ELECTRON COOLER DRIVEN TRANSVERSE RESONANCES

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Abstract

The cylindrical transverse charge profile of the electron beam in an electron cooler causes a linear tune shift of the ion beam when it is fully immersed in the electron beam. Near the edge of the electron beam, however, the electric field is highly non-linear and will perturb the motion of the ion beam. This regime is of interest for accelerators that use electron cooling to improve accumulation, especially after injection when the ions' beam size is larger than that of the electrons. In this report we analyze the magnitude of this effect and calculate the resonance driving terms of the electric field created by a cylindrical electron beam.

1 INTRODUCTION

The radial distribution of the electron beam in an electron cooler is almost constant over the cross section of the electron beam and almost zero outside. The resulting electric field rises linearly inside the electron beam and drop like $1/r$ outside, resulting in a sharp edge at the border of the electron beam. The interaction of the ion beam with the electric field of the electron beam is conceptually similar to a strong-weak beam-beam effect in a collider. In the electron cooler the sharp edge of the electric field may drive high order resonances more strongly than the rather smooth beam-beam kicks from round gaussian beams. In ref. 1 it is suggested that these resonances are responsible for the observed "electron heating" in CELSIUS and we try to add a facet to its interpretation.

If we assume a constant transverse charge density we can calculate the linear tune shift the ion beam experiences to be

$$\Delta\nu_x = \frac{r_p}{2\pi ec}\frac{\beta_x l}{b^2}\frac{I_e}{\gamma\beta^3} \tag{1}$$

where r_p is the classical proton radius, β_x is the beta function at the location of the cooler, l its length, b the radius of the electron beam, I_e the electron current. β and γ are the normalized speed and energy of the ion beam. In CELSIUS [1], for protons at injection energy of 48 MeV we calculate $\Delta\nu_x = 0.038 I_e/\mathrm{A}$.

Moreover, the electron beam attracts the positively charged ion beam towards its center and thus provides a focussing force which will increase the tune in the center of the ion distribution, provided good alignment between electron and ion beam. Ions in the tail will experience a smaller tune shift and will thus have lower tune values.

In the following sections we will investigate the nonlinear tune shift and the resonance structure of a slightly refined model to the hard edge electron beam by smoothing the edges.

2 CHARGE DISTRIBUTION AND FIELD

The radial distribution of the electrons is parametrized by

$$\psi(r) = \mathrm{erfc}\left(\frac{r-b}{a}\right) \tag{2}$$

where b determines the size of the electron beam and a provides a cutoff parameter to smooth the edge. It determines the width over which the electron distribution decays to zero. Using Gauss' Law $2\pi RE(R) = \int_0^R 2\pi r\psi(r)dr$ the electric field arising from this charge distribution is calculated in ref. 2 with the result

$$R\,E(R) = a^2\left(F(z_1) - F(z_2)\right) + ab\left(G(z_1) - G(z_2)\right) \tag{3}$$

with $F(z) = \int_z^\infty t\,\mathrm{erfc}(t)dt = \left(1/4 - z^2/2\right)\mathrm{erfc}(z) + z\,e^{-z^2}/2\sqrt{\pi}$ and $G(z) = \int_z^\infty \mathrm{erfc}(t)dt = e^{-z^2}/\sqrt{\pi} - z\,\mathrm{erfc}(z)$ and $z_1 = -b/a$ and $z_2 = (R-b)/a$. The distribution $\psi(r)$ and electric field $E(r)$ for $b = 1$ and $a = 0.1$ are shown in Fig. 1. On the same graph the distribution and electric field of a gaussian with the same slope at the

