Spin Tune Shifts and Closed Orbit Distortions

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Abstract

The application of analytical method for spin calculation is described. The calculation of both orbital and spin motion is based on Lie operators technique. The spin tune shift due to spin resonances and rms energy shift is estimated using this technique. The comparison with the experimental data is presented. The computer code SPINLIE was used for simulation of the rms energy shifts for colliders with low (VEPP 2M), medium (VEPP-4M) and high (HERA-e) energies.

1 SPIN TUNE SHIFTS

As it was shown in [1],[2] the solution of the BMT equation

$$\frac{d\vec{s}}{d\theta} = [\vec{W}, \vec{s}] \qquad (1)$$

can be found using Lie operators tecnique. Here $\vec{s}$ and $\vec{W}$ are spin and its precision frequency corespondingly and θ is the azimuth over the ring. The BMT frequency $\vec{W}$ can be divided into the $\vec{W}^{(0)} + \vec{\omega}$. Here $\vec{\omega}$ is a small correction due to errors.

The solution of this equation can be written as a map for spin vector from initial azimuth θ_0 to final θ:

$$\vec{s}(\theta) = S(\theta_0, \theta)\vec{s}(\theta_0).$$

The map S is found as the expansion of the exponential Lie operators product:

$$S(\theta_0, \theta) = e^{:\vec{\mathcal{W}}^{(0)}(\theta_0,\theta)\vec{s}:} \cdot e^{:\vec{\mathcal{W}}^{(r)}(\theta_0,\theta)\vec{s}:} = $$
$$= S^{(0)}(\theta_0, \theta) \cdot e^{:\vec{\mathcal{W}}^{(r)}(\theta_0,\theta)\vec{s}:}.$$

Here $S^{(0)}(\theta_0, \theta)$ is the usual rotation matrix. It describes the spin vector rotation due to $\vec{W}^{(0)}(\theta)$ part of BMT frequency. The expression for vector $\vec{\mathcal{W}}^{(r)}(\theta_0, \theta)$ is [2] [1]:

$$\vec{\mathcal{W}}^{(r)}(\theta_0, \theta) = \int_{\theta_0}^{\theta} d\theta' \left(S^{(0)}(\theta_0, \theta')\right)^{-1} \vec{\omega}(\theta') + $$
$$+\frac{1}{2}\int_{\theta_0}^{\theta} d\theta' \int_{\theta_0}^{\theta'} d\theta'' \cdot \qquad (2)$$
$$\cdot \left[\left(S^{(0)}(\theta_0, \theta'')\right)^{-1} \vec{\omega}(\theta''), \left(S^{(0)}(\theta_0, \theta')\right)^{-1} \vec{\omega}(\theta')\right] + \dots$$

[1] For brivety, we will consider part of $\vec{\omega}$ only, which is independent from betatron amplitudes. The influence of the betatron and synchrotron resonances is eliminated due to it. More complete results will be given in [3].

Let us take into account two terms of the expansion of $e^{:\vec{\mathcal{W}}^{(r)}\vec{s}:}$ only:

$$e^{:\vec{\mathcal{W}}^{(r)}\vec{s}:} =: E : + : \vec{\mathcal{W}}^{(r)}\vec{s} : + \frac{1}{2} : \vec{\mathcal{W}}^{(r)}\vec{s} :^2 .$$

One can find the following expression for the map S in this case:

$$S_{ij} = S^{(0)}_{ik}\left[\delta_{kj} + e_{kjm}\mathcal{W}^{(r)}_m + \frac{1}{2}e_{klm}\mathcal{W}^{(r)}_l e_{mnj}\mathcal{W}^{(r)}_n\right] =$$
$$= S^{(0)}_{ik}\left[\delta_{kj} + e_{kjm}\mathcal{W}^{(r)}_m + \frac{1}{2}\left(\mathcal{W}^{(r)}_k\mathcal{W}^{(r)}_j - \mid \vec{\mathcal{W}}^{(r)} \mid^2 \delta_{kj}\right)\right].$$

