

# Emittance growth and beam halo due to incoherent random phenomenoms

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## Abstract

Random phenomenoms like coulomb scattering or synchrotron radiation increases the transverse emittance of a beam and can give a halo. Most of the analytical estimates of the effects are done by supposing a long transfer line (many betatron oscillations) and by taking into account the RMS part of the phenomenom (Gaussian model). A more general theory has been developed to calculate the real evolution of the particle density in phase space. It remains applicable if the phenomenoms are not Gaussian or if the beam line is short. In particular, the beam profile can be derived from this theory, showing that the Gaussian model must be corrected by a "halo function". This function shows explicitly the influence of the transport functions and of the high order moments of the elementary phenomenom. Simulations confirm the theory.

## 1. INTRODUCTION

The problem of calculating a beam profile is well known and precise enough for most of the machines. For example, for electron storage rings, one can suppose, with a good approximation, that the beam is gaussian in any direction of the phase space. It is then possible to define a region containing, for example, 99.994% of the particules by taking 4 standard deviations (or more) in the appropriate direction. *Grosso modo*, one can suppose that the beam density is Gaussian and that the "contribution" of the line, due to random phenomenoms, is also Gaussian. Then, the resulting density is also Gaussian in each direction. In this paper, we shall consider a transfer line where particles are subject to incoherent random phenomenoms like synchrotron radiation or coulomb scattering. We suppose that the input beam profile density  $n_0(x)$  is known, as well as the laws of the random phenomenom. We want to calculate the beam profile density at the line exit, namely,  $n_1(x)$ . Mathematically speaking, we have to find a density function  $p$ , such as:

$$n_1(x) = n_0 * p_x(x) \quad (1)$$

where \* denotes the classical convolution product. As they are probability density functions, we have also:

$$\int_{-\infty}^{+\infty} n_0(\xi) d\xi = \int_{-\infty}^{+\infty} n_1(\xi) d\xi = \int_{-\infty}^{+\infty} p(\xi) d\xi = 1 \quad (2)$$

This study presents similarities with [1]. For Coulomb scattering, it can be compared to [2]. The main difference is that the aim of the calculations is to be more general than the calculation of an *equilibrium profile* inside a storage ring.

*Nota bene*: it must be noted that the space available in these proceedings is too small to have complete and detailed calculations. These can be found in [3].

Let  $\mathbf{X}$  be a vector in a n-dimension *phase space* and let  $\mathbf{K}$  be also a n-dimension vector. For  $\phi$  being a real function of  $\mathbf{X}$ , we define its Fourier transform by:

$$\hat{\phi}(\mathbf{K}) = \int_{\mathbb{R}^n} e^{i\mathbf{K}\mathbf{X}} \phi(\mathbf{X}) d^n\mathbf{X} \quad (3)$$

The Dirac function in a n-dimension space will be denoted  $\delta$ . More generally, this paper will deal with distributions rather than functions, especially for Fourier transforms: for example, the Fourier transform of the constant function 1 is well defined and equal to  $\delta$ .

## 2. MATHEMATICAL MODEL

### 2.1 Elementary contribution

This study can be compared to [1]. Let  $\bar{\lambda}$  be the mean number of interactions per particle and per meter. Let  $D(\mathbf{X},s)$  be the probability density in phase space of the elementary phenomenom. On a elementary  $ds$  length, the number of interactions follows a Poisson law with a mean number of interactions  $\zeta = \bar{\lambda} ds$ . Consequently, the relative amount of particles which do not interact is:

$$(\zeta^0/0!) e^{-\zeta} = 1 - \zeta = 1 - \bar{\lambda} ds \quad (4)$$

The proportion of particles having  $m$  ( $m>1$ ) interactions is  $(\zeta^m/m!) e^{-\zeta}$  is in  $(ds)^2$  and is of higher order. So, if  $N$  is the beam density in phase space, we can write:

$$N(\mathbf{X},s+ds) = N(s) * v(\mathbf{X},s) \quad (5)$$

where:

$$v(\mathbf{X},s) = (1 - \bar{\lambda} ds)\delta + \bar{\lambda} D(\mathbf{X},s) ds \quad (6)$$

The  $\delta$  function corresponds to the particles who did not interact.  $D$  describes one interaction. The integral of  $v$  over the whole space must be 1, as it is a probability density. If  $T_{S \leftarrow s}$  is the transport matrix of the **perturbation** (see 2.5) from the  $s$  abscissa to the end of the line (abscissa  $S$ ), then the elementary contribution of the segment  $[s,s+ds]$  is, transported at exit:

$$v_S(\mathbf{X},s) = (1 - \bar{\lambda} ds)\delta + \bar{\lambda} D(T_{S \leftarrow s}\mathbf{X},s) ds \quad (7)$$

