A METHOD FOR FINDING 4D SYMPLECTIC MAPS WITH REDUCED CHAOS

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Abstract

We have previously proposed a method for finding integrable, four-dimensional symplectic maps. The method relies on solving for parameter values at which the linear stability factors of the fixed points of the map have the values corresponding to integrability. We suggest that this method be applied to accelerator lattices in order to increase dynamic aperture. We have now implemented a numerical scheme for the practical application of this method to accelerator lattices. Results will be presented.

1 INTRODUCTION

Accelerator design is a member of a class of problems in which one has freedom in choosing the Hamiltonian, and for various reasons one would like to have a system that is either completely chaotic or uniformly integrable. Another examples occurs in the design of fusion confinement devices. Here we summarize how one might find symplectic maps with reduced chaos, and, specifically, how this might be applied to accelerator lattices to increase dynamic aperture. This problem is related to the problem of determining whether a dynamical system is chaotic [1-7].

The basic ideas for finding integrable systems [8] were first applied using fixed-point indicators of integrability [9] in the context of finding itegrable three-dimensional toroidal magnetic fields. Such systems are Hamiltonian of one and a half degrees of freedom [10] and, hence, correspond to two dimensional symplectic maps. There are no such systems known rigorously to be integrable while having no internal current and nonzero winding number. Application of methods similar to those discussed here allowed one to find systems with greatly reduced chaotic regions and, therefore, larger confined plasma volume. A series of designs were found [11].

Subsequently we applied these ideas to the similar problem of the dynamics in the uncoupled horizontal dynamics of an accelerator lattice [12]. As this is also a system of one and a half degrees of freedom, the method carried over in a straightforward manner and was successful, insofar as only one transverse degreee of freedom was concerned, i.e., for trajectories with initial conditions (momentum and position) entirely in the horizontal plane (an invariant plane for the lattices we considered). However, an examination of the dynamics of the four dimensional system (by considering trajectories with initial conditions out of the horizontal plane) found that the dynamics was worse. That is, the four dimensional volume of confined initial conditions was found to be smaller even though the uncoupled horizontal dynamics was improved. Thus, there remains a need to determine a scheme for optimization of 4D lattice dynamics.

2 RESONANCES, FIXED POINTS AND NONLINEAR DYNAMICS

Chirikov has related chaotic dynamics to resonances [13]. The paradigm for 4D symplectic maps is shown in Fig. 1. Resonance lines are found from a resonance condition, $l_1\omega_1(\mathbf{J})+l_2\omega_2(\mathbf{J})=m\Omega$, familiar in the tune diagram of accelerator physics. Resonance widths are calculated by the single-resonance approximation, which gives each of the widths of the lines.

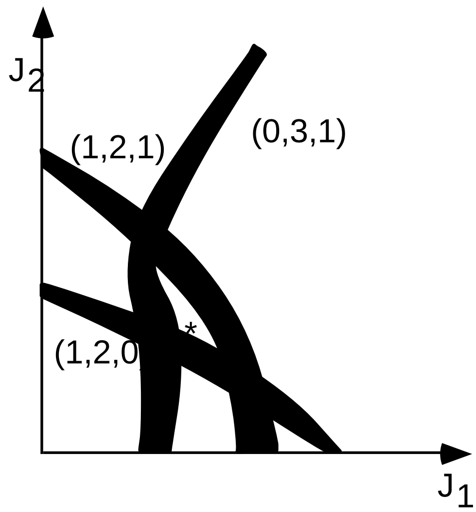

Fig. 1. Action plane chowing resonances and their widths. An invariant torus is a point in this plane such as the one indicated by the asterisk.

Analyzing the resonances for a nonlinear system is problematic, as typically the underlying integrable system is not known. Instead, we consider the fixed points. At each of the resonance crossings (locations in tune space where two resonances operate), there are, at least, 2^L fixed points, where L depends on the resonance indices. From the fixed points we will be able to deduce the resonance strengths.

The linear stability of fixed points of four dimensional symplectic maps has been studied extensively by Howard and MacKay [14]. The linearized motion near an N^{th} order fixed point is governed by the tangent map, the derivative of the N-times composed map. The tangent map **M** is represented by a symplectic matrix. The linear stability is determined by its eigenvalues. For symplectic matrices, if λ is an eigenvalue, then so are $1/\lambda$, λ^*, and $1/\lambda^*$. Thus, eigenvalues come in complex conjugate pairs on the unit circle ($\lambda=1/\lambda^*$), inverse pairs on the real line ($\lambda=\lambda^*$), or complex quadruplets in other parts of the complex plane.

