Analytical Formulae for the Resonant Frequency Changes due to Opening Apertures on Cavity Walls

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Abstract

Based on the perturbation method, the resonant frequency changes due to apertures on a cavity wall have been investigated, and analytical formulae have been derived. The dispersion relation of a periodic disk-loaded slow wave structure, which relates the group velocity explicitly to the shapes and sizes of coupling apertures, is established. Limited by the length of the paper the comparisons with the unmerical and experimental results have been omitted [1].

1 INTRODUCTION

The aim of this paper is to find out analytically the resonant frequency changes due to apertures on cavity walls, such as apertures for pumping, tuning and coupling etc., by the perturbation method. Based on the same method the analytical dispersion relation of a periodic disk-loaded slow wave structure is established, which relates the group velocity and other properties of this slow wave structure to the shapes, sizes, positions of the coupling apertures and cavity geometries. Another important quantity concerning a cavity in the microwave engineering is the coupling coefficient β between a cavity and a waveguide. An analytical formula for this coupling coefficient β has been established, verified and shows itself in ref. [2].

2 PERTURBATION THEORY

Slater's perturbation formula [3] which relates the resonant frequency change of a lossless resonant cavity to the perturbation on the boundaries of this cavity, states that

$$\omega^2 = \omega_0^2(1 + \frac{1}{2U}\int_{\Delta v}(\mu_0 H^2 - \epsilon_0 E^2)dv) \qquad (1)$$

where ω_0 is the resonant frequency before perturbation, ω is the resonant frequency after perturbation, U is the total energy stored in the cavity, Δv is the small volume change on the boundaries, and E, H are the electric and magnetic fields in this small volume with the values equal to those before perturbation. Eq. 1, can be rewriten as follows also [4]

$$\omega^2 = \omega_0^2(1 + \frac{2}{U}(\Delta W_m - \Delta W_e)) \qquad (2)$$

where ΔW_e and ΔW_m are the time-average electric and magnetic energies stored in the perturbation volume.

3 FREQUENCY CHANGE DUE TO APERTURES

According to ref. [5] it is known that an aperture on the cavity wall can be made equivalent to some combination of electric and magnetic dipoles (if the dimension of this aperture is small compared with the wavelength), such as

$$P = -\frac{\pi l_1^3(1-e_0^2)}{3E(e_0)}\epsilon_0 E_0 \qquad (3)$$

$$M_1 = \frac{\mu_0 \pi l_1^3 e_0^2}{3(K(e_0) - E(e_0))}H_1 \qquad (4)$$

$$M_2 = \frac{\mu_0 \pi l_1^3 e_0^2(1-e_0^2)}{3(E(e_0) - (1-e_0^2)K(e_0))}H_2 \qquad (5)$$

$$e_0 = (1 - l_2^2/l_1^2)^{1/2} \qquad (6)$$

where ϵ_0 is the permittivity of vacuum, μ_0 is the permeability of vacuum, P and M_1, M_2 are the electric and magnetic dipole moments, respectively. E_0 is the electric field perpendicular to the surface of the ellipse. H_1 and H_2 are the magnetic fields parallel to the major and minor axis of this ellipse. l_1 and l_2 are the lengths of demi- major and minor axis, respectively (see Fig. 1). $K(e_0)$ and $E(e_0)$ are complete elliptic integrals of the first and second kinds [6].

It should be mentioned that the apertures discussed above have no volumes, but only have elliptic surfaces. Since the apertures on the cavity wall can be equivalent to electric and magnetic dipoles as expressed in eqs. 3-5, ΔW_e and ΔW_m can be calculated by imagining that these electric and magnetic dipoles interact with applided driving electromagnetic fields. According to Bethe's theory [7], these driving fields in the center of the aperture are the halves of those values of E_0 and $H_{0,1}$ which are the electric and magnetic fields at the center of the aperture before being perturbed. Remembering to take the time-average of the energy changes due to these electromagnetic dipoles, we have

$$\Delta U_e = -\frac{1}{2}\mathbf{P}\cdot\mathbf{E}' = \frac{\pi l_1^3(1-e_0^2)}{12E(e_0)}\epsilon_0 E_0^2 = -\Delta W_e \qquad (7)$$

