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BEAMS WITH THREE-FOLD ROTATIONAL SYMMETRY: A THEORETICAL STUDY * †

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Abstract

Beams from ECR ion sources have 3-fold transverse rotational symmetry induced by the ECR sextupole. The symmetry imposes equality constraints among transverse beam moments, which can be derived using a theoretical framework we developed. Since the constraints are solely a consequence of the rotational symmetry of external fields, they hold for a multi-species beam with arbitrary composition and space charge intensity. These constraints provide a new tool to analyze phase space properties of ECR beams and their impact on low-energy transport. We prove that, regardless of their triangulated spatial density profile, beams with 3-fold rotational symmetry have the same RMS emittance and Twiss parameters along any transverse direction. These counter-intuitive results are applied to the FRIB Front End to show how symmetry arguments challenge long-standing assumptions and bring clarity to the beam dynamics.

INTRODUCTION

In theoretical studies, a beam is commonly assumed to have inherited the rotational symmetry of the source or beam line. The symmetry arises from the fact that the system is identical under different rotated transverse coordinate systems, which is true in the idealized case when there is no misalignment and when all elements are perfect.

Two symmetries that often occur are: 1) continuous rotational symmetry in a beam line consisting of solenoids or einzel lens; and 2) 2-fold discrete rotational symmetry in quadrupole transport. In the subsequent discussion, we denote a rotational symmetry by the notation of its respective symmetry group. A beam with n-fold discrete rotational symmetry is said to have C_n symmetry or is called a C_n beam. SO(2) refers to continuous rotational symmetry, which is often called axisymmetry.

Beams extracted from ECR ion sources have C_3 symmetry imposed by the ECR sextupole. As an example, simulation results of an ECR beam at the extraction plane are shown in Fig. 1 where a rotation by $2\pi/3$ would leave the beam distribution unchanged. Concerning the the transverse RMS emittances ε_x and ε_y, three questions can be asked which are central to the quality and transport of such an ECR beam:

- Are ε_x and ε_y different due to the triangulated spatial distribution?
- Do ε_x and ε_y depend on the orientation of the transverse coordinate system (with respect to the sextupole)?
- Do ε_x and ε_y change upon x-y coupling in solenoid transport?

We perform a theoretical analysis on 2nd order moments of C_3 beams to prove that the answers to all three questions are negative. These results are counter-intuitive and contradict conventional assumptions. The significant role they can play in clarifying ECR beam dynamics is demonstrated at the Facility for Rare Isotope Beams (FRIB) Front End.

MOMENT CONSTRAINTS FROM ROTATIONAL SYMMETRY

This section examines the consequences of rotational symmetry as a property of the beam's phase space distribution. We show that rotational symmetry imposes constraints on transverse beam moments and briefly overview a framework for deriving such constraints. A thorough treatment is presented in a manuscript in preparation [1].

Define transverse beam moments

$$\langle x^{b_1} x'^{b_2} y^{b_3} y'^{b_4} \rangle \equiv \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,x',y,y') x^{b_1} x'^{b_2} y^{b_3} y'^{b_4} dx dx' dy dy'}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,x',y,y') dx dx' dy dy'}$$

where $F(x,x',y,y')$ is the distribution function in transverse phase space and b_1, b_2, b_3, b_4 are non-negative integers. F can refer to the phase space distribution of a single species or multiple species. The moment is said to be of k-th order with $k = b_1 + b_2 + b_3 + b_4$. The definition assumes all 1st order moments vanish - it must be true when the beam has non-trivial rotational symmetry (see proof in [1]) which is the subject of this study.

If a beam has rotational symmetry, there are angles θ for which the beam moments are invariant under the phase space transformation:

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} \mapsto \begin{pmatrix} \tilde{x} \\ \tilde{x}' \\ \tilde{y} \\ \tilde{y}' \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} \quad (1)$$

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For C_n symmetry, $\theta = 2k\pi/n$, whereas for an SO(2) beam, θ is arbitrary. For every beam moment, invariance under rotation by θ is equivalent to:

$$\left\langle x^{b_1} x'^{b_2} y^{b_3} y'^{b_4} \right\rangle = \left\langle \tilde{x}^{b_1} \tilde{x}'^{b_2} \tilde{y}^{b_3} \tilde{y}'^{b_4} \right\rangle \tag{2}$$

which constitutes a constraint equation on beam moments because the right hand side can be expanded using Eq. (1). The class of Eq. (2) contains all information on how rotational symmetry constrains beam moments, but the information does not easily reduce into clean expressions since the expanded equations contain many terms. To simplify the analysis, we developed a method to derive the constraints imposed by rotational symmetry using complex coordinates.