### 2.2 Contribution of the whole line

The total contribution of the line, taken at the exit, is the (infinite) convolution product of the  $v_S(\mathbf{X},s)$  densities. The simplest way to calculate it is to consider the second characteristics of  $v_S(\mathbf{X},s)$ , which gives an integral over the whole line. By this way, if  $N_0$  is the entrance density in phase space, we obtain the density at exit  $N_1$  by:

$$N_1(x) = N_0 * p(x) \quad (8)$$

with: 
$$\hat{p}(\mathbf{K}) = \exp \left[ \bar{\lambda} \int_0^S (\hat{D}_0(\mathbf{K}, s) - 1) ds \right] \quad (9)$$

and 
$$D_0(\mathbf{X}, s) = D(\tilde{T}_{S \leftarrow s} \mathbf{X}, s) \quad (10)$$

The second characteristics of  $p(\mathbf{X})$  is  $q(\mathbf{K})$  such as:

$$q(\mathbf{K}) = \bar{\lambda} \int_0^S (\hat{D}_0(\mathbf{K}, s) - 1) ds \quad (11)$$

The calculation of  $\hat{D}_0(\mathbf{K}, s)$  is made with the variable change  $\mathbf{U} = \tilde{T}_{S \leftarrow s} \mathbf{X}$ . The Jacobian  $\alpha_s = |\det \tilde{T}_{S \leftarrow s}|$  is the oscillation damping factor from  $s$  to  $S$ . Then, if " $\sim$ " denotes matrix transposition, we get:

$$\hat{D}_0(\mathbf{K}, s) = \alpha_s \hat{D}(\tilde{T}_{S \leftarrow s}^T \mathbf{K}) \quad (12)$$

If we suppose that  $D$  is independant of  $s$ , we get finally:

$$q(\mathbf{K}) = \bar{\lambda} \int_0^S \alpha_s \hat{D}(\tilde{T}_{S \leftarrow s}^T \mathbf{K}) ds - \bar{\lambda} S \quad (13)$$

In the following, we shall suppose  $\alpha_s = 1$ .

### 2.3 Beam profile

We have got the density function  $p$  in phase space, which corresponds to the emittance obtained after transport of a zero input emittance. For  $\mathbf{X} = [x_1 \dots x_n]$ , the "generated profile" is the marginal density of  $p$ , namely  $p_x(x_1)$  such as:

$$p_x(x_1) = \int_{-\infty}^{+\infty} p(x_1, x_2, \dots, x_n) dx_2 \dots dx_n \quad (14)$$

Writing  $p$  as the inverse Fourier transform of  $\hat{p}$ , and working with distributions rather than functions, we obtain easily:

$$p_x(x_1) = \frac{1}{(2\pi)} \int_{-\infty}^{+\infty} \hat{p}(a, 0, 0, \dots) e^{-iax_1} da \quad (15)$$

or: 
$$\hat{p}_x(a) = \hat{p}(a, 0, 0, \dots) \quad (16)$$

This gives a simple result : the second characteristics of the beam profile in  $x_1$  is the second characteristics of the total density, where all the terms which do not correspond to  $x_1$  (in the order of writing) have been made equal to zero.

### 2.4 Development of $q_x(k_1) = q_x(k_1, 0, \dots)$ in series

Let us write  $\mathbf{X} = [x_1 \dots x_n]$  and  $\mathbf{K} = [k_1 \dots k_n]$ .

$$\hat{D}(\tilde{T}_{S \leftarrow s}^T \mathbf{K}) = \int_{\mathbb{R}^n} \exp \left[ i \tilde{T}_{S \leftarrow s}^T \mathbf{K} \mathbf{X} \right] D(\mathbf{X}) d^n \mathbf{X} \quad (17)$$

$$\left. \frac{\partial^m \hat{D}(\tilde{T} \mathbf{K})}{\partial k_1^m} \right|_{\mathbf{K}=0} = i^m \int_{\mathbb{R}^n} \left( \tilde{T}_{11} x_1 + \dots + \tilde{T}_{n1} x_n \right)^m D(\mathbf{X}) d^n \mathbf{X} \quad (18)$$

$$\left. \frac{\partial^m \hat{D}(\tilde{T} \mathbf{K})}{\partial k_1^m} \right|_{\mathbf{K}=0} = i^m \sum (\text{Transport})(\text{Moments of } D) \quad (19)$$

So, we can write  $q_x(k_1)$  as:

$$q_x(k_1) = \bar{\lambda} \sum_{m=1}^{+\infty} \frac{i^m}{m!} C_m k_1^m \quad (20)$$

We have shown the decoupling of the transport terms and of the moments of the fundamental interaction. According to 2.3, only the terms  $k_1$  (corresponding to  $x_1$ ) must be considered. We can forget all the others if we are only interested by the profile along  $x_1$ .

