Computation of the Dynamic Aperture of a One Dimensional Model of a Sextupole Nonlinearity, using Analytical Tools

Massimo Giovannozzi
CERN CH-1211 Geneva 23
and Armando Bazzani, Graziano Servizi, Ezio Todesco, Giorgio Turchetti
Department of Physics, University of Bologna, Via Irnerio 46, Bologna, I-40126

Abstract

We consider the conservative Hénon mapping, which is the one dimensional model of a FODO cell with a sextupole nonlinearity. We analyse the well known problem of estimating the dynamic aperture of such a model with analytical tools. A complete solution can be found by drawing the stable and unstable manifolds which emanate from the fixed point of the map; the envelope of these manifolds exactly reproduces the boundary of stability of the model. Some numerical examples are given.

1 INTRODUCTION

The computation of the dynamic aperture of a magnetic lattice using analytical tools is an open problem in accelerator physics. The standard numerical approach, based on tracking, has the main drawback of giving no theoretical understanding of the mechanism which drives the stability boundary. On the other hand, also in the very simple case of a single sextupole nonlinearity in the one dimensional case (i.e., the Hénon map [1], [2]), a reliable analytical estimate of the dynamic aperture is not available.

In this paper we show that the stability boundary of the Hénon map is determined by the envelope of the invariant manifolds that emanate from the hyperbolic fixed point of the map itself [3]: such manifolds, through eteroclinic intersections with the invariant manifolds of the fixed cycles, explore all the chaotic region, reaching the border of stability of the map. The same mechanism should rule the dynamic aperture of generic one dimensional polynomial maps; in the two dimensional case a further analysis is required, since also the definition of dynamic aperture itself is not well posed.

In Section 2 we introduce the Hénon map, showing that it is the Poincaré section of a one dimensional magnetic lattice with a single sextupole nonlinearity; in Section 3 we define and compute the fixed points. In Section 4 we outline the method, giving an analytical proof for the case $\omega \to 0$ and giving numerical evidence for the generic case. Open problems and conclusions are given in Section 5.

2 THE MODEL

We consider a one dimensional model of a magnetic lattice, whose hamiltonian reads:

$$H(\hat{p},\hat{x};s) = \frac{\hat{x}^2+\hat{p}^2}{\beta(s)} + \sum_{n=2}^{+\infty} \frac{k_n(s)}{(n+1)!}\sqrt{\beta(s)}^{\,n+1}\hat{x}^{n+1}. \quad (2.1)$$

where $(\hat{x},\hat{p})$ are the Courant–Snyder coordinates, β the beta function and $k_n(s)$ the value of the non integrated gradient of the magnetic field.

We consider a lattice made up of N identical cells of lenght L, each cell having the phase shift ω, containing a sextupole in the kick approximation located at $s = jL$, $j \in \mathbf{Z}$, whose integrated gradient is K_2. In this case the hamiltonian reads:

$$H(\hat{p},\hat{x};s) = \frac{\hat{x}^2+\hat{p}^2}{\beta(s)} + \sum_{j\in\mathbf{Z}} \frac{\delta(s-jL)K_2}{6}\sqrt{\beta(s)}^{3}\hat{x}^3. \quad (2.2)$$

We compute the Poincaré map of this cell, i.e. a function which gives the phase space position of a particle immediately before the second sextupole placed at $s = L$ as a function of the phase space coordinates of the particle immediately before the first sextupole ($s = 0^-$); we define

$$\begin{cases} \hat{x}^- = \hat{x}(0^-) & \hat{x}^+ = \hat{x}(0^+) & \hat{x}' = \hat{x}(L^-) \\ \hat{p}^- = \hat{p}(0^-) & \hat{p}^+ = \hat{p}(0^+) & \hat{p}' = \hat{p}(L^-). \end{cases} \quad (2.3)$$