$$\Delta U_m = \Delta U_{m,1} + \Delta U_{m,2} = -\Delta W_m \qquad (8)$$

$$\Delta U_{m,1} = \frac{1}{2}\mathbf{M_1}\cdot\mathbf{H'_1} = \frac{\mu_0 \pi l_1^3 e_0^2}{12(K(e_0) - E(e_0))}H_1^2 \qquad (9)$$

$$\Delta U_{m,2} = \frac{1}{2}\mathbf{M_2}\cdot\mathbf{H'_2} = \frac{\mu_0 \pi l_1^3 e_0^2(1-e_0^2)}{12(E(e_0) - (1-e_0^2)K(e_0))}H_2^2 \qquad (10)$$

where $E' = E_0/2$, $H'_{1,2} = H_{1,2}/2$, E_0 and $H_{1,2}$ are the electric and magnetic fields at the center of the aperture before being perturbed. By combining eqs. 2, and 7-10 we get the resonant frequency change due to an open aperture on the cavity wall. Sometimes a cavity is perturbed as shown in Fig. (2), where the frequency change also depends on the distance z as in the case of a frequency tuner. If the hole is a circular one, the frequency change will be

$$\omega^2 = \omega_0^2(1+\frac{2\Delta U_e}{U}(1-e^{-2\alpha_1 z})-\frac{2\Delta U_m}{U}(1-e^{-2\alpha_2 z})) \quad (11)$$

with ΔU_e, ΔU_m as expressed in eqs. 7 and 8 $z \geq 0$ and with α_1, α_2 expressed as follows:

$$\alpha_1 = \frac{2\pi}{\lambda}((\frac{\lambda}{\lambda_{c1}})^2-1)^{1/2}, \alpha_2 = \frac{2\pi}{\lambda}((\frac{\lambda}{\lambda_{c2}})^2-1)^{1/2} \quad (12)$$

where λ is the wavelength in free space, $\lambda_{c1} = 2.62a$ is the cutoff wavelength of TM_{01} mode wave, $\lambda_{c2} = 3.41a$ is that of TE_{11} mode wave, and a is the radius of the circular pipe. The necessity of there existing two factors $(1-e^{-2\alpha_1 z})$ and $(1-e^{-2\alpha_2 z})$ in eq. 11 can be proved easily and it is omitted here. To show the applications of eq. 11, two examples will be given here. First example is shown in Fig. (2a) where a circular aperture is opened. Since there is almost no magnetic field where the aperture is located, eq. 11 reduce to

$$\omega^2 = \omega_0^2(1+\frac{2\Delta U_e}{U}(1-e^{-2\alpha_1 z})) \quad (13)$$

or

$$\delta\omega = \omega_0\frac{a^3\epsilon_0 E_0^2}{6U}(1-e^{-2\alpha_1 z}) \quad (14)$$

It is known consequently that

$$\frac{d\omega}{dz} = \omega_0\frac{a^3\epsilon_0\alpha_1 E_0^2}{3U}e^{-2\alpha_1 z} \quad (15)$$

The second example is shown in Fig. (2b). Since there is no electric field where the circular aperture is located, eq. 11 reduces to

$$\omega^2 = \omega_0^2(1-\frac{2\Delta U_m}{U}(1-e^{-2\alpha_2 z})) \quad (16)$$

or

$$\delta\omega = -\omega_0\frac{a^3\mu_0 H_0^2}{3U}(1-e^{-2\alpha_2 z}) \quad (17)$$

and consequently

$$\frac{d\omega}{dz} = -2\omega_0\frac{a^3\mu_0\alpha_2 H_0^2}{3U}e^{-2\alpha_2 z} \quad (18)$$

Numerical results from Superfish have been compared with the theoretical results calculated by eqs. 14 and 15, and the LAL (Orsay) RF Gun [8][9] experimental results [10] have been compared with those calculated by eq. 17 and 18.

