using eigenmodes of a dielectric waveguide, since no such structure exists in the physical system.

ELECTRON BEAM FLUID MODEL AND COUPLED EXCITATION EQUATIONS

A linear plasma fluid model for a cold e-beam (negligible energy spread) can be used to describe the small-signal excitation in an FEL interaction[4]. A relativistic e-beam in an FEL experiences transverse oscillations driven by an interaction with a periodic structure. This motion drives an axial ponderomotive force that modulates the axial electron velocity such that, to first-order, the axial velocity of a cold beam within a static undulator can be expanded as $v_z(\underline{r},t) = v_{z0} + \mathrm{Re}[\tilde{v}_{z1}(\underline{r})e^{-i\omega t}]$ where $v_{z0} = \beta_z c$ is the d.c. component and $\tilde{v}_{z1}$ is the perturbation oscillating at signal frequency ω. Longitudinal variations in the velocity like the half-frequency modulation found in planar undulator systems are ignored here. The velocity modulation $\tilde{v}_{z1}$ develops a density bunching modulation that is similarly described in a linear model as $\mathsf{n}(\underline{r},t) = \mathsf{n}_0(r_\perp) + \mathrm{Re}[\tilde{\mathsf{n}}_1(\underline{r})e^{-i\omega t}]$ where $\mathsf{n}_0(r_\perp)$ is the transverse density profile of the e-beam. The a.c. component of the longitudinal current density results from both the axial velocity and density perturbations and is $\tilde{J}_{z1}(\underline{r}) = -e[\mathsf{n}_0(r_\perp)\tilde{v}_{z1}(\underline{r}) + v_{z0}\tilde{\mathsf{n}}_1(\underline{r})]$. If the transverse divergence of the current perturbation is assumed small $\nabla_\perp \cdot \tilde{J}_{\perp 1} \ll \partial\tilde{J}_{z1}/\partial z$, the continuity equation can be written as $d\tilde{J}_{z1}/dz = -i\omega e\tilde{\mathsf{n}}_1(\underline{r})$. The transverse component of the current density that excites the signal wave is written in terms of the density perturbation as

$$\underline{\tilde{J}}_\perp(\underline{r}) = -\frac{1}{2}e\tilde{\mathsf{n}}_1(\underline{r})\underline{\tilde{v}}_{\perp w}e^{-ik_w z}. \tag{6}$$

where $\underline{\tilde{v}}_{\perp w}$ is the transverse velocity vector and k_w is the axial wavenumber of the periodic undulator lattice.

From the expressions for the current density and the relativistic force equation for the axial velocity perturbation, the density bunching can be expressed as a second order differential equation[4]. It is useful to define the density bunching parameter

$$\tilde{\mathbf{i}}_q(z) = \frac{e}{8\mathcal{P}_q}e^{-i\frac{\omega}{v_{z0}}z}\int\int \tilde{\mathsf{n}}_1(\underline{r})\underline{\tilde{v}}_{\perp w}\cdot\underline{\tilde{\mathcal{E}}}^*_{\perp q}(r_\perp)d^2\mathbf{r}_\perp. \tag{7}$$

By combining the density modulation equation from Ref [4] with Eqs (4) and (7) we obtain a coupled form for the mode excitation evolution equations:

$$\frac{d}{dz}C_q(z) = \tilde{\mathbf{i}}_q(z)e^{i\theta_q z} - i\sum_{q'}\kappa^d_{q,q'}C_{q'}(z)e^{-i\Delta k_{z;qq'}z}$$

$$\Big[\frac{d^2}{dz^2} + \theta^2_{pr}\Big]\tilde{\mathbf{i}}_q(z) = i\sum_{q'}Q_{q,q'}C_{q'}(z)e^{-i\theta_{q'}z} \tag{8}$$

where $\theta_q = \omega/v_{z0} - (k_{zq} + k_w)$ is the characteristic detuning parameter for a given mode. The coupling between the e-beam with transverse density distribution function $f(r_\perp)$ and the FEL radiation field is given by the parameter $Q_{q,q'} = \theta_p^2\kappa_{q,q'}$ in Eq (8) where $\theta_p^2 = e^2\mathsf{n}_0/\gamma\gamma_z^2\epsilon_0 m_e v_{z0}^2$ and the e-beam mode-coupling coefficient $\kappa_{q,q'}$ is

