EVALUATING HIGH ORDER RESONANCES USING RESONANT NORMAL FORMS *

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Abstract

Resonant normal forms allow to study various aspects of resonances up to high orders. We apply these techniques to evaluate resonances in four phase space variables. The input is a truncated one-turn map derived from standard tracking codes. A code automatically finds fixed line locations in phase space for resonances up to a desired order. The island widths and the island tunes of these resonances are calculated as well. As a check, it is shown to which extent results from first order perturbation theory can be reproduced and how well the predictions of resonant normal form agree with tracking simulations.

1 INTRODUCTION

The comprehension of the relation between resonances, nonlinearities, tuneshifts and dynamic aperture in four-dimensional betatronic motion is a very difficult task. The dynamic aperture is usually determined through numerical integration based on tracking [1, 2]. Theoretical methods on the other hand, i.e. the perturbative theory based either on Hamiltonian flows [3, 4, 5] or on symplectic mappings [6, 7, 8, 9], provide a lot of analytical information on the detuning and on the features of the resonances.

In the case of unstable resonances, the dynamic aperture is usually determined by the hyperbolic resonant orbits, i.e. fixed lines [10, 11, 12, 13]. The stable resonances, on the other hand, feature families of islands that do not limit the stability domain, and therefore there is no direct relation with the dynamic aperture. Several studies have shown, however, that the analytical indicators extracted through perturbative tools can be well-correlated with the dynamic aperture: for instance, a minimisation of the amplitude-dependent detuning has been used to cure the effect of the systematic errors [14], and the correction of resonant driving terms has been proposed to sort the random errors [5]. In a recent study concerning magnet sorting strategies to optimise the dynamic aperture [15], a systematic analysis of the correlations of the analytical quality factors with the short-term dynamic aperture has been carried out for an LHC-like cell lattice.

During the past years, arbitrary order codes have been developed to compute perturbative series (normal forms) of a generic truncated one-turn map [6, 7, 9]. More recently, a code has been developed to analyse the interpolating Hamiltonian of the resonant normal form [16], and to provide at arbitrary order all the features of the resonance [17]. In this paper we give a check of the analytical results of this code through tracking for a four-dimensional model of LHC that includes all the errors and imperfections. We show that for unstable resonances one can determine the position of the hyperbolic fixed lines, and that the contribution of the higher orders can be very significant. Moreover, we analyse some stable resonances, finding a good agreement between the analytical and the numerical evaluations of the island area.

2 PERTURBATIVE APPROACHES TO NONLINEAR MOTION

The betatronic motion is described by an Hamiltonian in a four-dimensional phase space (x, p_x, y, p_y) with a periodic dependence on the azimuthal coordinate s. The quadratic part corresponds to Hill equations and the solutions can be written in terms of the Courant-Snyder coordinates.

2.1 *Classical perturbative theory*

Using the method of the variation of constants, one substitutes the linear solution in the complete Hamiltonian, where now the constants of the linear motion $\epsilon_x, \epsilon_y, \varphi_x, \varphi_y$ become the new phase space variables, and one obtains a new Hamiltonian H_1. Then one can apply the perturbative approach [3] and transform H_1 to a simpler form that can be either dependent on the emittances only (nonresonant theory), or also on a linear combination of angles (resonant theory). The perturbative parameters are usually chosen to be the gradients of the nonlinear elements. The first order approximation [4] corresponds to neglecting in H_1 all the terms that depend on the angles with the exception of the resonant combination that shall be analysed: this approach provides very simple and useful explicit formulas. However, the results may not be accurate enough when the higher orders are relevant.

2.2 *Normal form theory*

The approach based on transfer maps and normal forms has some significant differences with respect to the previous approach. First of all, one selects a particular section of the machine and only the intersections of the trajectories with that section are considered. Moreover, the perturbative parameter is the distance to the closed orbit, i.e. the contributions are ordered according to the powers of the coordinates

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and not to those of the gradients. The perturbative construction is based on two steps. Firstly, an exactly symplectic map is associated to each element, and the one-turn map is built as the composition (and truncation) of all the maps of the lattice. The truncated map contains all the interference terms between the nonlinearities up to the truncation order. Secondly, the one-turn map is transformed to another more symmetric map, the normal form, which is written as the map at integer times of an interpolating time-independent Hamiltonian. The usefulness of this Hamiltonian is twofold: it provides the integrals of motion and all the analytical information about tuneshifts and resonances.