Let us investigate one turn transformation $\theta \to \theta + 2\pi$. The matrix $S^{(0)}(\theta, \theta+2\pi)$ determines the spin tune ν_0 for particles on the equilibrium orbit ($\cos 2\pi\nu_0 = \frac{Sp(S^{(0)})-1}{2}$) such that [2]:

$$\begin{aligned} S^{(0)}_{ij}(\theta, \theta+2\pi) &= \cos 2\pi\nu_0\delta_{ij} + \sin 2\pi\nu_0 e_{ijk}n^{(0)}_k + \\ &+ (1-\cos 2\pi\nu_0)n^{(0)}_i n^{(0)}_j, \end{aligned}$$

where $\vec{n}^{(0)}$ is a periodical eigen vector (on azimuth θ) of the matrix $S^{(0)}(\theta, \theta+2\pi)$. The spin tune $\nu(\theta)$ for nonequilibrium particles is determined by the trace of one turn matrix $S(\theta, \theta+2\pi)$ and after some transformations one can find:

$$\cos 2\pi\nu = \cos 2\pi\nu_0 - \frac{1}{4}(1+\cos 2\pi\nu_0) \mid \vec{\mathcal{W}}^{(r)} \mid^2 -$$
$$- \sin 2\pi\nu_0\left(\vec{n}^{(0)}\vec{\mathcal{W}}^{(r)}\right) - \frac{1}{4}(1-\cos 2\pi\nu_0)\left(\vec{n}^{(0)}\vec{\mathcal{W}}^{(r)}\right)^2.$$

Therefore the value $\Delta\nu = \nu - \nu_0$ equals:

$$\begin{aligned} 2\pi\Delta\nu(\theta) &\approx \left(\vec{n}^{(0)}(\theta)\vec{\mathcal{W}}^{(r)}(\theta)\right) + \frac{1}{4} \mid \vec{\mathcal{W}}^{(r)}(\theta) \mid^2 ctg\pi\nu_0 - \\ &- \frac{1}{4}\left(\vec{n}^{(0)}(\theta)\vec{\mathcal{W}}^{(r)}(\theta)\right)^2 tg\pi\nu_0. \end{aligned}$$

The transversal perturbation (relatively vector $\vec{n}^{(0)}$) is the main part of the value $\vec{\mathcal{W}}^{(r)}$ [4] [2]. So, averaging over the ring we receive the final result:

$$2\pi\overline{\Delta\nu} = \overline{\mathcal{W}^{(r)}_{\parallel}} + \frac{1}{4}\overline{\left|\mathcal{W}^{(r)}_{\perp}\right|^2} ctg\pi\nu_0. \qquad (3)$$

Let us introduce the spectrum:

$$\left(S^{(0)}(\theta_0, \theta)\right)^{-1}\omega_{\perp}(\theta)e^{i\nu_0\theta} = \frac{1}{2\pi}\sum_k w_k e^{ik\theta + i\psi(\theta_0)}, \qquad (4)$$

[2] $\mathcal{W}^{(r)}_{\parallel}$ is proportional to betatron amplitudes, but after averaging over the ring it becomes the second order on betatron amplitudes.

then after short calculations one can receive:

$$\begin{aligned}\mathcal{W}_{\perp}^{(r)}(\theta) &\approx \int_{\theta}^{\theta+2\pi} d\theta' \left(S^{(0)}(\theta,\theta')\right)^{-1} \omega_{\perp}(\theta') = \\ &= \frac{1}{2\pi}\sum_k w_k e^{i\psi(\theta)} \int_{\theta}^{\theta+2\pi} d\theta' e^{i(k-\nu_0)\theta'} = \cdots = \\ &= \sum_k w_k \frac{\sin\pi(\nu_0-k)}{\pi(\nu_0-k)} e^{i\tilde{\psi}(\theta)}.\end{aligned}$$

Therefore

$$\begin{aligned}\overline{\left|\mathcal{W}_{\perp}^{(r)}\right|^2} &= \sum_k \overline{|\vec{w}_k|^2}\left(\frac{\sin\pi(\nu_0-k)}{\pi(\nu_0-k)}\right)^2 + \\ &+ \sum_{k,k\neq k'} \overline{(\vec{w}_k\vec{w}^*_{k'})}\frac{\sin\pi(\nu_0-k)}{\pi(\nu_0-k)}\frac{\sin\pi(\nu_0-\nu_{k'})}{\pi(\nu_0-\nu_{k'})} \approx \\ &\approx \sum_k \overline{|\vec{w}_k|^2}\Big(\frac{\sin\pi(\nu_0-k)}{\pi(\nu_0-k)}\Big)^2.\end{aligned}$$