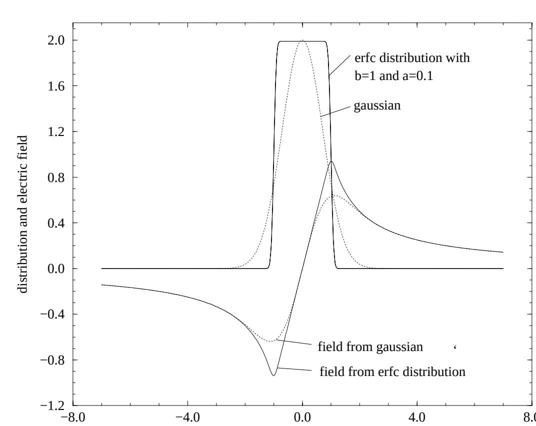

Figure 1: *Radial charge distribution and associated field for gaussian (dotted lines) and erfc-distribution (solid lines).*

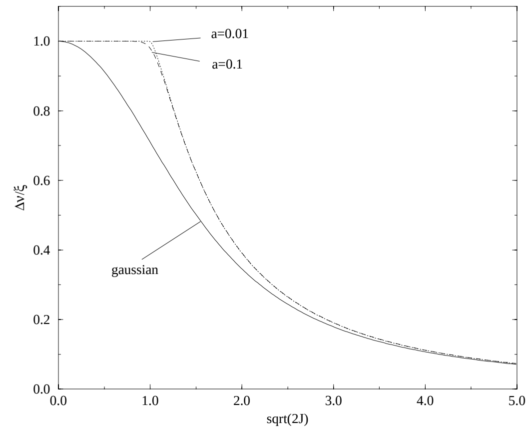

Figure 2: *Non-linear tune shift as a function of the amplitude of the oscillation* $\sqrt{2J}$*. The curve from the gaussian is shown solid, the one for* $a = 0.1$ *dashed, and for* $a = 0.01$ *dotted.*

origin are shown for comparison. We clearly observe that both distributions have the same slope at the origin and that the field from the erfc-distribution has enhanced inflection points near the edge of the distribution. For large radii r both fields decay as $1/r$. It is also noteworthy that the total charge in the beam for a gaussian and an erfc-distribution with small a are equal, provided the associated tune shifts are equal. Another fact is that the rms of the gaussian with the same tune shift as the erfc-distribution is about $1/\sqrt{2}$ times the beam size b of the erfc-distribution.

3 TUNE SHIFT AND FOOTPRINT

In order to calculate the non-linear tune shift we mimic the calculations in ref. 3. We start with the differential equation for the transverse betatron motion with a single located non-linear force $S(\eta)$ with unit slope at the origin

$$\frac{d^2\eta}{d\theta^2} + \nu^2\eta = -4\pi\nu\xi S(\eta)\delta_p(\theta) \ . \tag{4}$$

Here we parametrized the strength of the perturbation in terms of the linear tune shift ξ at the origin, θ is the azimuthal variable and runs from zero to 2π. As independent variables we have chosen those of normalized phase space $\eta,\ \eta'$ which are related to those of phase space $x,\ x'$ and of action angle variables J, ϕ by $x = \sqrt{\beta_x}\eta = \sqrt{2J\beta_x}\cos(\phi)$ and $\eta' = -\nu\sqrt{2J}\sin(\phi)$. For the tune shift we get after some manipulations

$$\Delta\nu(J) = \frac{2\xi}{\sqrt{2J}}\int_0^{2\pi}\frac{d\phi}{2\pi}\cos\phi S(\sqrt{2J}\cos\phi) \ . \tag{5}$$

As a check we calculate the tune shift for a constant linear force $S(\eta) = \eta$. It is easy to see that in this case the tune shift is given by ξ. For a general kick it is easiest to evaluate the integral numerically. Inserting the erfc-distribution's electric field we generate the data shown in Fig. 2 where we plot $\Delta\nu(J)/\xi$ as a function of $\sqrt{2J}$ for $a = 0.1$ and 0.01 and for the gaussian beam-beam kick. We clearly see that the tune shift for the erfc-distribution stays constant up to the electron beam's radius at $\sqrt{2J} = 1$. The only difference lies in the sharper roll-over to the inverse decay outside the beam.

