The Frequency Behaviour of the Longitudinal Coupling Impedance of a Periodic Array of Diaphragms

G. DÔME
SL Division, CERN, CH-1211 Geneva 23, Switzerland
L. PALUMBO
Dip. di Energetica, Università La Sapienza Via Scarpa 14, I-00161 Rome, Italy
and INFN-LNF, Via E. Fermi 40, I-00044 Frascati, Italy
V. G. VACCARO
Dip. di Scienze Fisiche, Università Federico II and INFN-Napoli,
Mostra d'Oltremare, Pad. 20, I-80125 Naples, Italy
L. VEROLINO
INFN-LNF, Via E. Fermi 40, I-00044 Frascati, Italy

Abstract

The diffraction problem of a charge travelling on the axis of circular apertures in a periodic array of perfectly conducting planes is described by a system of dual integral equations. The system can be transformed into a single Fredholm integral equation of the second kind whose free term and kernel are continuous functions. This integral equation can be solved numerically up to some frequency which increases with γ but stays always finite. A tentative asymptotic expression is proposed for the longitudinal coupling impedance, whose real part decays as $\omega^{-3/2}$; this expression seems to represent qualitatively the general behaviour of the computed impedance.

1 INTRODUCTION

Let us consider a particle of charge Q travelling at constant velocity $\vec{v}$ on the axis of circular holes (radius a) in a set of parallel planes (step L). We shall use cylindrical coordinates whose $\hat{z}$ axis passes through the center of the apertures and is perpendicular to the plane of the screens. We shall assume that the charge moves in the positive $\hat{z}$ direction. One of these plates, we shall call 0−plate, coincides with the reference plane $z = 0$.

The charge moving with uniform velocity in vacuum radiates only because of the optical inhomogeneities present near its path. The radiation is due to the diffraction of the field at the edges of the holes. The field created by the charge in the presence of the screens will interact with the charge itself so that, together with the phenomenon of radiation, we should find a decrease of the particle velocity. Such a radiation problem is very difficult to solve and therefore a simplifying assumption will be made: we shall suppose that the charge moves at *constant velocity* during its flight. The constant velocity can be imagined as being maintained by an external source. The result will be a good approximation provided that the velocity of the charge does not change significantly during the interaction with the screen. This assumption can be considered to be realistic when dealing with ultrarelativistic charges.

The problem will be treated as a *boundary-value problem* for Maxwell's equations: we have the radiation condition at infinity, the condition on the tangential component of the electric field on the screen, and the edge condition for the discontinuity at the edge of the holes (the last condition will ensure the uniqueness of the solution); therefore we shall write a system of dual integral equations, involving Bessel functions, for the surface current density on the screen. By means of an appropriate auxiliary function this system can be rewritten as a single Fredholm integral equation of the second kind with a continuous kernel; the analytical solution of this new equation is not easy to obtain, while a numerical treatment can give satisfactory results, at least for not too high frequencies.

2 STATEMENT OF THE PROBLEM

The diffraction problem is described by the field $(\vec{E}_0, \vec{H}_0)$ travelling with the charge itself and by the radiation $(\vec{E}, \vec{H})$ from the plates, which has a travelling wave character. Accordingly, we can represent all the fields and/or potentials as the superposition of two terms: a term generated by the charge in free space and a term created by the presence of the screens, which together must satisfy the boundary conditions; these are generally of mixed type (on the screen and on the hole) and lead to two integral equations. We thus write

$$\vec{E}_t = \vec{E}_0 + \vec{E}\,,$$
$$\vec{H}_t = \vec{H}_0 + \vec{H}\,.$$

It is worth noting that, owing to the symmetry of the problem, the induced currents on the screens are directed radially and the only components of the field are E_r, E_z, H_φ. Moreover since the induced currents are orthogonal to the edge, the Meixner or edge condition requires that the components of the electric field orthogonal to the edge diverge as $d^{-1/2}$, where d is the distance from the edge.

We shall consider the distribution of induced currents on the plates as the unknown of the problem; the first integral equation states that these currents have to be zero in the holes. The component of the radiated electric field along the surface of the plates is set equal to the negative component of the primary electric field due to the charge; this implies another integral equation where the unknown function is the space time transformed current. It has already been shown [1], [2] that this problem can be formulated [1] as the following system of **dual integral equations**[3], [4]

$$\int_0^\infty uF(u)J_1(ur)du = 0 \qquad 0 \le r \le a, \tag{1}$$

$$\int_0^\infty uF(u)\sqrt{u^2-k^2}S(u,0)J_1(ur)du = jAK_1(\kappa r) \qquad r > a, \tag{2}$$

where $\kappa = k/(\beta\gamma)$, $A = Qk^2/(\pi\beta^2\gamma)$ and k, the wave number, has to be considered as being complex with a small imaginary part, such that $\mathrm{Im}(k) < 0$,

$$S(u,0) = \frac{\sinh\left(L\sqrt{u^2-k^2}\right)}{\cosh\left(L\sqrt{u^2-k^2}\right) - \cos\left(kL/\beta\right)}, \tag{3}$$

and $F(u)$ is the Hankel transform of the current density flowing on the plate $z = 0$, that is

$$F(u) = \int_0^\infty rJ_1(ur)J_r^0(r;k)dr. \tag{4}$$

All these results can also be used in the case of a single screen, where, because we take $\mathrm{Im}\sqrt{k^2-u^2} < 0$, we have

$$\lim_{L\to\infty} S(u,0) = 1 \qquad \text{for any } u.$$

There are no analytical methods to find an exact solution of the system (1) and (2) and direct numerical solution suffers convergence and stability problems [3]. Nevertheless it is possible to reformulate the problem in order to get a single Fredholm integral equation of the second kind, introducing an *ad hoc* representation of the unknown $F(u)$, so that one of the two equations of the system is automatically satisfied.

