ON TRIPLE FOCUSING DIPOLE MAGNETS

D. Bernard and A. E. Specka
LPNHE, Ecole Polytechnique, 91128 Palaiseau, France
IN2P3 & CNRS

Abstract

We present a general, paraxial study of triple focusing (i.e., stigmatic and non-dispersive) index-free dipole magnets. The transcendental equations which describe such magnets lead to a second-degree polynomial equation. The two real solutions of this equation correspond to magnets having either one or no intermediate focal point in the vertical direction. The first-order optical properties of the physical solutions are studied.

1 INTRODUCTION

After the focusing properties of inclined entrance and exit faces of dipole magnets were discovered, the possibility of designing double focusing (DF, i.e., stigmatic) deflecting magnets became apparent [1, 2]. In general, these magnets also disperse the beam; that is, particles with different momenta entering on a common trajectory follow different trajectories after leaving the magnet.

It is obvious that the dispersion increases inside a magnet up to a certain deflection angle of the particles, and then decreases. Therefore, it is possible to design triple focusing (TF, i.e., stigmatic and non-dispersive) dipole magnets.

Several examples of such magnets have already appeared in the litterature [3, 4, 5, 6]. In each case, the choice of a particular configuration has led to a simple solution of the equations that define TF magnets. The magnet described in [3, 4] has a deflection angle of 270^{o} and a parallel incident beam. In [6], a solution is obtained in the particular case of symmetric magnets. In [5], a solution is obtained through a numerical computation. No complete study has been presented yet, as "*the transcendental equations are quite unwieldy*" [5].

More recently, interest in TF magnets has been renewed for their use in bunch compression [7, 8], since the time of flight through the magnet depends on the momentum of the incident particle.

TF magnets are also useful when the path length of the particles must be minimized. The present work has, in fact, been triggered by the need for non-dispersive and stigmatic injection of an electron beam into a gas vessel for the plasma wave acceleration experiment at the Ecole Polytechnique [9]. In this experiment, the use of a combination of three magnets would have resulted in too long a path in the gas and thus to an intolerable amount of scattering of the injected electrons before they reach the plasma.

We present a general study of triple focusing, dipole (zero index) single magnets[10]. We use Enge's paraxial formalism in the approximation of a small air gap with the same notation as in [11].

2 PARAXIAL DESCRIPTION OF TF MAGNETS

A dipole magnet is usually described by the following parameters [11]: the straight drift length A (B) before (after) the magnet, the radius of curvature R of the particles, the angle α (β) of the central ray with respect to the normal on the entrance (exit) face of the dipole, and the deflection angle φ. All quantities of dimension length can be normalized to the radius of curvature, in which case they are denoted by a lower case letter, e.g. $a = A/R$. The normalized drift lengths are called *arms* here. Hence, the set (a,b,φ,α,β) defines a dipole magnet.

This parametrisation might seem counter-intuitive. In practice, one would rather position a source with a given beam direction in front of a magnet with given dipole angle ω, thus fixing a and α. The deflection and exit angles φ and β would then be determined by ω, a, and α and the position of the exit face of the magnet. On the contrary, it appears here as if one designs the magnet around a given central trajectory.

Let (O, x, y, z) be the moving frame of the central ray, in which (Ox) is the tangent, (Oy) the normal, (Oz) the bi-normal, and s is the curvilinear coordinate. The plane (Oxy) is called the median or horizontal plane of the magnet, and the direction of the magnetic field (Oz) is called the vertical direction. The angles are written $y' = \mathrm{d}y/\mathrm{d}x$, and the momentum deviation $\Delta p/p$ is denoted δ. In the paraxial approximation, the particle coordinates in phase space $(y_2, y'_2, z_2, z'_2, \delta)$ at $s = s_2$ are linear functions of the entrance coordinates in phase space at $s = s_1$. The elements of the transfer matrices are written in short form, e.g. $(y|y) = \mathrm{d}y_2/\mathrm{d}y_1$.

In this notation, a TF magnet is described by the system of equations:

$$(y|y') = 0, \qquad (z|z') = 0, \qquad (y|\delta) = 0, \qquad (1)$$

where each matrix element depends on the five parameters defining the dipole.

The transfer matrices for transport inside the magnet are given in [11]. The transfer matrices from the object to the image are simply obtained as the product of the matrices of the drift length b, of the magnet, and of the drift length a.

