Orbit Analysis of the Spiral Inflector for Cyclotrons

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Abstract

A set of non-linear differential equations has been derived to describe the ion motion in a spiral inflector by using a coordinate system moving with the central trajectory. In this formulation, an electric potential has been introduced instead of treating the electric field directly. The differential equations together with the potential can be used to get the ion-optical properties of the spiral inflector with various forms of the electrodes and to study approximately the effects of the edge of the electrodes and those of the space charge. The same formulation can be shown to be valid also in the cases of non-uniform magnetic field when the field distribution is given and the trajectory is known numerically.

1. INTRODUCTION

The spiral inflector has been known since the early stage of the vertical injection of the ions into cyclotrons [1], and it is now widely used for large and small cyclotrons. The analyses of the ion orbit have been made in the simple cases of the so-called untilted and tilted electrodes [2,3]. The method used so far employed has tried to construct the electric field produced by the electrodes, which satisfies the Maxwell equations [3].

In order to design an inflector for the SF cyclotron at the Institute for Nuclear Study, University of Tokyo, we investigated the ion motion for both the central and the non-central cases. We examined the central orbit from a point of view that the electric field acts like a magnetic field and vice versa in the meaning that both bends the ions into the direction perpendicular to the their velocity. It has been shown that a simple solution exists which includes the types already known as its special cases of the integration constants in addition to a new possibility.

For the non-central ions, we have derived a set of simultaneous differential equations by using a coordinate system moving with the central trajectory. It can be solved by numerical integration. This formulation introduces an electric potential to describe the electric field produced by the inflector electrodes instead of treating the electric field directly. The potential makes it possible to calculate the optical properties as well as the transmission efficiency and the numerical data for machining the electrode surfaces.

A method has been devised to expand the electric potential around the reference trajectory in power series of the deviation from the trajectory to satisfy the Laplace equation starting from a simple polynomial, although the Laplace equation in our case has a somewhat complicated form due to the twist of the reference trajectory. As a numerical example, design of an inflector for the SF cyclotron is discussed.

2. ION ORBITS IN A SPIRAL INFLECTOR

2.1 Central Orbits in a Uniform Magnetic Field

An ion with mass m and charge e is moving with velocity v in a magnetic field $\boldsymbol{B}$ and an electric field $\boldsymbol{E}$ produced by a pair of the inflector electrodes. The electric field should be always perpendicular to the ion velocity. Let's consider three unit vectors, $\boldsymbol{t}$, $\boldsymbol{e}$, and $\boldsymbol{n}$, of the coordinate system moving with the central orbit, C, where $\boldsymbol{t}$ is the tangent to C, $\boldsymbol{e}$ is perpendicular to $\boldsymbol{t}$ and $\boldsymbol{n} = \boldsymbol{t} \times \boldsymbol{e}$. We take the length along the orbit $s=vt$ as the independent variable, and write the electric and magnetic fields as

$$\boldsymbol{E} = \frac{mv^2}{e}\left(K\boldsymbol{e} - F\boldsymbol{n}\right)$$

and

$$\boldsymbol{B} = \frac{mv}{e}\,\boldsymbol{b},$$

where K and F are some functions of s. Then the equations of motion can be written in a form

$$\boldsymbol{t}' = K\boldsymbol{e} - F\boldsymbol{n} + (\boldsymbol{t} \times \boldsymbol{b}), \tag{1}$$

where the prime designates the differentiation with respect to s. It can be shown that the three unit vectors must satisfy the following equations,

$$\boldsymbol{e}' = -K\boldsymbol{t} + G\boldsymbol{n} + (\boldsymbol{e} \times \boldsymbol{b}) \tag{2}$$

and

$$\boldsymbol{n}' = -G\boldsymbol{e} + F\boldsymbol{t} + (\boldsymbol{n} \times \boldsymbol{b}), \tag{3}$$

together with Eq.(1), where G is another function of s. It should be remarked that the electric field acts like a magnetic field in a spiral inflector so far as the central trajectory is concerned since the kinetic energy is constant. This fact is embodied in these equations.

The system of equations has been found to have a simple analytical form when the magnetic field is uniform and $K = K_e$, $F = K_0 \sin(\alpha)$, $G = -K_0 \cos(\alpha)$ and $\alpha = \alpha_0 + K_e s$, where K_0, K_e and α_0 are constants. The solution has three parameters and includes both the untilted ($K_0 = 0$) and tilted ($K_e \neq 0$, $K_0 \neq 0$, $\alpha_0 = 0$) cases already known. It should be remarked that the usual tilted inflector has an electric field which increases as s. In addition to them, the solution contains the cases of the tilted electrodes with both constant ($K_e = 0$, $K_0 \neq 0$, $\alpha_0 \neq 0$) and decreasing ($K_e \neq 0$, $K_0 \neq 0$, $\alpha_0 = \pi/2$) electric field strengths. Thus, it gives more flexibility for matching the inflector orbit to the cyclotron acceleration orbit and better optical properties.