If  $D$  acts along one particular direction  $x_r$  (ex: synchrotron radiation, which gives only a random change of  $\Delta p/p$ , see 2.5), only the moments

$$\mu_m = \int_{\mathbb{R}^n} x_r^m D(\mathbf{X}) d^n \mathbf{X} \quad (21)$$

and the  $\tilde{T}_{r1}$  terms appear, leading to:

$$q_x(k_1) = \bar{\lambda} \sum_{m=1}^{+\infty} \frac{i^m}{m!} \mu_m \tilde{T}_{r1}^m k_1^m \quad (22)$$

For other phenomenons like Coulomb scattering,  $D$  acts along the transverse  $x'$  and  $y'$  directions, but with the same law, leading to no too much complicated results.

### 2.5 Example: synchrotron radiation.

We consider a structure like an electron recirculating accelerator (ex: ELFE arcs [4][5]) where synchrotron radiation gives an emittance growth (no damping).

Let us consider the  $(x, x', \Delta p/p)$  space. The elementary phenomenon is a random change  $\varepsilon$  of  $\Delta p/p$  only,  $x$  and  $x'$  remaining constant. So, we can write:

$$D(\mathbf{X}) = \delta_x \delta_{x'} \varepsilon \left( \frac{\Delta p}{p} \right) \quad (23)$$

If  $\eta$  is the dispersion function, if the  $U_{ij}$  terms are the transport functions, the transfer matrix of the perturbation is:

$$T = \begin{bmatrix} U_{11} & U_{12} & \eta U_{11} + \eta' U_{12} \\ U_{21} & U_{22} & \eta U_{21} + \eta' U_{22} \\ 0 & 0 & 1 \end{bmatrix} \quad (24)$$

Supposing that the odd moments are zero and using formula (22), we obtain a generalization of [4], equation (25):

$$q_x(k_1) = \bar{\lambda} \sum_{m=1}^{+\infty} \frac{(-1)^m}{(2m)!} \int_0^S (\eta U_{11} + \eta' U_{12})^{2m} ds \mu_{2m} k_1^{2m}$$

### 2.6 Beam profile model

We deal only with the contribution of the line  $p_x$ . If the mean number of interactions  $\lambda = \bar{\lambda} S$  is low, the proportion  $\chi = e^{-\lambda}$  particles will not interact. So,  $p_x$  will not be a function but a distribution, as in 2.1:

$$p_x(x) = \chi \delta + (1-\chi) \phi(x) \quad (26)$$

where  $\phi$  is a probability density. From:

$$\hat{p}_x(k_1) = \exp(q_x(k_1)) = \chi + (1-\chi) \hat{\phi}(k_1) \quad (27)$$

we deduce that we must do the inverse Fourier transform of the function:

$$\hat{\phi}(k_1) = \frac{\exp[q_x(k_1)] - \chi}{1 - \chi} \quad (28)$$

$\chi$  and  $q_x$  being known. The inverse Fourier transform of  $\hat{p}_x$  cannot work from the numerical point of view.

### 2.7 Inverse Fourier transform. Halo function.

Let  $Q_n(x) = e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$  be the  $n$ th Hermite polynomial.

Let  $h_n(x) = e^{-x^2/2} Q_n(x)$ . These functions are invariant by Fourier transform, that is:

$$\int_{-\infty}^{+\infty} e^{isx} h_n(s) ds = (-i)^n \sqrt{2\pi} h_n(x) \quad (29)$$

For  $q_x(a) = -\frac{\sigma_\phi^2}{2} a^2 + \dots$ , it can be shown that the standard deviation of  $\phi$  is given by  $\sigma_\phi^2 = (1 - \chi) \sigma^2$  and that:

$$\hat{\phi}(a) = \exp\left[-\frac{\sigma_\phi^2}{2} a^2 + \Xi(a)\right] \quad (30)$$