Using the notations $t_\alpha = \tan\alpha$ and $t_\beta = \tan\beta$, the system of equations (1) then reads:

$$
\begin{aligned}
0 &= \sin\varphi + (\cos\varphi + t_\alpha \sin\varphi)a + (\cos\varphi + t_\beta \sin\varphi)b \\
&\quad + \Big((t_\alpha t_\beta - 1)\sin\varphi - (t_\alpha + t_\beta)\cos\varphi\Big)ab \qquad \text{(2a)} \\
0 &= \varphi + (1 - t_\alpha\varphi)a + (1 - t_\beta\varphi)b \\
&\quad + (t_\alpha t_\beta \varphi - t_\alpha - t_\beta)ab \qquad \text{(2b)} \\
0 &= 1 - \cos\varphi + b\sin\varphi + b(1 - \cos\varphi)t_\beta \qquad \text{(2c)}
\end{aligned}
$$

This system of equations fixes three of the five parameters which describe the dipole.

3 SOLUTION OF THE SYSTEM OF EQUATIONS

The system of equations (2) is transcendental only in φ and otherwise linear in a, b, t_α, and t_β. Therefore, we found it convenient to solve the system of equations (2) for the entrance and exit angles α and β.

First, we solve (2c) for t_β:

$$t_\beta = -\left(\frac{1}{b} + \cot\left(\frac{\varphi}{2}\right)\right) \qquad (3)$$

Next, we substitute the expression obtained for t_β in (2b) and solve for b:

$$b = -2\frac{a + (1 - at_\alpha)\varphi}{\Big(a + (1 - at_\alpha)\varphi\Big)\cot\left(\frac{\varphi}{2}\right) + 1 - at_\alpha} \qquad (4)$$

Then, we insert (3) and (4) into (2a) and solve for t_α:

$$t_\alpha = -\left(\frac{1}{a} + \cot\left(\frac{\varphi}{2}\right)\right) \qquad (5)$$

Finally, we substitute the expression obtained for t_α in (4):

$$ab\cot\left(\frac{\varphi}{2}\right)\left(\varphi\cot\left(\frac{\varphi}{2}\right) + 2\right) + 2(a+b)\left(\varphi\cot\left(\frac{\varphi}{2}\right) + 1\right) + 4\varphi = 0 \qquad (6)$$

The system of equations [3,5,6] is explicitly symmetric in entrance and exit, i.e., invariant under the exchange $(a,\alpha) \leftrightarrow (b,\beta)$, obviously because of the symmetry under time reversal.

Introducing the parameter $\kappa = \varphi\cot\left(\frac{\varphi}{2}\right)$ and the arm ratio $g = b/a$, and substituting $u = a/\varphi$, we re-write equation (6) in a more compact form:

$$g\kappa(\kappa+2)u^2 + 2(g+1)(\kappa+1)u + 4 = 0 \qquad (7)$$

This second-degree polynomial equation in u has two solutions which depend on the two free parameters g and κ ($\kappa < 2$ for $0 < \varphi < 2\pi$).

We are looking for solutions of (7) describing physical magnets, i.e., magnets with positive arms and therefore $g > 0$, and u real and positive. The ratio g being positive, the discriminant of equation (7) $\Delta' = (\kappa+1)^2(g-1)^2 + 4g$ is always positive. Therefore, equation (7) always has two real solutions u_+ and u_-. In order to study their sign, we form their product p and their sum s from the coefficients of (7):

$$p = \frac{4}{g\kappa(\kappa+2)} \qquad\qquad s = -\frac{2(g+1)}{g}\cdot\frac{(\kappa+1)}{\kappa(\kappa+2)} \qquad (8)$$

We give the sign of p and s, and thus of u_+ and u_- in the following table:

κ	$-\infty$ $\quad$ -2	-1 $\quad$ 0	2
φ	360^o $\quad$ 232.5^o	180^o	0
p	$+$	$-$	$+$
s	$+$	$-$ | $+$	$-$
$\{0\}$	$u_+ > 0$	$u_+ > 0$	$u_+ < 0$
$\{1\}$	$u_- > 0$	$u_- < 0$	$u_- < 0$

Physical solutions only exist in the interval $\pi < \varphi < 2\pi$, corresponding to negative values of κ. We obtain exactly one physical solution in the interval $\pi < \varphi < \varphi_0$, where φ_0 is defined by $\varphi_0\cot\left(\frac{\varphi_0}{2}\right) = \kappa_0 = -2$, that is, $\varphi_0 \approx 232.5^o$, and two physical solutions in the interval $\varphi_0 < \varphi < 2\pi$.