Now one can finaly write the following formula after substitution of this expression into (3):

$$2\pi\overline{\Delta\nu} = \overline{\mathcal{W}_{\parallel}^{(r)}} + \frac{1}{4}\frac{1}{tg\pi\nu_0}\sum_k \overline{|\vec{w}_k|^2}\left(\frac{\sin\pi(\nu_0-k)}{\pi(\nu_0-k)}\right)^2. \quad (5)$$

For *numerical estimations* of spin tune shift it is useful to connect harmonic spectrum (4) with polarization level $\mathcal{P}$ [5]:

$$\begin{aligned}\mathcal{P} &= \frac{8}{5\sqrt{3}}\frac{< k^3(n_y - \gamma\frac{\partial n_y}{\partial\gamma}) >}{\left\langle |k|^3\left[1-\frac{2}{9}n_s^2+\frac{11}{18}\left(\gamma\frac{\partial\vec{n}}{\partial\gamma}\right)^2\right]\right\rangle} \approx \\ &\approx \frac{8}{5\sqrt{3}}\frac{< k^3 >}{\left\langle |k|^3\left[1+\frac{11}{18}\left(\gamma\frac{\partial\vec{n}}{\partial\gamma}\right)^2\right]\right\rangle}.\end{aligned}$$

Here $k = \frac{eB_z}{\gamma mc^2}$ is the orbit curvature, $\vec{n}$ and $\gamma\frac{\partial\vec{n}}{\partial\gamma}$ are periodical spin and spin-orbit coupling vectors for nonequilibrium particle. Let us write equation for periodical solution $\vec{n}$ [2]:

$$\vec{n}(\theta) = \vec{n}^{(0)}(\theta) + \left[\widetilde{\mathcal{W}^{(r)}}(\theta), \vec{n}^{(0)}(\theta)\right],$$

where

$$\widetilde{\mathcal{W}^{(r)}}(\theta) = \sum_{n=0}^{\infty}\left[\left(S^{(0)}(\theta,\theta+2\pi)\right)^{-1}\right]^n \vec{\mathcal{W}}^{(r)}(\theta).$$

Taking into acount that part of $\widetilde{\vec{\mathcal{W}}}^{(r)}$ paralel to $\vec{n}^{(0)}$ is not important, one can find part of $\widetilde{\vec{\mathcal{W}}}^{(r)}_{\perp}$:

$$\begin{aligned}\widetilde{\mathcal{W}_{\perp}^{(r)}}(\theta) &= \sum_{n=0}^{\infty} e^{-2\pi i n\nu_0}\int_{\theta}^{\theta+2\pi} d\theta'\left(S^{(0)}(\theta,\theta')\right)^{-1}\omega_{\perp}(\theta') = \\ &= \frac{1}{1-e^{-2\pi i\nu_0}}\frac{1}{2\pi}\sum_k w_k e^{i\psi(\theta)}\int_{\theta}^{\theta+2\pi} d\theta' e^{i(k-\nu_0)\theta'}.\end{aligned}$$

Then:

$$\begin{aligned}\left(\gamma\frac{\partial\vec{n}}{\partial\gamma}\right)^2 &= \gamma^2\left|\frac{\partial}{\partial\gamma}\widetilde{\mathcal{W}_{\perp}^{(r)}}(\theta)\right|^2 = \cdots = \\ &= \frac{\nu_0^2}{4\sin^2\pi\nu_0}\Big[\sum_k |\vec{w}_k|^2\frac{sinc^2\pi(\nu_0-k)}{(\nu_0-k)^2} + \\ &+ \sum_{k,k'\neq k}(\vec{w}_k\vec{w}^*_{k'})\frac{sinc\pi(\nu_0-k)}{(\nu_0-k)}\frac{sinc\pi(\nu_0-k')}{(\nu_0-k')}\Big] \approx \\ &\approx \frac{\nu_0^2}{4\sin^2\pi\nu_0}\sum_k |\vec{w}_k|^2\frac{sinc^2\pi(\nu_0-k)}{(\nu_0-k)^2} = \\ &= \frac{\nu_0^2}{4\pi^2}\sum_k\frac{|\vec{w}_k|^2}{(\nu_0-k)^4}.\end{aligned}$$

It is necessary to note that the last formula valid only for isolated integer spin resonances. Now one can express the spin tune shift $\overline{\Delta\nu}$ through polarization level $\mathcal{P}$ near integer spin resonances:

$$\frac{\overline{\Delta\nu}}{\nu} = \frac{9}{11}\frac{\frac{8}{5\sqrt{3}}-\mathcal{P}}{\mathcal{P}}\frac{(k-\nu_0)^2}{\nu_0^3}\sin 2\pi\nu_0. \quad (6)$$

Unfortunately we have experimental result for the evaluation of $\overline{\Delta\nu}$, which was obtained for the collider VEPP-4 only [6]. It is as follows: for $E \approx 5$ Gev ($\nu_0 \approx 11$; it is an area near Υ-mesons family) and level polarization up then 20 % the value of the $\frac{\overline{\Delta\nu}}{\nu} \approx 2\cdot 10^{-6}$. It is good agreement with formula (6).