4 FREQUENCY CHANGE DUE TO COUPLING BETWEEN CAVITIES

Now we consider two cavities coupled by an aperture on the common wall. Aiming at explaining physics, a simple case is dicussed and shown in Fig. (3a), where the coupling is conducted only by a circular aperture (an electric dipole). Since there is coupling between the two cavities, the energy change in the first cavity due to electric dipole will be as follows (assuming that the electromagnetic fields in both cavities oscilate with the same frequency):

$$\Delta W_{e,1} = \frac{1}{2}\mathbf{P_1}\cdot\mathbf{E'_1} - \frac{1}{2}\mathbf{P_1}\cdot\mathbf{E'_2} \quad (19)$$

where $\mathbf{P_1}$ is the dipole moment corresponding to first cavity, $\mathbf{E'_1} = 1/2\mathbf{E_1}$, $\mathbf{E'_2}$ is the electric field of the second cavity seen by the electric dipole of the first cavity, with $\mathbf{E'_2} = 1/2e^{-\alpha_1 d}\mathbf{E_2}$, d is the thickness of the common wall where the aperture is located, and $\mathbf{E_1}$, $\mathbf{E_2}$ are the electric fields at the center of the aperture in the two cavities when the aperture is replaced by an ideal metallic boundary. Therefore according to eq. 2, one can get the frequency change of the first cavity as

$$\begin{aligned}\omega_1^2 &= \omega_{0,1}^2(1-\frac{2\Delta W_{e,1}}{U}) \\ &= \omega_{0,1}^2(1+\frac{1}{3}a^3\epsilon_0\frac{\mathbf{E_1}\cdot\mathbf{E_1}}{U} - \frac{1}{3}a^3\epsilon_0\frac{\mathbf{E_1}\cdot\mathbf{E_2}}{U}e^{-\alpha_1 d}) \\ &= \omega_{0,1}^2(1+\frac{1}{3}a^3\epsilon_0\frac{E_1^2}{U} - \frac{1}{3}a^3\epsilon_0\frac{E_1E_2cos\theta}{U}e^{-\alpha_1 d}) \quad (20)\end{aligned}$$

where θ is the phase difference between $\mathbf{E_1}$ and $\mathbf{E_2}$. As for the second cavity one could follow the same procedure accordingly. If the two cavities are coupled magnetically through a circular aperture as shown in Fig. (3b), the frequency change of the first cavity will be expressed as follows:

$$\begin{aligned}\omega_1^2 &= \omega_{0,1}^2(1+\frac{2\Delta U_m}{U}) \\ &= \omega_{0,1}^2(1-\frac{2}{3}a^3\mu_0\frac{\mathbf{H_1}\cdot\mathbf{H_1}}{U} + \frac{2}{3}a^3\mu_0\frac{\mathbf{H_1}\cdot\mathbf{H_2}}{U}e^{-\alpha_2 d}) \\ &= \omega_{0,1}^2(1-\frac{2}{3}a^3\mu_0\frac{H_1^2}{U} + \frac{2}{3}a^3\mu_0\frac{H_1H_2cos\theta}{U}e^{-\alpha_2 d}) \quad (21)\end{aligned}$$

where θ is the phase difference between $\mathbf{H_1}$ and $\mathbf{H_2}$. If this coupling aperture is located where electric and magnetic fields both are not vanishing, the total frequency change caused by electric and magnetic dipoles could be evaluated by combining eqs. 20 and 21 according to eq. 2.

5 DISPERSION RELATION OF SLOW WAVE STRUCTURE

As a practical application of eqs. 20 and 21 we consider a periodic disc-loaded accelerator structure as shown in Fig. (4). According to Floquet's theorem it is known that $\theta = \beta_0 D$, where β_0 is the foundamental wave number, and D is the space periodicity of the periodic structure.

We consider first the case of electrical coupling structure. According to eq. 20 we have

$$\omega^2 = \omega_0^2\left(1 + \frac{N}{3}a^3\epsilon_0\frac{E_1^2}{U} - \frac{N}{3}a^3\epsilon_0\frac{E_1E_2cos(\beta_0 D)}{U}e^{-\alpha_1 d}\right) \quad (22)$$

where N is the number of the coupling apertures on the wall of each cavity (assuming that the physical conditions for these N apertures are same). If $\beta_0 D = \pi/2$ ($\pi/2$ mode), then

$$\omega_{\pi/2}^2 = \omega_0^2\left(1 + \frac{N}{3}a^3\epsilon_0\frac{E_1^2}{U}\right) \quad (23)$$

Usually $|(\omega_0 - \omega_{\pi/2})/\omega_{\pi/2}| \ll 0$, eq. 22 can be rewritten as

$$\omega^2 = \omega_{\pi/2}^2\left(1 - \frac{N}{3}a^3\epsilon_0\frac{E_1E_2cos(\beta_0 D)}{U}e^{-\alpha_1 d}\right) \quad (24)$$