$$\kappa_{q,q'} = \frac{\epsilon_0}{8\mathcal{P}_q}(k_{zq} + k_w)\int\int f(r_\perp)\tilde{\mathcal{E}}_{pm,q'}\underline{\tilde{v}}^w_\perp\cdot\underline{\tilde{\mathcal{E}}}^*_{\perp q}d^2\mathbf{r}_\perp. \tag{9}$$

The radiation field and the undulator field are assumed to be polarization matched. The axial ponderomotive field is $\tilde{\mathcal{E}}_{pm,q}(r_\perp) = \frac{1}{2}[\underline{\tilde{v}}_{\perp q} \times \underline{\tilde{\mathcal{B}}}^*_{\perp w} + \underline{\tilde{v}}^*_{\perp w} \times \underline{\tilde{\mathcal{B}}}_{\perp q}]\cdot\hat{e}_z$ where $\tilde{v}_{\perp q}$ is the transverse electron velocity due to the Lorentz force of the q^{th} mode of the signal field, $\tilde{\mathcal{B}}_{\perp w}$ is the transverse magnetic field of the undulator, $\tilde{v}_{\perp w}$ is the transverse velocity due to the undulator field and $\tilde{\mathcal{B}}_{\perp q} = \mu_0\tilde{\mathcal{H}}_{\perp q}$. The term $JJ = [J_0(\alpha) - J_1(\alpha)]^2$ can be included in the coupling parameter $\kappa_{q,q'}$ for a strong planar undulator ($JJ = 1$ for a helical undulator geometry), where J_0 and J_1 are the first and second order Bessel functions and $\alpha = \omega cK^2/(8\gamma^2 v_{z0}^2 k_w)$ where $K = e|\tilde{\mathcal{B}}_{\perp w}|/mck_w$ is the undulator parameter. The relativistic factor is $\gamma = \gamma_z\sqrt{1+K^2/2}$ with $\gamma_z^2 = 1/(1-\beta_z^2)$. The effect of the longitudinal space-charge in the beam is represented by the finite-width beam parameter $\theta_{pr} = \bar{r}\theta_p$ where $\theta_p = \omega_{p0}/v_{z0}$ is the longitudinal space-charge parameter (plasma wavenumber) of a uniformly distributed electron beam profile used in a 1D model. The plasma reduction factor satisfies $|\bar{r}| \leq 1$, and can be calculated numerically for a specific e-beam geometry[7]. In the limit that $\lambda\beta_z \ll r_0$ where r_0 is the e-beam radius, one can make the approximation $\bar{r} \simeq 1$.

The first equation in Eqs (8) describes the excitation of the mode amplitude C_q of a virtual dielectric waveguide eigenmode due to the density perturbation and transverse wiggling motion of the electrons throughout the FEL interaction. The second equation in (8) highlights the evolution of the modal density bunching parameters through the e-beam coupling to the expansion modes.

The initial conditions for Eqs (8) specify the operating characteristic of the FEL. For example, when operating as a single-pass amplifier (seeded FEL) there is no initial density and velocity modulation $\tilde{\mathbf{i}}_q(0), d\tilde{\mathbf{i}}_q(z)/dz|_{z=0} = 0$ and the initial seed field is non-zero $C_q(0) \neq 0$. Alternately, for a self-amplified spontaneous emission FEL (SASE), the amplified shot noise can be related to the pre-bunching conditions $\tilde{\mathbf{i}}_q(0) \neq 0, d\tilde{\mathbf{i}}_q(z)/dz|_{z=0} = 0$ and the input signal field vanishes $C_q(0) = 0$.