A major advantage of the map approach is that arbitrary order codes for the computation of both the one-turn map and the interpolating Hamiltonian can be built. An overview of the normal form theory can be found in [9]; the specific cases of four-dimensional mappings is treated in [11, 16].

3 APPLICATIONS TO LHC

3.1 The model

We considered the LHC lattice version 4, with all the normal systematic errors and both chromatic and systematic correctors. The integer part of the linear tune is set to 63 in both planes. The fractional part of the tune was fixed to different values to check the reliability of the resonant perturbative tools.

3.2 Resonance (3,0)

We fixed the linear tune to $Q_x = 63.3333$ and $Q_z = 63.31$, close to the resonance $3Q_x = 190$. The nonlinear emittance ρ_2 is the second invariant, and there is a family of hyperbolic fixed lines (i.e. fixed points $\times$ 1D tori) that limit the stability boundary. Using the lowest order approximation, the position of the fixed line is independent of ρ_2; this is equivalent to the first order approach in classical perturbative theory [4]. In Fig. 1 we plot the position of the hyperbolic fixed lines evaluated through resonant normal forms at order 5 in the space of the nonlinear emittances ρ_1 and ρ_2. The two solid lines correspond to the maximum and to the minimum distance to the origin of the separatrix in the plane given by a fixed ρ_2; the first order results of classical perturbative theory are plotted for comparison (dotted line). The higher orders provoke a collapse of the hyperbolic structures on $\rho_1 = 0$ for positive ρ_2. In Tab. 1 we give a numerical check of the position of one of the hyperbolic fixed lines (indicated by a solid circle in Fig. 1). The first order classical perturbative theory using HARMON [1, 2] is compared to the normal form results at different orders, and to a numerical search of the hyperbolic fixed line based on tracking. The lowest order normal form agrees with HARMON, but both are a factor two larger than the value obtained through tracking. However, taking the normal form to order 5 this difference reduces to a mere 4%.

Table 1: Position of one of the fixed lines for resonance (3,0)

	x	x'	z	z'
Harm.	0.0263	0.0000	0.0599	0.000
NF-2	0.0257	-0.0026	0.0599	0.000
NF-5	0.0117	-0.0012	0.0599	0.000
Track.	0.0113	-0.0011	0.0599	0.000

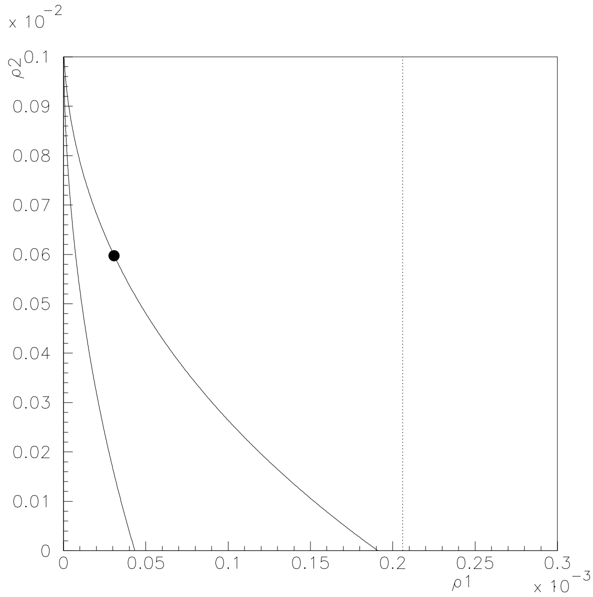

Figure 1: Minimum and maximum distance (solid lines) of the separatrix due to resonance $(3, 0)$ compared with first order classical perturbative theory (dotted line) in the space of nonlinear emittances

3.3 Resonance (1,2)

The same analysis was carried out for the coupled resonance $Q_x + 2Q_z = 190$: we fixed the linear tune to $Q_x = 63.28$ and $Q_z = 63.3599$. In this case the second invariant is given by $2\rho_1 - \rho_2$, and one can prove [13] that there is a family of hyperbolic fixed lines that arise from the resonance. Fig. 2 shows the minimum and the maximum distance of these fixed lines from the origin (solid lines). In Tab. 2 we give the value of the intersection of the invariant manifold of one of the fixed lines (indicated by a solid circle in Fig. 2) in the plane $p_x = p_z = 0$. The disagreement between normal form at order 5 and tracking is less than 1%. In this case the first order approximation is already very good.