Let us write:  $\Phi(a) = \exp\left[\Xi\left(\frac{a}{\sigma_\phi}\right)\right]$  (31)

and:  $\hat{\psi}(a) = \hat{\phi}\left(\frac{a}{\sigma_\phi}\right) = \exp\left[-\frac{a^2}{2}\right] \Phi(a)$  (32)

which is a "normalization" of  $\hat{\phi}$ . Then:

$$\phi(x) = \frac{1}{\sigma_\phi} \psi\left(\frac{x}{\sigma_\phi}\right) \quad (33)$$

We develop  $\Phi(a)$  on a Hermite polynomial basis:

$$\Phi(a) = \sum_{n=0}^{+\infty} \beta_n Q_n(a) \quad (34)$$

The orthogonality of  $Q_n$  as well as:

$$\int_{-\infty}^{+\infty} h_n^2(x) dx = 2^n n! \sqrt{2\pi} \quad (35)$$

leads to:

$$\beta_n = \frac{\int_{-\infty}^{+\infty} e^{-a^2} \Phi(a) Q_n(a) da}{2^n n! \sqrt{2\pi}} \quad (36)$$

The numerator can be computed by quadrature [6] also by using Hermite polynomials. Then, we obtain (37):

$$\phi(x) = \frac{1}{\sigma_\phi} \psi\left(\frac{x}{\sigma_\phi}\right) = \frac{1}{\sigma_\phi \sqrt{2\pi}} \exp\left[-\frac{x^2}{2\sigma_\phi^2}\right] \sum_{n=0}^{+\infty} (-i)^n \beta_n Q_n\left(\frac{x}{\sigma_\phi}\right)$$

If only the even terms are not zero, we get (38):

$$\phi(x) = \frac{1}{\sigma_\phi \sqrt{2\pi}} \exp\left[-\frac{x^2}{2\sigma_\phi^2}\right] \sum_{n=0}^{+\infty} (-1)^n \beta_{2n} Q_{2n}\left(\frac{x}{\sigma_\phi}\right)$$

Writing:  $\mathcal{H}_\ell(x) = \sum_{n=0}^{+\infty} (-1)^n \beta_{2n} Q_{2n}\left(\frac{x}{\sigma_\phi}\right)$ , halo function (39)

and:  $\mathcal{G}(x) = \frac{1}{\sigma_\phi \sqrt{2\pi}} \exp\left[-\frac{x^2}{2\sigma_\phi^2}\right]$ , Gaussian density (40)

the contribution of the line to the profile density is:

$$p_x(x) = e^{-\lambda} \delta + (1 - e^{-\lambda}) \mathcal{G}(x) \mathcal{H}_\ell(x) \quad (41)$$

This gives the classical Gaussian contribution for large  $\lambda$ .  $\mathcal{H}_\ell(x)$  is a correction function of the classical formulation.

## 3. SIMULATION

We have simulated an obvious non-Gaussian phenomenon in phase space  $(x, x')$  where each interaction acts like a random change of  $x'$  (uniform on a given interval) of odd moments zero. The transfer line is modeled by supposing an harmonic betatron oscillation. The input "beam" has a Dirac emittance ( $x=x'=0$  for all particles). For  $\lambda=0.5$  interactions (mean value) and for one betatron oscillation only, we get the profile given figure 1, with a good precision up to 4 standard deviations, when the Gaussian model fails rapidly.

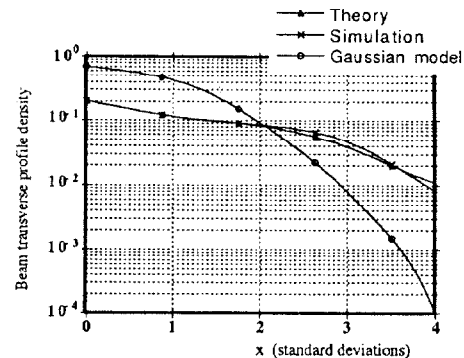


Figure 1. Comparison of profiles

## 6 REFERENCES

- [1] K.Hirata and K.Yokoya: non gaussian distribution of electron beams due to incoherent stochastic processes. KEK-91-212.
- [2] T. Raubenheimer: Emittance growth due to beam-gas scattering. KEK 1991.
- [3] J.M. De Conto: détérioration de faisceau le long d'une ligne de transport. ISN report 93-120.
- [4] P. Audy, G. Leleux, A. Tkatchenko: Détérioration de l'émission transverse due au rayonnement synchrotron dans une ligne de transport. LNS/GT/90-03.
- [5] The ELFE project, an Electron Laboratory For Europe, edited by J. Arvieux and E. De Sanctis. Società Italiana di Fisica, Editrice Compositori, Bologna 1992.
- [6] Abramovitz et Stegun: handbook of mathematical functions.