The two branches of solutions correspond respectively to magnets having either no or exactly one intermediate focal point in the vertical direction, and are therefore denoted $\{0\}$ and $\{1\}$. Magnets of both classes also have an intermediate focal point in the horizontal plane. As the dispersion at this point is large, momentum selection may be done easily using collimating slits.

We calculate the parameters a, b, α, β, and the dipole angle ω analytically as functions of φ and g using equations (3),(5), and (7), and present them in figure 1. The symmetry between entrance and exit appears clearly in figure 1a. The dipole angle $\omega = \varphi + \alpha + \beta$ is obviously symmetric under exchange of g and $1/g$.

Figure 1b shows the above parameters as functions of φ for arm ratios greater than unity[1]. One observes that the arms are longer on branch $\{1\}$ than on branch $\{0\}$. The magnet described in references [3, 4] is indicated by an open square. Ray tracing of example magnets for branches $\{0\}$ and $\{1\}$ are presented in reference[10].

4 OPTICAL PROPERTIES OF THE SOLUTIONS

Magnification The horizontal and vertical magnifications are given by $m_y \equiv (y|y)$ and $m_z \equiv (z|z)$ and are equal to $m_y = g$ and $m_z = -g(\kappa+1) - \frac{2}{u}$ [10].

Obviously, m_z is positive for branch $\{1\}$, and negative for branch $\{0\}$, depending directly on the existence of an intermediate focal point in the vertical direction.

Of particular interest is the case where the image shows no first-order distortion, that is, where the horizontal and vertical magnifications have the same absolute value: $m_z = \pm m_y$, that is, $m_z = \pm g$. Using the above expression, we

[1] The graphs for arm ratios smaller than unity may be obtained by interchanging a and b, and α and β.

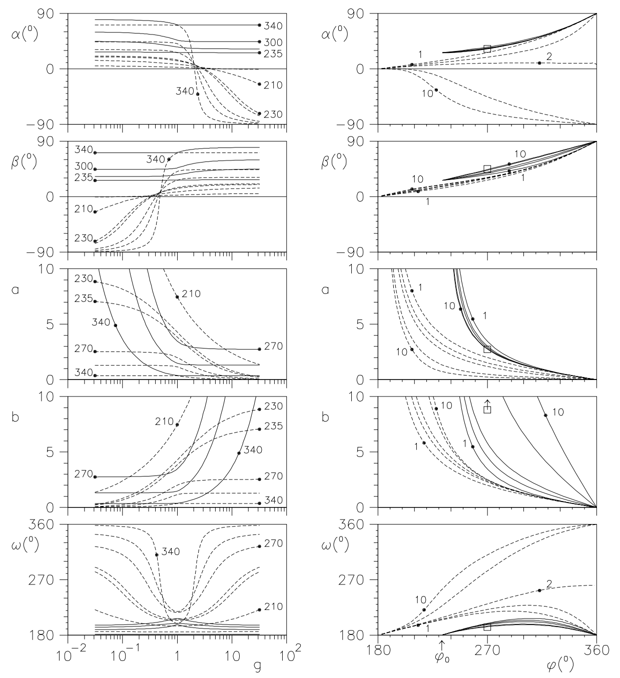

Figure 1: Variation of a,b,α,β and ω with g and φ. Solid: branch $\{1\}$, dashed: branch $\{0\}$.

obtain $u = -2/g(\kappa + 1 \pm 1)$. Substituting this expression for u in equation (7), we obtain $g = 1$. Hence, the condition for identical magnifications in the horizontal and vertical directions is independent of φ.

Angular dispersion The angular dispersion is $(y'|\delta) = \sin\varphi + (1 - \cos\varphi)t_\beta$, that is, $(y'|\delta) = (\cos\varphi - 1)/b$ here, and is negative for $\varphi < 360^{\text{o}}$. It is therefore impossible to design strictly achromatic, i.e., non-dispersive and angle achromatic, DF magnets with finite arm lengths.