If we define two complex conjugate pairs composed of transverse phase space coordinates:

$$\begin{pmatrix} w \\ \overline{w} \\ w' \\ \overline{w'} \end{pmatrix} \equiv \begin{pmatrix} x + iy \\ x - iy \\ x' + iy' \\ x' - iy' \end{pmatrix} \tag{3}$$

these complex coordinates transform under rotation by θ as follows:

$$\begin{pmatrix} w \\ \overline{w} \\ w' \\ \overline{w'} \end{pmatrix} \mapsto \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 \\ 0 & e^{-i\theta} & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} w \\ \overline{w} \\ w' \\ \overline{w'} \end{pmatrix} \tag{4}$$

We can construct a k-th order complex moment $\left\langle w^{a_1} \overline{w}^{a_2} w'^{a_3} \overline{w'}^{a_4} \right\rangle$ where a_1, a_2, a_3, a_4 are non-negative integers and $a_1 + a_2 + a_3 + a_4 = k$. The real and imaginary parts of $\left\langle w^{a_1} \overline{w}^{a_2} w'^{a_3} \overline{w'}^{a_4} \right\rangle$ each comprises sums of physical k-th order beam moments. Upon a rotation by θ, the complex moment undergoes the following transformation in accordance with Eq. (4):

$$\left\langle w^{a_1} \overline{w}^{a_2} w'^{a_3} \overline{w'}^{a_4} \right\rangle \mapsto e^{i(a_1 - a_2 + a_3 - a_4)\theta} \left\langle w^{a_1} \overline{w}^{a_2} w'^{a_3} \overline{w'}^{a_4} \right\rangle$$

If rotation by θ is a symmetry of the beam, $\left\langle w^{a_1} \overline{w}^{a_2} w'^{a_3} \overline{w'}^{a_4} \right\rangle$ remains unchanged upon the transformation which gives:

$$\left\langle w^{a_1} \overline{w}^{a_2} w'^{a_3} \overline{w'}^{a_4} \right\rangle = e^{i(a_1 - a_2 + a_3 - a_4)\theta} \left\langle w^{a_1} \overline{w}^{a_2} w'^{a_3} \overline{w'}^{a_4} \right\rangle \tag{5}$$

Eq. (5) is the key equation that efficiently generates equality constraints among beam moments due to symmetry. For every 4-tuple (a_1, a_2, a_3, a_4) where $e^{i(a_1 - a_2 + a_3 - a_4)\theta} \neq 1$, $\left\langle w^{a_1} \overline{w}^{a_2} w'^{a_3} \overline{w'}^{a_4} \right\rangle = 0$ which give two constraints:

$$\mathrm{Re}\left(\left\langle w^{a_1} \overline{w}^{a_2} w'^{a_3} \overline{w'}^{a_4} \right\rangle\right) = 0 \tag{6}$$

$$\mathrm{Im}\left(\left\langle w^{a_1} \overline{w}^{a_2} w'^{a_3} \overline{w'}^{a_4} \right\rangle\right) = 0 \tag{7}$$

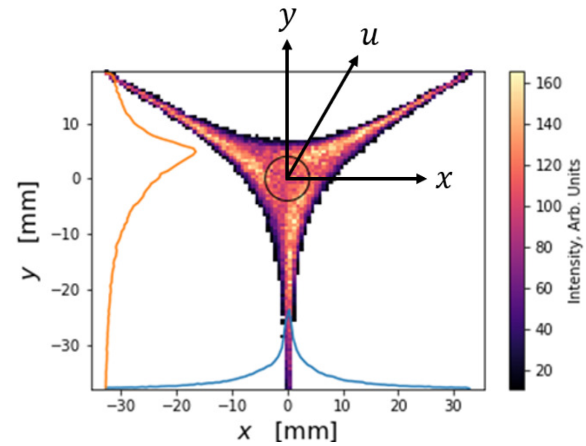

Figure 1: Simulated spatial distribution of Ar^{9+} ions at the extraction plane of the ARTEMIS [2] ECR ion source at FRIB. The inner circle indicates the extraction aperture. Simulation results courtesy of the code developed by Vladirmir Mironov and his colleagues at JINR [3, 4].