In order to calculate the tune footprint of the erfc distribution we write down the equations of motion for the coupled system in x and y

$$\frac{d^2\eta_x}{d\theta^2} + \nu_x^2\eta_x = -4\pi\nu_x\xi S(r)\frac{\eta_x}{r}\delta_p(\theta) \tag{6}$$

with $r^2 = 2J_x\cos^2\phi_x + 2J_y\cos^2\phi_y$ and η, η', ϕ, and J now carry subscripts x or y. After some algebra we obtain

$$\frac{d\phi_x}{d\theta} = \nu_x + 4\pi\xi\cos^2\phi_x\frac{S(r)}{r}\delta(\theta) \tag{7}$$

and a similar equation for the vertical coordinate. We now keep only the constant part of the delta-function, average over phases ϕ_x and ϕ_y, and reach

$$\begin{aligned}\Delta\nu_x(J_x, J_y) \ &= \ 2\xi\int_0^{2\pi}\frac{d\phi_x}{2\pi}\int_0^{2\pi}\frac{d\phi_y}{2\pi}\cos^2\phi_x \\ &\quad \times\frac{S(\sqrt{2J_x\cos^2\phi_x + 2J_x\cos^2\phi_x})}{\sqrt{2J_x\cos^2\phi_x + 2J_x\cos^2\phi_x}}\end{aligned} \tag{8}$$

and a similar equation for x and y exchanged. The double integral can be easily integrated numerically and we can plot the tune shifts $\Delta\nu_x$ and $\Delta\nu_y$ for different action variables J_x and J_y, as shown in Fig. 3.

The left figure shows the tune footprint for a round gaussian beam and the right figure that of an erfc-distribution. The rms beam size of the gaussian corresponds to $1/\sqrt{2}$ times the radius b of the erfc-distribution, such as those shown in Fig. 1. The lines correspond to amplitudes $\sqrt{2J_{x/y}} = 1, 2, 3, 4 \times b$. Comparing the footprints we learn that for small amplitudes $\sqrt{2J_{x/y}} \leq b$ the tune shift stays close to the top right corner which corresponds to the full linear tune shift. Another important observation is that the

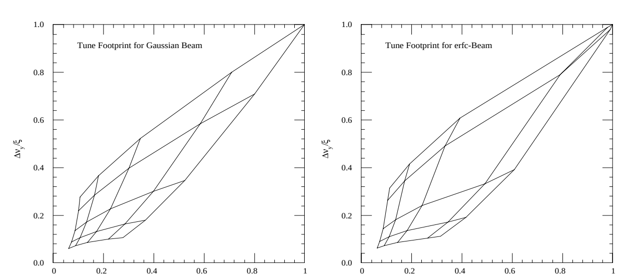

Figure 3: *Tune footprint of a round gaussian beam with* $\sigma = b/\sqrt{2}$ *(left) and a beam with erfc-distribution with* $a = 0.1$ *(right). The lines correspond to particle amplitudes of* $\sqrt{2J_{x/y}} = 1, 2, 3, 4 \times b$.

tune footprint of the erfc-distribution is wider than of the gaussian, which means that it is more likely to hit resonances. The driving terms of these resonances will be addressed in the next section.

4 RESONANCES

The resonance driving terms can be evaluated by following ref. 3 again. To this end we replace the periodic delta function by its Fourier expansion $\delta_p(\theta) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-im\theta}$. For $dJ/d\theta$ and $d\phi/d\theta$ we obtain the expressions

$$\frac{dJ}{d\theta} = 2\xi\sqrt{2J}\sin\phi\, S(\sqrt{2J}\cos\phi)\sum_{m=-\infty}^{\infty} e^{-im\theta} \quad (9)$$

$$\frac{d\phi}{d\theta} = \nu + \frac{2\xi\cos\phi}{\sqrt{2J}} S(\sqrt{2J}\cos\phi)\sum_{m=-\infty}^{\infty} e^{-im\theta}\,.$$