SUPERMODE MATRIX SOLUTION

In the high-gain regime of an FEL the radiation field is optically guided towards the beam axis. After a sufficiently long interaction length, prior to the onset of saturation, a balance is reached between the natural diffraction effects of the radiation and the focusing effects of the gain-guiding. The FEL settles into a period of steady exponential growth, characterized by a radiation field profile that propagates self-similarly which can be descried by a specific, fixed combination of expansion modes called the *supermode*. In order to find the system supermodes, it is enough to find the characteristic solutions to Eq (8). These are the combinations of expansion modes that propagate with a distinct wavenumber k_{SM}, and can be found by looking for a solution of the form

$$\underline{\tilde{E}}_{SM}(\underline{r}) = \Big[\sum_q b_q\ \underline{\tilde{\mathcal{E}}}_q(r_\perp)\Big]e^{ik_{SM}z} \tag{10}$$

where the mode amplitude coefficients b_q are constants and the z-dependence is contained solely in the exponential term. The excitation equations for this supermode expansion can be found by transforming the mode amplitude coefficents $C_q(z)$ via:

$$C_q(z) = b_q e^{i(k_{SM}-k_q)z}. \tag{11}$$

Plugging this expression into the coupled excitation evolution equations (8), a single equation is obtained for the supermode coefficients:

$$(\theta_{pr}^2 - \theta_{SM}^2)\Big[(k_{SM}-k_{zq})b_q + \sum_{q'}\kappa^d_{q,q'}b_{q'}\Big] = \sum_{q'}Q_{q,q'}b_{q'}, \tag{12}$$

where $\theta_{SM} = \theta + k - k_{SM}$ is the supermode detuning parameter, and θ is the detuning parameter for a 1D model ($k_{zq} = k$). Since we anticipate that the supermode propagates with a slightly modified axial wavenumber from that of free-space, we define $k_{SM} = k + \delta k$ and Eq (12) can be written in matrix form

$$\Bigg[\Big[(\delta k - \theta)^2 - \theta_{pr}^2\Big][\underline{\underline{\mathbf{I}}}\delta k + \underline{\underline{\kappa}}^d - \underline{\underline{\Delta k}}] + \underline{\underline{Q}}\Bigg]\underline{b} = \underline{0} \tag{13}$$

where $\underline{\underline{\Delta k}}$ is diagonal with matrix elements given by $\Delta k_q = k_{zq} - k$.

Solutions to the determinant equation $|[(\delta k - \theta)^2 - \theta_{pr}^2][\underline{\underline{\mathbf{I}}}\delta k + \underline{\underline{\kappa}}^d - \underline{\underline{\Delta k}}] + \underline{\underline{Q}}| = 0$ result in $3N$ solutions for δk, where N is the number of expansion modes. Each δk can then be inserted in Eq (13) to find a non-trivial solution (if one exists) for the mode amplitude vector $\underline{b}$, the components of which are the expansion coefficients of a supermode of the FEL system. From Eq (10) and the definition of k_{SM} it can be seen that the solution for δk with the largest imaginary component drives the highest gain, and dominates over the rest of the supermodes. This value, δk_{SM}, used in solving Eq (13) will yield the coefficients of the dominant supermode.

We note that when $\underline{\underline{\Delta k}}, \underline{\underline{\kappa}}^d = 0$, as in the case of an expansion into a free-space mode basis, the matrix equation in (13) reduces to a generalized matrix form of the canonical FEL cubic equation. Further, in the limit of a large transverse beam profile, $Q_{q,q'} \to Q$ and Eq (13) is the familiar 1D FEL cubic equation for the mode independent beam coupling parameter Q.

LAGUERRE-GAUSSIAN MODE EXPANSION

The choice of the refractive index in Eq (3) determines the form of the basis expansion used in the excitation equations (8). For quadratic refractive index of the form $n^2(r_\perp) = n_0^2[1 - 2\Delta(r/a)^2]$ with $\Delta \ll 1$, the constituent VDE expansion basis consists of a complete, orthogonal set of either Hermite-Gaussian (HG)[9] or Laguerre-Gaussian (LG) functions[10]. Since both HG and LG modes also occur as solutions to the paraxial wave equation, this choice for the refractive index identifies a connection between the VDE method and a description of the FEL system using paraxial, diffracting modes of free-space. The intrinsic advantage of the guided mode expansion formalism is the ability to efficiently model the signal field during high-gain over many Rayleigh lengths with only a few modes.