From 0^+ to L^- the solution is trivial, since the nonlinearity is zero and the Courant Snyder coordinates simply rotate by the phase shift ω:

$$\begin{pmatrix} \hat{x}' \\ \hat{p}' \end{pmatrix} = R(\omega)\begin{pmatrix} \hat{x}^+ \\ \hat{p}^+ \end{pmatrix}, \quad (2.4)$$

where $R(\omega)$ denotes a two dimensional rotation matrix. Integrating around the kick one has:

$$\hat{x}^+ - \hat{x}^- = \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} \frac{\partial H(s)}{\partial \hat{p}}ds = 0$$

$$\hat{p}^+ - \hat{p}^- = -\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} \frac{\partial H(s)}{\partial \hat{x}}ds = -\frac{K_2}{2}(\hat{x}^-)^2\sqrt{\beta(0)}^{3}. \quad (2.5)$$

Therefore the map of the lattice model reads

$$\begin{pmatrix} \hat{x}' \\ \hat{p}' \end{pmatrix} = R(\omega)\begin{pmatrix} \hat{x}^- \\ \hat{p}^- - K_2\sqrt{\beta(0)}^{3}(\hat{x}^-)^2/2 \end{pmatrix}. \quad (2.6)$$

We can get rid of the constants β and K_2 by scaling the coordinates and the momenta by a factor $\frac{2}{K_2\sqrt{\beta(0)}^3}$: in the new variables (x',p') and (x,p) the map is:

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = R(\omega)\begin{pmatrix} x \\ p - x^2 \end{pmatrix}, \quad (2.7)$$

which explicitely reads

$$x' = x\cos\omega + (p - x^2)\sin\omega$$
$$p' = -x\sin\omega + (p - x^2)\cos\omega; \qquad (2.8)$$

such a map is the simplest non trivial nonlinear map, and was first proposed by Hénon [1].

Every map obtained as a Poincaré section of a hamiltonian is symplectic; in the one dimensional case such a condition is equivalent to the area–preserving constraint:

$$\frac{\partial x'}{\partial x}\frac{\partial p'}{\partial p} - \frac{\partial x'}{\partial p}\frac{\partial p'}{\partial x} = 1. \qquad (2.9)$$

3 FIXED POINTS

The qualitative behaviour of a map $(x', p') = F(x, p)$ is determined by its fixed points and cycles; a fixed point is defined by the equation

$$(x_0, p_0) = F(x_0, p_0), \qquad (3.1)$$

i.e. is invariant under the application of the map; a fixed cycle of order m is a set of m points which is invariant under the application of the map iterated m times. The fixed points and cycles are classified according to the eigenvalues $(\lambda_1,\ \lambda_2)$ of the linearized map: if they are distinct, complex conjugate of modulus one, the point is elliptic (like, for instance, the origin in the Hénon map); if they are distinct, real, and $\lambda_1\lambda_2 = 1$, the point is hyperbolic; if $\lambda_1 = \lambda_2 = 1$ the point is parabolic. Other possibilities are excluded by the area–preserving character of the map (2.9).

The Hénon map has two fixed points: one is the origin and the other one can be easily computed by solving equation (3.1). After eliminating the $p + x^2$ term we obtain $p = -x\tan(\omega/2)$ and then $x = 2\tan(\omega/2)$. As a consequence the fixed point (x_0, p_0) reads

$$x_0 = 2\tan\frac{\omega}{2}, \quad p_0 = -2\tan^2\frac{\omega}{2}. \qquad (3.2)$$

This point is always hyperbolic for $\omega \neq 0$; such a property can be proved by computing the trace of the map linearized around (x_0, p_0), which is $2[1 + 2\sin^2(\omega/2)] > 2$ for $\omega \neq 0$.