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THREE-FOLD ROTATIONAL SYMMETRY

We employ the theoretical framework developed in the previous section to derive 2nd order moment constraints imposed by C_3 symmetry. For $\theta = 2\pi/3$, one can use Eq. (5) to find three (actually six, but three of them are redundant, see [1]) 2nd order complex moments that must vanish. They generate six unique constraints:

$$\begin{aligned} \mathrm{Re}\left(\langle ww \rangle\right) = 0 \quad &\Rightarrow \quad \langle xx \rangle = \langle yy \rangle \\ \mathrm{Re}\left(\langle ww' \rangle\right) = 0 \quad &\Rightarrow \quad \langle xx' \rangle = \langle yy' \rangle \\ \mathrm{Re}\left(\langle w'w' \rangle\right) = 0 \quad &\Rightarrow \quad \langle x'x' \rangle = \langle y'y' \rangle \\ \mathrm{Im}\left(\langle ww \rangle\right) = 0 \quad &\Rightarrow \quad \langle xy \rangle = 0 \\ \mathrm{Im}\left(\langle ww' \rangle\right) = 0 \quad &\Rightarrow \quad \langle xy' \rangle = -\langle x'y \rangle \\ \mathrm{Im}\left(\langle w'w' \rangle\right) = 0 \quad &\Rightarrow \quad \langle x'y' \rangle = 0 \end{aligned} \tag{8}$$

It is known in literature [5], and we can easily derive using the above framework, that 2nd order moments of an SO(2) beam also obey the set of constraints in Eq. (8). This renders a C_3 beam effectively axisymmetric in the following sense: in terms of 2nd order moments, C_3 and SO(2) beams obey the same constraints and are thus indistinguishable. Only higher order moments can tell them apart.

The indistinguishability entails that 2nd order moments of C_3 and SO(2) beams must have the exact same properties. This argument enables us to prove a counter-intuitive theorem regarding the 2nd order moments of C_3 beams:

Theorem 1. *Take $u(\phi) = x\cos\phi + y\sin\phi$ to be a rotated coordinate. If a beam has C_3 symmetry, the equalities*

$$\begin{aligned} \langle uu \rangle &= \langle xx \rangle \\ \langle uu' \rangle &= \langle xx' \rangle \\ \langle u'u' \rangle &= \langle x'x' \rangle \end{aligned} \tag{9}$$

hold for any ϕ.

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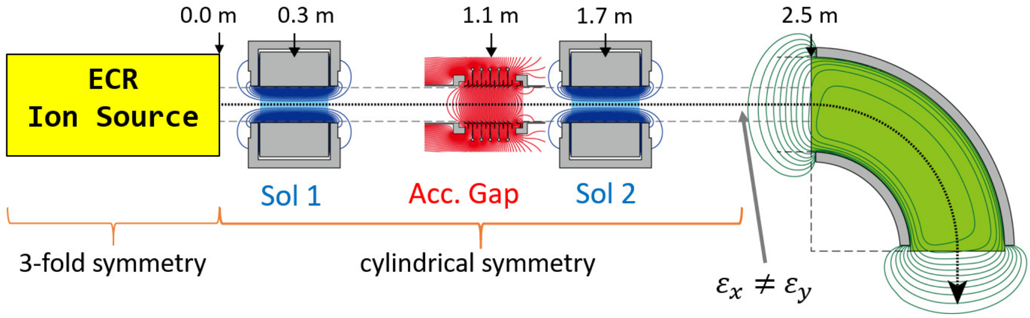

Figure 2: Schematic of the ARTEMIS beam line at the FRIB Front End. Ideally, the ECR ion source has C_3 symmetry while the transport line consisting of two solenoids and an electrostatic acceleration gap is axisymmetric. Image based on original from Ref. [6].