For geometries that are largely axisymmetric over the interaction length, LG modes provide a convenient working basis to model the FEL radiation. They have the form

$$\tilde{\mathcal{E}}_{\perp;p,l}(r,\phi) \propto e^{-il\phi}e^{-\frac{r^2}{w_0^2}}\Big(\frac{r\sqrt{2}}{w_0}\Big)^{|l|}L_p^{|l|}\Big(\frac{2r^2}{w_0^2}\Big) \tag{14}$$

where L_p^l is an Associated Laguerre polynomial. The mode index q takes on two values (p, l) corresponding to the radial and azimuthal mode numbers, respectively. The argument of the LG polynomials contain the parameter defined as $w_0^2 \equiv 2a/kn_0\sqrt{2\Delta}$ which sets a characteristic waist size in terms of the refractive index parameters.The field modes in Eq (14) are thus identical to free-space LG fields that satisfy the paraxial wave equation, when the free-space modes are evaluated at the waist. This waist definition specifies a characteristic dielectric profile form for a quadratic dielectric, in which a free-space Laguerre-Gaussian mode with waist size w_0 will propagate as a guided eigenmode.

It is noted that the LG modes of Eqn (14), as well as those of a free-space system, possess an axial field component for both the electric and magnetic fields for modes with $|l| > 0$. The magnitude of the axial electric field component is on the order of λ/w_0 by comparison to the principle transverse component, and can be neglected, validating the approximation made in deriving the amplitude evolution equation (4)[11].

It is instructive to cast the associated Rayleigh range for a free-space paraxial wave in terms of the dielectric parameters $z_R = kw_0^2/2 \equiv a/n_0\sqrt{2\Delta}$, which yields a refractive index $n^2(r) = n_0^2 - (r/z_R)^2$. The axial wavenumber $k_{zq} = k_{z;p,l}$ is given by

$$k_{z;p,l}^2 = k^2n_0^2 - \frac{4}{w_0^2}(2p + l + 1). \tag{15}$$

The refractive index on axis can be taken as $n_0^2 \simeq 1$ and the FEL signal field can be expanded into a sum of LG guided modes given by Eq (14), each with wavenumber $k_{z;p,l}$. The expansion waist w_0 is arbitrary, but can be chosen to be on the order of the e-beam size to facilitate efficiency in modeling the beam evolution with a finite number of modes included in the expansion[12].