4 COMPUTATION OF THE DYNAMIC APERTURE

4.1 *Definition of dynamic aperture*

The dynamic aperture is defined as the border of the global stability domain, i.e. the minimum amplitude where one can have an unbounded motion. In the one dimensional case this quantity is well defined, since the invariant curves are a topological barrier to the motion: every particle which starts inside an invariant curve will remain there forever. Polynomial maps of finite order like the Hénon map always have a finite dynamic aperture (except some cases which are totally non generic) as there is a distance to the origin where the higher order nonlinearities are dominant and therefore one has a fast escape to infinity.

4.2 *Case $\omega \approx 0$*

The problem of the computation of the dynamic aperture can be solved by analytical tools when the frequency of the Hénon map is close to zero. Using the same techniques we have been discussed in section 2 one can show that Hénon map (2.8) can be seen as the exact Poincaré section of the following time dependent hamiltonian:

$$H(x, p; s) = \omega\frac{p^2 + x^2}{2} - \frac{x^3}{3}\sum_{j\in\mathbf{Z}}\delta(s - j). \qquad (4.1)$$

If we perform the scaling

$$\begin{cases} x = \omega\xi & p = \omega\eta \\ H = K\omega^3 & t = s\omega, \end{cases} \qquad (4.2)$$

which preserves the motion equations, the new hamiltonian K reads

$$K(\xi, \eta; t) = \frac{\eta^2 + \xi^2}{2} - \frac{\xi^3}{3}\omega\sum_{j\in\mathbf{Z}}\delta(t - j\omega); \qquad (4.3)$$

the advantage of such a rescaling is that in the limit $\omega \to 0$ the distribution $\omega\sum_{j\in\mathbf{Z}}\delta(t - j\omega)$ tends to one, and one recovers a time independent integrable hamiltonian

$$\lim_{\omega\to 0} K(\xi, \eta; t) = \frac{\eta^2 + \xi^2}{2} - \frac{\xi^3}{3}. \qquad (4.4)$$

Since a time independent one dimensional hamiltonian is always integrable, its dynamic aperture is given by the location of the unstable fixed point $\xi = 1$ and its related manifold, which satisfies

$$3(\eta^2 + \xi^2) - 2\xi^3 = 1, \qquad (4.5)$$

and transformed to the original coordinates reads

$$3\omega(p^2 + x^2) - 2x^3 = \omega^3. \qquad (4.6)$$

4.3 *Generic case*

When the linear frequency is far enough from zero, we cannot reduce the map to the time independent hamiltonian (4.4), and a simple analytical estimate of the dynamic aperture cannot be given. Nevertheless the situation is not different as there is a strong numerical evidence that also in this case the stability boundary is driven by the stable and unstable manifold of the hyperbolic fixed point of the map.

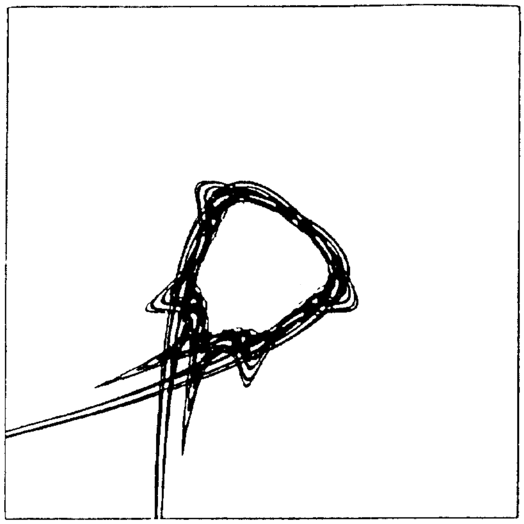

Figure 1. Invariant manifolds of the hyperbolic fixed point of the Hénon map with $\omega = .21$