3.4 Higher order resonances

In the case of resonances of order higher than four, the motion is stable in generic cases creating a one-parameter fam-

Table 2: Position of one of the separatrices for resonance (1,2)

	x	x'	z	z'
NF-2	0.0360	0.0000	0.0505	0.0000
NF-5	0.0353	0.0000	0.0496	0.0000
Track.	0.0354	0.0000	0.0496	0.0000

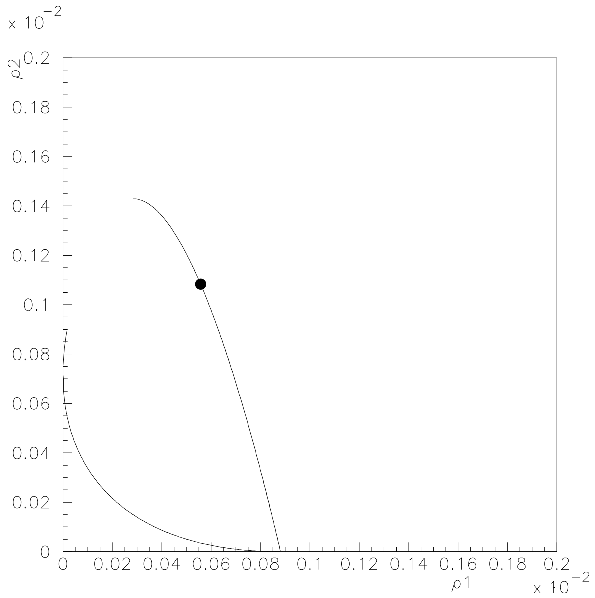

Figure 2: Minimum and maximum distance of the separatrix due to resonance $(1, 2)$ in the space of nonlinear emittances

ily of islands. The island width is changing with the second invariant, and is therefore strongly dependent on the position in phase space. The list of the analysed resonances and of the selected linear tunes is given in Tab. 3. In the last two columns we give the value of the width of the island in the physical space (x, z) for a fixed value of the second invariants. In all cases the normal form series were truncated at two orders higher than the first significant order, i.e. order 6 for the $(5, 0)$ and $(1, -4)$ respectively and order 10 for the $(9, 0)$ resonance. The agreement of the widths can be considered very satisfactory. It must be pointed out that it cannot be predicted, a priori, when the higher orders of the map are relevant to determine the width of a resonance to the stated precision. For instance, the width of the island evaluated for the $(9, 0)$ resonance at order 8 is 0.45, i.e. more than three times the value computed at order 10, which agrees so well with tracking. For all resonances, studied in this report, we found that when the normal form series were truncated at two orders higher than the first significant order the accuracy of the computation was always increased. However, we could not use much higher orders so as to avoid divergences of the perturbative series. We also checked the position of the elliptic and of the hyperbolic fixed lines and we found that the normal form predictions were accurate.

Table 3: Comparison of some island widths

Resonance	Q_x	Q_z	Tracking	NF
$(5, 0)$	63.203	63.310	0.176 ± 0.001	0.178
$(1, -4)$	63.280	63.315	0.156 ± 0.017	0.137
$(9, 0)$	63.220	63.310	0.012 ± 0.001	0.014

4 CONCLUSIONS

The implemented normal form tool has proved to be very useful to determine the network of resonances and the global dynamics in phase space. The higher order effects can be very relevant, and are automatically take into account by the normal form codes [16]. The agreement with tracking is excellent and the perturbative methods were shown to be effective for all the analysed resonances (i.e. up to 9-th order resonances). The higher orders are sometimes very relevant, and have always increased the accuracy of the computations when the normal form series were truncated at two orders higher than the first significant order.

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