3 AUXILIARY FUNCTION

There are many possibilities for choosing an auxiliary function to reduce a system of dual integral equation to a single integral equations [3], [4]. A possible choice for obtaining a Fredholm integral equation of the second kind with continuous kernel is to put [5]

$$F(u)\sqrt{u^2-k^2}S(u,0) = Q\frac{jk}{\pi\beta}\left[\frac{u}{u^2+\kappa^2} - \int_0^a p(x)\sin(ux)dx\right], \tag{5}$$

where $p(x)$ is some auxiliary function which is continuous, together with its first derivative, in the closed interval $[0,a]$. Then equation (2) is automatically satisfied, while equation (1) becomes

$$p(r) = T(r) + \frac{1}{2}\int_0^a \left[G(|x-r|) - G(x+r)\right]p(x)dx \qquad 0 \le r \le a, \tag{6}$$

where the kernel and the free term are continuous functions, respectively given as

$$G(y) = \frac{2}{\pi}\int_0^\infty [1-N(u)]\cos(uy)du, \tag{7}$$

$$T(r) = e^{-\kappa r} - \frac{2}{\pi}\int_0^\infty \frac{u[1-N(u)]}{u^2+\kappa^2}\sin(ur)du, \tag{8}$$

where, for brevity, we put

$$N(u) = \frac{u}{\sqrt{u^2-k^2}S(u,0)}. \tag{9}$$

The function $N(u)$ has only simple poles; so we have the possibility of developing the relevant quantities in a form suitable for numerical calculations. Therefore, making use of the expansion [6]

$$N(u) = \frac{u}{L}\sum_0^\infty \epsilon_n \frac{1-(-1)^n\cos(kL/\beta)}{u^2-k^2+(n\pi/L)^2}, \tag{10}$$

(ϵ_n is the Neumann's symbol), we can compute [2] the kernel and the free term of the previous integral equation.

Equation (6) has been solved numerically for different values of the frequency, of γ, and of the ratio L/a. Limitations of computers impose limitations on the highest frequency which can be computed. The main results can be summarized as:

- the upper frequency increases with γ, but stays always finite;
- the auxiliary function $p(r)$ is a smooth function, even when the kernel and the free term have very rapide oscillations (at high frequencies).

Finally it can be shown that with expression (5) for $F(u)$, the edge condition for E_r at $r = a-$ is automatically satisfied.

4 LONGITUDINAL COUPLING IMPEDANCE

The longitudinal coupling impedance is defined by [7]

$$Z_{\parallel}(k) = -\frac{1}{Q}\int_0^\infty E_z(r=0,z;k)e^{jkz/\beta}dz, \tag{11}$$

where E_z is the radiated field. Because in our case we have a periodic structure, the last integrand is periodic on z with period L. This means that we can redefine an impedance per cell as

$$Z_{\parallel}(k) = -\frac{1}{Q}\int_0^L E_z(r=0,z;k)e^{jkz}dz, \tag{12}$$

[1] It will be assumed in the following that the order of integration in repeated integrals, and the orders of differentiation and integration, can be reversed as necessary without explicit justification

[2] In equations (7) and (8), we must again assume that $\mathrm{Im}(k) < 0$.

where the integral is now taken over a single cell. This impedance can be rewritten [2] as a function of the unknown $F(u)$, obtaining

$$Z_{\parallel}(k) = \frac{\zeta_0}{Q\beta} \int_0^{\infty} \frac{u^2}{u^2+\kappa^2} F(u)du\,, \qquad (13)$$

where $\zeta_0 = 120\pi\Omega$ is the characteristic impedance of free space. It is worth noting that this expression is formally identical to the one found [1], [2] for the case of a single screen.

Equation (13) enables us to compute the impedance in terms of the auxiliary function $p(x)$; we get

$$Z_{\parallel}(k) = \frac{jk\zeta_0}{2\beta^2}\left[\frac{2}{\pi}\int_0^{\infty}\frac{u^2 N(u)}{(u^2+\kappa^2)^2}du - \int_0^a p(x)T(x)dx\right]\,, \quad (14)$$

where the first integral can be computed by means of the expansion (10) for the function $N(u)$. This is the equation we used for numerical calculations.