2 RMS ENERGY SHIFTS

As it is known [4] the energy shift ΔE appears due to taking into account finite beam rms sizes. This energy shift is connected with the quadratic nonlinearity of guide magnetic field ($s = \frac{e}{\gamma mc^2}\frac{\partial^2 B_z}{\partial x^2}$):

$$\frac{\overline{\Delta E}}{E} = \frac{1}{2 < k\psi >}\left[2\epsilon\left\langle s\beta\psi\right\rangle + \overline{\left(\frac{\Delta E}{E}\right)^2}\left\langle s\psi^3\right\rangle\right], \quad (7)$$

where $\frac{\overline{\Delta E}}{E}$ is energy shift, which is averaged over the beam distribution, β and ψ are a horizontal beta and dispersion functions; ϵ and $\overline{\left(\frac{\Delta E}{E}\right)^2}$ are a horizontal beam emittance and rms beam energy spread; $< \ldots >$ is an averaging over the ring and overline denote averaging over the beam distribution.

Energy shift appears also if one takes into account the real i.e. distorsion orbit which is connected with magnet element misalignments (especialy quadrupoles) and using the kickers for its correction. Therefore, it is necessary to add [3] new terms into (7):

$$\frac{\overline{\Delta E}}{E} = \cdots + \frac{1}{2 < k\psi >}\Big[< gD_y\Delta y > - < s\psi\Delta y^2 > \Big], \quad (8)$$

where $g = \frac{e}{\gamma mc^2}\frac{\partial B_z}{\partial x}$ is a quadrupole strenght, D_y is a vertical displacements of the quadrupoles and Δy is the vertical distorted orbit.

Spin tune is shifted with energy and all these terms (7,8) do not "destroy" the connection between spin tune and beam energy. The terms from (7) give the spin tune spreading which is averaged over the beam in contrast to the terms from (8), which describe the coherent energy and spin tune shifts. In the Table 1 the numerical results of simulation for $\overline{\Delta\nu/\nu}$ for the colliders with the different electron energies are presented.

	VEPP-2M	VEPP-4M	HERA-e
ν	1.5	11	60.5
Δy_{max}, mm	1.7	1.2	3.2
Δy_{rms}, mm	0.5	0.8	0.8
RMS D_y, mm	0.3	0.3	0.3
$\frac{<s\psi\epsilon\beta>}{<k\psi>}$	$0.28 \cdot 10^{-4}$	$0.10 \cdot 10^{-6}$	$0.30 \cdot 10^{-5}$
$\frac{<s(\Delta E/E)^2\psi^3>}{2<k\psi>}$	$-.54 \cdot 10^{-6}$	$0.27 \cdot 10^{-7}$	$0.15 \cdot 10^{-5}$
$\frac{<gD_y\Delta y>}{2<k\psi>}$	$0.22 \cdot 10^{-7}$	$0.24 \cdot 10^{-6}$	$0.27 \cdot 10^{-6}$
$\frac{<s\psi\Delta y^2>}{2<k\psi>}$	$-.14 \cdot 10^{-4}$	$0.14 \cdot 10^{-6}$	$-.19 \cdot 10^{-5}$

Table 1: $\overline{\Delta\nu/\nu}$ simulation for different colliders.

Beside that the usage of vertical bumps for the harmonic compensation of the spin-orbit coupling can produce the local distorsion of the vertical orbit. 8-closed orbit bumps are used for HERA-e, as an example. The orbit was distorted inside eight regular arcs with sextupoles on this basis (the total number of arcs is 192) and inside these arcs $\Delta y \approx 1$ cm. Under this conditions the spin tune from these bumps will be six times as much as the shift which is connected with the rest of the sextupoles (the rezults presented in the table 1 were received without taking into consideration these bumps).

3 REFERENCES

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