The eigenvalues can be found by first defining the stability index [15],

$$\rho = \lambda + 1/\lambda \ , \tag{1}$$

for each inverse pair. Given the stability indices, which can be complex, one can solve for the inverse pair of eigenvalues. The stability indices are the roots of a polynomial,

$$Q(\rho) \equiv \rho^2 - A\rho + B - 2 = 0, \tag{2}$$

where

$$A \equiv \mathrm{Tr}(M) \tag{3}$$

and

$$B \equiv \left\{[\mathrm{Tr}(M)]^2 - \mathrm{Tr}\left(M^2\right)\right\}\big/2 \tag{4}$$

Our interest is in obtaining maps that are nearly integrable and, furthermore, have invariant surfaces that are simply nested tori. (We do not want island structures.) Away from the origin, integrable maps preserve angular separation $\delta\theta$: two orbits on the same torus separated by some angles initially are always separated by those angles as the angles increase linearly in time. This implies that each canonical pair has one eigenvalue that is unity. As the generalized eigenvalues of each canonical pair are inverses, this implies that all of the eigenvalues are unity. Thus, the stability index must be 2, and the values of A and B for integrable systems are

$$A = 4 \tag{5a}$$

and

$$B = 6. \tag{5b}$$

Hence, to reduce the resonance widths at the crossings we require the conditions (5).

A different condition holds if the trajectory corresponds to having zero action in one of the planes. In this case, the fixed point has only one iterate in that plane, i.e., it appears to be period-one in that phase plane. In this case, one of the oscillation pairs has arbitrary tune - corresponding to the central tune ν_y at nonzero J_x or vice versa. In this case one must block-diagonalize the tangent-map matrix and demand that the trace of one of the 2×2 blocks be equal to two.

3 OPTIMIZATION: WORK IN PROGRESS

We have now developed an accelerator tracking code [16] that is capable of finding fixed points and the stability indices of those fixed points. Our results show that the limit of the dynamic aperture of the coupled motion in ALS occurs where there are several period-six fixed points. We now seek to reduce the indicators of chaos. Doing so should reduce the resonance widths at the crossing, so that an action plane like that shown in Fig. 2 results.

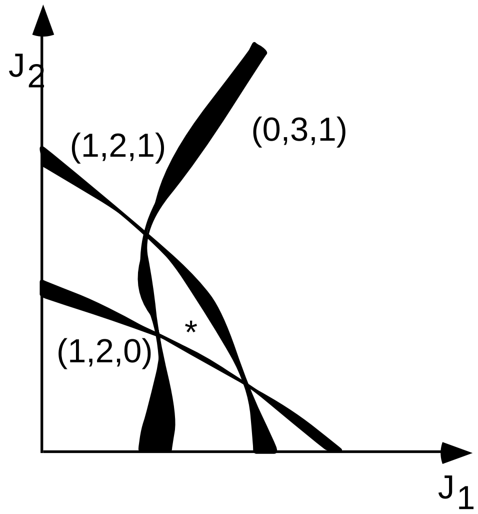

Fig. 2. Action plane after reduction of the chaos indicators at the fixed points.

However, at this point we do not have optimization software implemented for reducing the indicators of chaos. We attempted to use COSY for optimization but were unsuccessful. We believe that rescursive optimization is necessary. One must find fixed points (one optimization, in the sense of many accelerator modeling codes) for each set of parameters. Then one must find the parameters (the second optimization) having the correct value of fixed point tangent map invariants.

4 DISCUSSION

Our hope is that such methods will allow those who need to develop accelerator lattices to control the chaos in these systems as much as they now control the linear dynamics. With this degree of control, large dynamic apertures may be achievable to have larger capture regions, to provide a greater region of recapture for scattered particles, and in strongly nonlinear synchrotron light sources [17] to produce light beams of greater brightness. Additionally, nonlinearity can have a stabilizing effect in at least two ways. The effect of perturbations on nonlinear systems is guaranteed to be small; nonlinear stabilization guarantees that the associated resonant regions have finite size, and the spread in tunes can make such systems less susceptible to collective instabilities [18], which are strongest when all particles have the same frequency and resonate together. Finally, control allows the opposite possibility: smaller dynamic apertures may be put in place to, for example, collimate the beam, i.e., reduce its emittance or energy spread.

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