5 ASYMPTOTIC EXPANSION

Using the numerical results we tested the validity of the asymptotic expansion [3] (dot-dashed in the figures)

$$\left[\frac{a}{L}\frac{2\pi}{\zeta_0}Z_{\parallel}(k)\right]^{-1} \approx \frac{1+j}{2}\sqrt{kL\pi} + j\frac{ka}{2} + \frac{H_1^{(2)}(ka)}{jH_0^{(2)}(ka)}\,, \quad (15)$$

valid for $\beta = 1$. The general behaviour of the impedance is well reproduced when $ka \gg L/a$. The first term of the expansion (15) is chosen to be consistent with the asymptotic formula given by Gluckstern [8]

$$\frac{a}{L}\frac{2\pi}{\zeta_0}\mathrm{Re}\left[Z_{\parallel}(k)\right] \approx 2\sqrt{\frac{L\pi}{a}}\frac{1}{(ka)^{3/2}}$$

$$\frac{a}{L}\frac{2\pi}{\zeta_0}\mathrm{Im}\left[Z_{\parallel}(k)\right] \approx -\frac{2}{ka}\,,$$

whose real part decays as $\omega^{-3/2}$.

6 CONCLUSION

We have considered a point charge travelling on the axis of a periodic array of circular holes in perfectly conducting infinite planes. The Hankel transform of the induced current in the planes is determined by a set of dual integral equations, whose solution is expressed in terms of an auxiliary function which satisfies a Fredholm integral equation of the second kind, and which can be computed by numerical methods. The longitudinal coupling impedance per cell of the array is then obtained by integration from the auxiliary function.

The numerical work was carried out on the CERN computers. It not only provided numerical values for the coupling impedance per cell as a function of frequency, but it also showed that the proposed asymptotic formula represents qualitatively the general behaviour of the coupling impedance, at least when $L/a \leq 1$, i.e. when the numerical computations can be carried out up to frequencies which are high enough to be in the asymptotic region.

[3] We are also testing a more complete formula valid for all β.

7 REFERENCES

[1] G. Dôme, E. Gianfelice, L. Palumbo, V.G. Vaccaro, L. Verolino, Longitudinal coupling impedance of a circular iris, Il Nuovo Cimento A **104** (1991) 1241.

[2] G. Dôme, L. Palumbo, V.G. Vaccaro, L. Verolino, A method for computing the longitudinal coupling impedance of circular apertures in a periodic array of infinite planes, Particle Accelerators **36** (1991) 161.

[3] I.N. Sneddon, Mixed boundary value problems in potential theory, North-Holland, Amsterdam, 1966.

[4] I.N. Sneddon, The use of integral transforms, McGraw-Hill, New York, 1972.

[5] N.N. Lebedev, I.P. Skal'skaya, Sov. Phys.-Tech. Phys. **7** (1963) 867.

[6] G.N. Watson, E.T. Whittaker, A course of modern analysis, Cambridge University Press, 1927.

[7] L. Palumbo, V.G. Vaccaro, CERN 87-03 (1987), 341.

[8] R.L. Gluckstern, Longitudinal impedance of a periodic structure at high frequency, Physical review D **39** (1989) 2780.

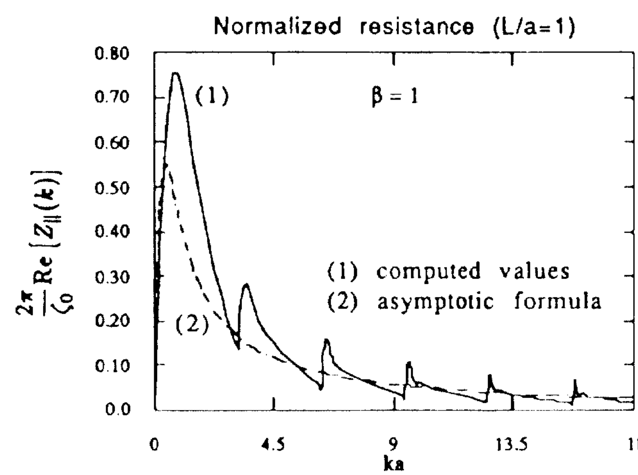

Figure 1: real part of the impedance

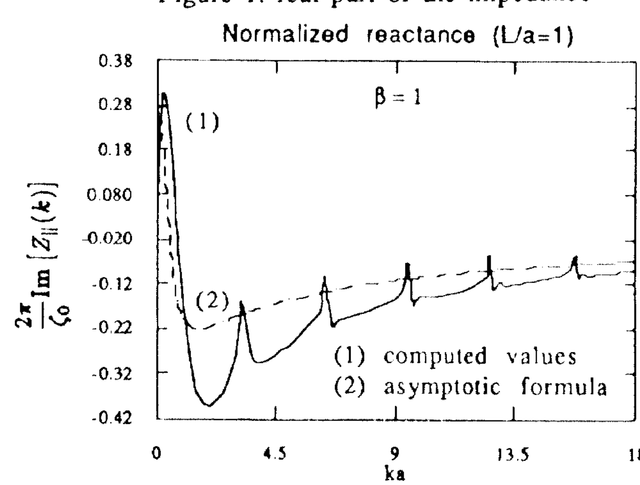

Figure 2: imaginary part of the impedance