It is very clear to see that eq. 14 is the dispersion relation of an electrically coupled periodic slow wave structure, and by comparing with that obtained from an equivalent circuit as shown in Fig. (4) [11],

$$\omega^2 = \omega_{\pi/2}^2(1 - kcos(\beta_0 D)) \quad (25)$$

where $k = 2C/(2C + C')$, we know that the coupling constant k in the classical dispersion relation, eq. (31), can be represented as follows:

$$k = \frac{N}{3}a^3\epsilon_0\frac{E_1E_2}{U}e^{-\alpha_1 d} \quad (26)$$

The group velocity of this electrically coupled slow wave structure is

$$v_g = \frac{d\omega}{d\beta_0} = \omega_{\pi/2}\frac{N}{6}a^3\epsilon_0\frac{\alpha_e DE_1^2 sin(\beta_0 D)}{U}e^{-\alpha_1 d} \quad (27)$$

where $\alpha_e = |E_2/E_1|, 1 \geq \alpha_e \geq 0$, and in an normal accelerator structure $\alpha_e = 1$.

If magnetic coupling is chosen as shown in Fig. (5), started from eq. 21 we get the dispersion relation of magnetic coupling structure:

$$\omega^2 = \omega_{\pi/2}^2\left(1 + \frac{2N}{3}a^3\mu_0\frac{H_1H_2cos(\beta_0 D)}{U}e^{-\alpha_2 d}\right) \quad (28)$$

where

$$\omega_{\pi/2}^2 = \omega_{0,1}^2\left(1 - \frac{2N}{3}a^3\mu_0\frac{H_1^2}{U}\right) \quad (29)$$

Compared with the equivalent circuit shown in Fig. (5) and the classical dispersion relation [11]

$$\omega^2 = \frac{\omega_{\pi/2}^2}{(1 - kcos(\beta_0 D))} \approx \omega_{\pi/2}^2(1 + kcos(\beta_0 D)) \quad (30)$$

where $k = M/L$, we know that

$$k = \frac{2N}{3}a^3\mu_0\frac{H_1H_2}{U}e^{-\alpha_2 d} \quad (31)$$

The group velocity of this magnetically coupled slow wave structure is

$$v_g = \frac{d\omega}{d\beta_0} = -\omega_{\pi/2}\frac{N}{3}a^3\mu_0\frac{\alpha_m DH_1^2 sin(\beta_0 D)}{U}e^{-\alpha_2 d} \quad (32)$$

where $\alpha_m = |H_2/H_1|, 1 \geq \alpha_m \geq 0$, and in an normal accelerator structure $\alpha_m = 1$.

If the aperture is an ellipse rather than a circle general formulae can be obtained by using eqs. 3-5 [1].

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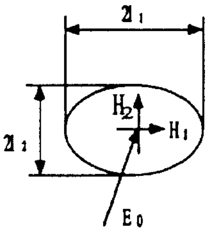

Fig.1

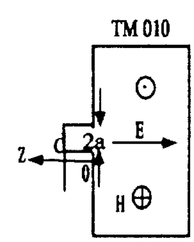

Fig.2a

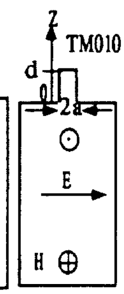

Fig.2b

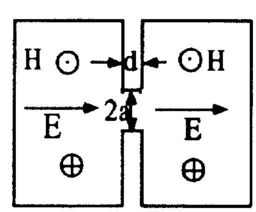

Fig.3a

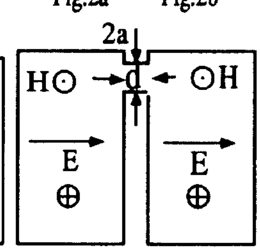

Fig 3b

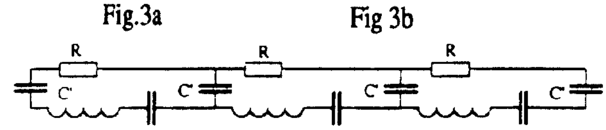

Fig. 4 L C L C L C

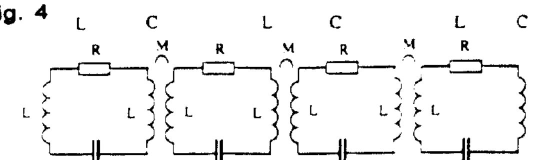

Fig. 5 C C C C