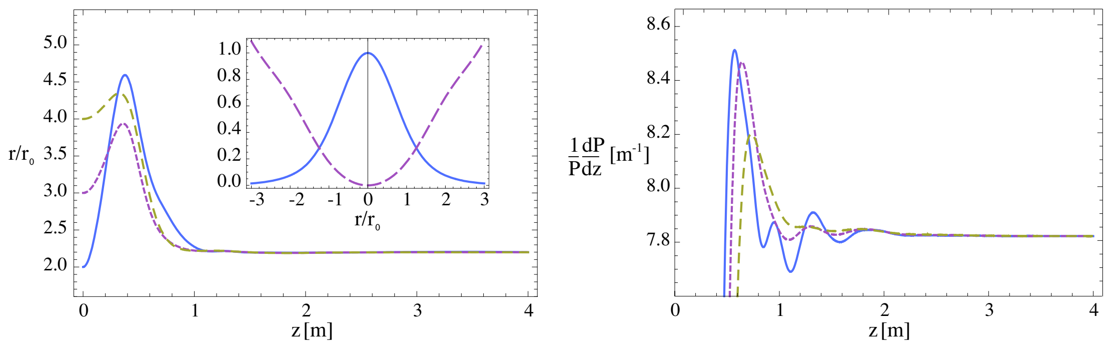

Figure 1: Left: Numerically modeled evolution of the radiation spot size throughout undulator for VISA FEL at $\lambda = 1064$nm for $\theta = 0$ detuning. Three spot sizes ($w_{s0} = 2r_0, 3r_0$, and $4r_0$) for a $P = 0.1\mu W$ gaussian seed are introduced at the undulator entrance with a $r_0 = 115\mu$m gaussian e-beam $f(r) = e^{-r^2/r_0^2}$. During the start-up period early on, the seed beam diffracts briefly until exponential gain develops. The self-focusing effects then guide the beam toward a fixed spot size, characteristic of the FEL supermode. Inset: Normalized field intensity (solid) and phase (dashed) profiles of supermode. Right: Normalized differential radiation power evolution for each input seed. The power slope for each input spot size evolves towards the same value after the supermode is established halfway through the 4 m VISA undulator.

SIMULATIONS

Results generated by the VDE method that uses an LG mode expansion basis to model the Visible to Infrared SASE or Seeded Amplifier (VISA) FEL currently in operation at Brookhaven National Laboratory are shown in Figure 1. The VISA FEL is ideal for investigations using an LG expansion since it has generated both hollow and spiral transverse em intensity patterns that are suggestive of single or multiply interfering LG modes[13]. The results in Figure 1 show the evolution of various injected radiation fields towards the supermode, both in the transverse spot sizes (left) and in the differential power curves (right) as a function of the longitudinal coordinate. Both the inwardly curved phase front (dashed line – inset) and the "pinched gaussian" intensity profile (solid line – inset) of the radiation at the undulator exit (z=4 m) display the signature characteristics of a gain-guided FEL supermode.

CONCLUSIONS

We have presented a description of the FEL electromagnetic signal field during high-gain through an expansion into a complete set of eigenmodes of a virtual dielectric waveguide. The FEL supermode can be found in the small-signal limit by finding the eigenfunction solutions to the coupled mode excitation equations derived from a cold e-beam fluid model. The choice of a quadratic index medium for the virtual dielectric yields a set of guided Hermite-Gaussian or Laguerre-Gaussian modes that relate to paraxial modes of free-space. Future work will involve comparisons of the VDE formulation with 1) experiments at VISA, where plans are underway to measure the phase and mode content of the em radiation[14] and, 2) numerical simulations using GENESIS 1.3 to verify and support the analytic and experimental results. The coupling to higher-order radial and azimuthal modes will also be investigated.

REFERENCES

[1] N. M. Kroll, P. L. Morton, and M. N. Rosenbluth, IEEE Journal of Quantum Electronics 17, 1436 (1981). G. T. Moore, Nuclear Instruments and Methods in Physics Research A 239, 19 (1985).

[2] E. T. Scharlemann, A. M. Sessler, and J. S. Wurtele, Phys. Rev. Lett. 54, 1925 (1985).

[3] M. Xie and D. A. G. Deacon, Nuclear Instruments and Methods in Physics Research A 250, 426 (1986).

[4] Y. Pinhasi and A. Gover, Phys. Rev. E 51, 2472 (1995).

[5] M. Xie, Nuclear Instruments and Methods in Physics Research A 445, 59 (2000). M. Xie, Nuclear Instruments and Methods in Physics Research A 445, 67 (2000).

[6] E. L. Saldin, E. A. Schneidmiller, and M. V. Yurkov, Optics Communications 97, 272 (1993).

[7] Y. Pinhasi and A. Gover, Phys. Rev. E 48, 3925 (1993).

[8] P. Sprangle, A. Ting, and C. M. Tang, Phys. Rev. Lett. 59, 202 (1987).

[9] A. Yariv, Optical Electronics in Modern Communications (Oxford University Press, USA, 1997)

[10] L. Yu, et al, Journal of Physics A 31, 9353 (1998). O. Georg, Applied Optics 21, 141 (1982).

[11] W. L. Power, et al Phys. Rev. A 52, 479 (1995).

[12] E. Hemsing, A. Gover, and J. Rosenzweig, unpublished.

[13] G. Andonian, et al, Proceedings of FEL 2006 Conference, 443-446, Berlin, Germany.

[14] E. Hemsing, et al, Proceedings of PAC 2007 Conference, 1263-1265, Albuquerque, New Mexico, USA.