Time of flight The variation λ of the time of flight of the particles is described in the same paraxial formalism by a development to first order around the central trajectory.

The coefficients describing the dependance of λ on position and angle in the vertical direction vanish for symmetry reasons.

In the horizontal plane, the coefficient describing the dependance of λ on angle is $(\lambda|y') = (1 - \cos\varphi)$ and is positive for $\varphi < 360^{\text{o}}$. The coefficient describing the dependance of λ on position is $(\lambda|y) = \sin\varphi + (1 - \cos\varphi)t_\alpha$, that is, $(\lambda|y) = (\cos\varphi - 1)/a$ here, and is negative. The coefficient describing the dependance of λ on momentum is $(\lambda|\delta) = \varphi - \sin\varphi$, and is positive.

We can see that it is impossible to design isochronous TF magnets, with finite arm lengths.

Special case of zero arm ratio ($g \to 0$) A case of practical interest is the deflection of a parallel beam with a large energy spread and its focusing on a small spot. This situation corresponds to the limit $g \to 0$, $u \to \infty$, and gu finite. We then read equation (7) as an equation in gu: $\kappa(\kappa+2)(gu)^2 + 2(\kappa+1)gu = 0$. The non-trivial solution (branch $\{1\}$) is: $gu = -2(\kappa+1)/\kappa(\kappa+2)$.

For the particular value $\varphi = 270^{\text{o}}$, that is, $\kappa = -\varphi$, we obtain $b \to 2(3\pi - 2)/(3\pi - 4) \approx 2.74$, $a \to \infty$, $\alpha \to \pi - \varphi/2$, that is, $\alpha \to 45^{\text{o}}$, and $\tan\beta \to 3\pi/2(3\pi - 2)$, that is, $\beta \to 32.4^{\text{o}}$. This magnet has been presented in [3, 4].

Special case of a unit arm ratio ($g = 1$) For $g = 1$, we get $a = b$ and $\alpha = \beta$: the magnet is symmetric. We then get unit absolute magnifications: $m_y = 1$ and $m_z = \pm 1$. The solution of equation (7) is particularly simple here, with $\Delta' = 4$, $u = -2/\kappa$ (branch $\{0\}$) and $u = -2/(\kappa+2)$ (branch $\{1\}$). The particular value $\varphi = 270^{\text{o}}$ gives $a = b = 2$, $\alpha = \beta = \arctan(1/2) \approx 26.6^{\text{o}}$ on branch $\{0\}$ used in reference [9].

5 CONCLUSION

We have presented a general study of TF, index-free, single dipole magnets. We have written the paraxial equations describing TF magnets in the approximation of a small air gap.

We have solved the system of three equations in five variables, which leads to a second-degree polynomial equation. The solutions describe a two dimensional surface which is parametrized by the angle of deflection of the particles and by the arms ratio.

Physical solutions exist in the interval $180^{\text{o}} < \varphi < 360^{\text{o}}$ only. We obtain one physical solution in the interval $180^{\text{o}} < \varphi < 232.5^{\text{o}}$, and two physical solutions in the interval $232.5^{\text{o}} < \varphi < 360^{\text{o}}$. The two branches of solutions correspond respectively to magnets having either none or only one intermediate focal point in the vertical direction.

6 REFERENCES

[1] W. G. Cross, Rev. Sci. Instr. 22 (1951) 717.

[2] M. Camac, Rev. Sci. Instr. 22 (1951) 197.

[3] H. A. Enge, Rev. Sci. Instr. 34 (1963) 385.

[4] Ref. [11], page 225.

[5] M. Komma, Nucl. Instr. and Meth. 154 (1978) 271.

[6] H. Ejiri et al., Nucl. Instr. and Meth. 134 (1976) 107.

[7] C. D. Johnson, J. Ströde, Internal Note CERN/PS/LP Note 91-42, CLIC NOTE 148, Sept. 1991.

[8] M. Borland, SLAC Report 402, Feb. 1991.

[9] F. Amiranoff et al., Nucl. Instr. and Meth. A **363** (1995) 494.

[10] D. Bernard, A. E. Specka, Nucl. Instr. and Meth. A **366** (1995) 43.

[11] H. A. Enge, *Deflecting Magnets*, in "Focusing of Charged Particles", edited by A. Septier, Academic Press, 1967.