The proof relies on the fact the three equalities in Eq. (9) always hold for an SO(2) beam which looks identical along any direction. Since the theorem only concerns 2nd order moments, and 2nd order moments of SO(2) and C_3 beams have the same properties, the theorem must hold.

Figure 1 shows a beam with C_3 symmetry at an ECR extraction plane and a u-axis that makes angle ϕ with the x-axis. What theorem 1 says is that, although the beam's projected distribution in u-u' phase space varies significantly with ϕ, all 2nd order phase space moments, and hence the RMS phase space ellipse, always remain identical.

An immediate consequence of theorem 1 is:

Corollary 1.1. *A C_3 beam has identical RMS envelope and emittance along any transverse direction*

In particular, the Twiss parameters and emittance in x of a C_3 beam are always equal to their y counterparts.

APPLICATION TO FRIB FRONT END

Corollary 1.1 can help answer long-standing questions about beam emittances in ECR beam extraction and transport. To illustrate the arguments, we describe how theoretical results on C_3 beams were applied to analyze beam dynamics at the FRIB Front End [7]. Figure 2 shows a schematic of the ARTEMIS [2] beam line and depicts the rotational symmetries of the respective segments. In the ideal case, the beam should have C_3 symmetry up to non-zero dipole fields illustrated by the green curves.

Phase space diagnostics are located downstream of the dipole and their measurements on the target species often show $\varepsilon_x \neq \varepsilon_y$. While it is possible for ε_x to increase due to dispersion generated by the dipole, and for ε_y to couple to ε_x via space charge, the sign and magnitude of the difference between ε_x and ε_y were found to depend on the solenoid strengths. Since $\varepsilon_y > \varepsilon_x$ also occurred, dispersion cannot be the sole reason for $\varepsilon_x \neq \varepsilon_y$ downstream of the dipole, and we conclude that $\varepsilon_x \neq \varepsilon_y$ upstream of the dipole as depicted in Fig. 2.

$\varepsilon_x \neq \varepsilon_y$ prior to the dipole in an ECR beam line is often deemed an unsurprising empirical phenomenon caused by the beam's triangulated spatial density profile and x-y solenoid coupling. However, we have proved corollary 1.1 which states that $\varepsilon_x = \varepsilon_y$ if the beam has C_3 symmetry. The statement is a consequence of symmetry alone, so it holds even in the presence of chromatic aberrations, radial field nonlinearities and multi-species space charge with arbitrary intensity and charge state distribution.

Instead, the contrapositive of corollary 1.1 enables us to deduce: there is only one fundamental cause for $\varepsilon_x \neq \varepsilon_y$ in an axisymmetric beam line downstream of an ECR ion source and it is broken symmetry. At FRIB, the argument motivated a search for the source of broken symmetry in the transport line. The first solenoid was found via 3D magnet simulations [8] to have strong multipole fields due to a non-optimal design of the current leads.

CONCLUSION

We proved that 3-fold rotational symmetry guarantees identical RMS emittances and beam envelopes along x and y (and indeed along any transverse direction). These results were derived using a theoretical framework that describes how rotational symmetry imposes constraints on beam moments. The results are counter-intuitive and disprove common assumptions that triangulated distribution or ideal x-y coupling in solenoids would render $\varepsilon_x \neq \varepsilon_y$ in ECR beams. The only fundamental cause of $\varepsilon_x \neq \varepsilon_y$ is broken symmetry, which may arise from misalignments, imperfect elements or neutralization effects from back-flowing electrons. Such arguments are readily applicable to ECR beam lines and successfully clarified beam dynamics at the FRIB Front End.

The theoretical framework we developed also allows us to derive constraints imposed upon 3rd or higher order moments by C_3 symmetry. Their results and implications are discussed in Ref. [1]. In addition to being analytic tools in beam dynamics, symmetry-imposed moment constraints may also serve as a benchmark for ECR simulation models via consistency checks between simulation results and moment constraints. Preliminary work in this regard showed that 2nd order moment constraints are in agreement with the models provided by Vladimir Mironov from JINR that are described in Ref. [3, 4].

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