We now consider the equation for ϕ and express the electric cooler profile function $S(\sqrt{2J}\cos\phi)$ as a Fourier series

$$\sqrt{2J}\cos\phi\, S(\sqrt{2J}\cos\phi) = J\sum_{n=0}^{\infty} A_n \cos n\phi \quad (10)$$

where the (action $J-$dependent) Fourier coefficients A_n can be easily calculated numerically by Fast Fourier Transform methods. The factor J is retained on the right hand side to make later formulae easier to write. Note that only cosine terms appear in that Fourier series, because any power series of $\cos\phi$ can be expressed in terms of $\cos n\phi$, only. After some algebra we obtain an equation for the resonant phase $\chi = p\phi - q\theta$ which reads

$$\frac{d\chi}{d\theta} \approx (p\nu - q) + p\xi A_0 + p\xi A_p \cos p\chi\,. \quad (11)$$

We recover the non-linear tune shift term A_0 which depends on the action variable J. We also see that the "instantaneous tune" $d\phi/d\theta$ varies slowly in the vicinity of a resonance p/q with amplitude given by ξA_p which coincides with the definition of the resonance width in ref. 3.

Turning back to the erfc-distribution we use the kick function $S(\eta)$ and calculate the resonance widths for $a = 0.1$. The resulting tune shift and resonance widths are shown in Fig. 4 in units of the linear tune shift parameter ξ, both for a gaussian and an erfc distribution. We see that there are no resonances excited within the electron beam ($\sqrt{2J} \leq b = 1$), but as soon as the ions oscillate with amplitudes close to the electron beam radius they feel a strong non-linearity which causes the resonance strengths to become large. Furthermore the resonance strength stay larger than those for a gaussian beam in the intermediate range. Only for large ($\sqrt{2J} \geq 4$) amplitudes does the erfc-resonance strength get smaller, as can be seen in Fig. 4. In summary: *in the range between one to three times the electron beam radius the resonance strengths are considerably larger for an erfc-distribution.* This implies that all even resonances (if the beams are aligned) are excited and

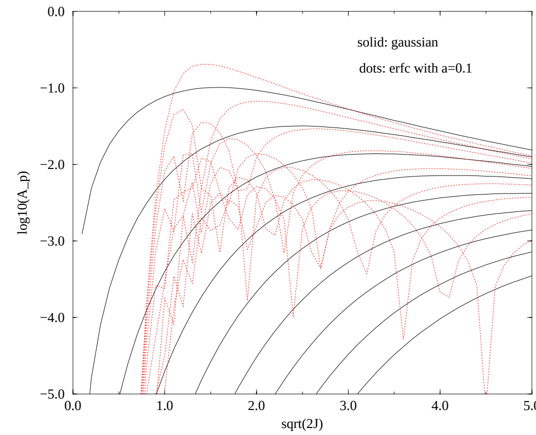

Figure 4: *The resonance width for $p = 4,\ldots,20$ for the gaussian (solid) and the erfc-distribution (dots) superimposed for comparison.*

a large number of islands can form. Increasing the electron current I_e will cause the islands to grow and since there is a multitude of islands, some will merge, causing chaotic dynamics (Chirikov's criterion) and particles will have a path to diffuse to larger amplitudes. It can, however, also cause particles which are damped by the electron cooler to be trapped in the islands, thus impeding the particle's path to smaller amplitudes. This phenomenon would reveal itself as increased damping time. As a final observation we report that the resonance strengths scale smoothly with the cutoff parameter a.

5 CONCLUSIONS

We calculated the non-linear tune shift, tune footprint, and resonance driving terms for an electric field distribution generated by a radial electron distribution given by eq. 2 and compared with a gaussian distribution. We found that the resonances are driven stronger near the edge of the distribution as documented in Fig. 4.

In order to understand "electron heating" better we have to extend the analysis to coupling resonances, which are excited due to the presence of the cooler's solenoid. Another field of interest is to calculate the width of the islands and determine "capture rates" for particles being trapped in those islands.

6 REFERENCES

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3. E. Keil, *Beam-Beam Dynamics*, in CERN 95-06, p. 539.