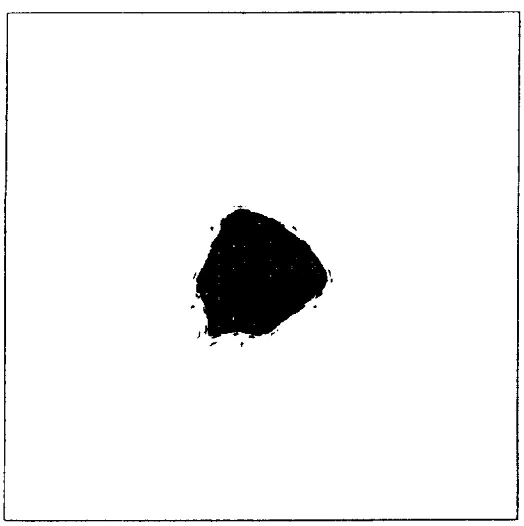

Figure 2. Stability domain of the Hénon map with $\omega = .21$ (direct tracking)

Such manifolds can be computed with a numerical method, iterating a given number of initial conditions which lie on the linear manifolds in the neighborhood of the fixed point; in Figure 1 we draw the stable and unstable manifolds with the related homoclinc tangle for ω close to resonance five; the comparison to the stability domain (Figure 2) shows an excellent agreement: the invariant manifolds of F enter the chaotic sea and reach the border of stability. Finally, in Figure 3 we plot the dynamic aperture of the Hénon map in its dependence on the linear frequency ω; the comparison between the tracking values (solid line) and the analytical estimate based on the above outlined method (squares) shows an impressive agreement.

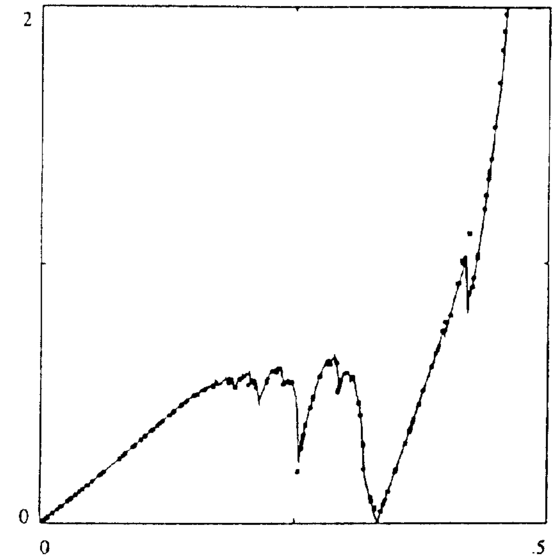

Figure 3. Dynamic aperture of the Hénon map versus ω: invariant manifolds (squares) and tracking (solid line)

5 OPEN PROBLEMS AND CONCLUSIONS

The outlined method allows to understand how the dynamic aperture of the Hénon map is given by the location of the invariant manifolds of the hyperbolic fixed point. The same picture should hold for a generic one dimensional map; in principle such a map has several hyperbolic fixed points: the dynamic aperture should be computed by drawing the invariant manifolds relative to all these points.

In two dimensional models the concept itself of dynamic aperture is not theoretically well-defined, since the invariant tori are not a topological barrier to the diffusion: in fact a particle which starts close to the origin could escape to infinity through the net of resonances (Arnold diffusion). Therefore the generalization of the proposed method is not trivial, and requires further investigation and a well defined dynamic aperture.

6 ACKNOWLEDGEMENTS

We want to acknowledge Prof. C. Simó for his worth suggestions; we also thank Walter Scandale and Frank Schmidt from the SL division for helpful discussions.

7 REFERENCES

[1] M. Henon, Quarterly of Applied Mathematics, vol 27, pp. 291–312, 1969.

[2] A. Bazzani, P. Mazzanti, G. Servizi, G. Turchetti, Il Nuovo Cimento, vol 102 B, pp. 51-80, 1988.

[3] A. Bazzani, M. Giovannozzi, G. Servizi, E. Todesco, G. Turchetti, Resonant normal forms, interpolating hamiltonians and stability analysis of area preserving maps, submitted to Physica D.