2.2 Non-Central Orbits

In the case of the non-central orbit, it is also convenient to introduce a length variable, $u = vt$. Note that v is the velocity of the reference ion, and therefore u is not the length along the trajectory but rather the transit time of the ion under consideration. Then the equations of motion of the non-central ions can be written just in the same form as that of the central trajectory.

Take a point Q on the orbit under consideration at time t. Find a plane containing Q and perpendicular to the tangent of C. Let the intersecting point of this plane with C be P, whose distance along the trajectory be s. The radius vector of Q in this plane with the origin at P be $\boldsymbol{q}$. In this way, s is a function of u. So long as a small region around C is considered, this correspondence is unique and we can define a single-valued function, $u = f(s)$. Once this correspondence is established, it is more convenient to use s as the independent variable rather than u, since all quantities concerning C are expressed as functions of s. This method is usually employed in the theory of particle accelerators in which the radius of curvature is negligible. In our case, however, the twist of the reference curve is not negligible and the tangential electric field appears in the equations for non-central orbits even if it does not exist on the central trajectory.

By definition,

$$\boldsymbol{q} = \boldsymbol{r} - \boldsymbol{r}_c \tag{4}$$

and

$$\boldsymbol{q} \cdot \boldsymbol{t} = 0, \tag{5}$$

where $\boldsymbol{r}$ is the radius vectors of the ion under consideration, and $\boldsymbol{r}_c$, that of the central orbit, respectively. Let's introduce another vector,

$$\boldsymbol{p} = \frac{d\boldsymbol{r}}{du} - \boldsymbol{t} \tag{6}$$

By differentiating Eq.(5) with respect to s, the differential equation for the function u is derived as

$$u' = \frac{\lambda}{1 + \boldsymbol{p} \cdot \boldsymbol{t}}, \tag{7}$$

where

$$\lambda = 1 - (\boldsymbol{t}' \cdot \boldsymbol{e})(\boldsymbol{q} \cdot \boldsymbol{e}) + (\boldsymbol{n}' \cdot \boldsymbol{t})(\boldsymbol{q} \cdot \boldsymbol{n}). \tag{8}$$

This function introduces main non-linearity into the equations, which arises from the fact that the non-central ion changes its energy along the trajectory and the path length is different from that of the central ion. Similarly, using the equations of motion, we have the simultaneous differential equations for $\boldsymbol{q}$ and $\boldsymbol{p}$,

$$\boldsymbol{q}' = u'\boldsymbol{p} + (u' - 1)\boldsymbol{t} \tag{9}$$

and

$$\boldsymbol{p}' = (u' - 1)[\boldsymbol{K}_c + (\boldsymbol{t} \times \boldsymbol{b}_c)] + u'[(\boldsymbol{K} - \boldsymbol{K}_c) + (\boldsymbol{t} \times (\boldsymbol{b} - \boldsymbol{b}_c))] + u'(\boldsymbol{p} \times \boldsymbol{b}) \tag{10}$$

where $\boldsymbol{K}_c$ and $\boldsymbol{K}$ are the electric curvatures at $\boldsymbol{r}_c$ and at $\boldsymbol{r}$ respectively, $\boldsymbol{b}_c$ and $\boldsymbol{b}$ are the magnetic curvatures there. In the case of uniform magnetic field we can put $\boldsymbol{b} = \boldsymbol{b}_c$.

It is easy to rewrite them for the components (q_e, q_n) for $\boldsymbol{q}$ and (p_e, p_n, p_t) for $\boldsymbol{p}$. The set of coupled equations Eqs. (1)-(10) can only be solved numerically for the purpose of study on the optical properties.

2.3. Non-Uniform Magnetic Field

In the case of a non-uniform magnetic field, simple analytical solution is no more possible. However, Eqs.(1)-(10) do not assume a uniform magnetic field. If the electric and magnetic distributions are given, i. e., the functions K, F, G and $\boldsymbol{b}$ as functions of space, we can solve for $\boldsymbol{e}$, $\boldsymbol{n}$, $\boldsymbol{t}$ and $\boldsymbol{r}_c$ simultaneously, which determines the reference trajectory. Eqs.(1)-(3) guarantee still in this case the orthogonality of the vectors $\boldsymbol{e}$, $\boldsymbol{n}$ and $\boldsymbol{t}$. Once the central trajectory is fixed, we can integrate Eqs.(4)-(10) to get the optical properties. It may be more convenient, however, to start from the uniform-field approximation and to proceed to necessary modification introduced by the non-uniformity.

3. Electric Potential and the Laplace Equation

In a curvilinear coordinate the Laplace equation takes somewhat complicated form due to the twist of the reference curve, the central orbit of the ions in this case[4]. The electric potential ϕ is a function of $\boldsymbol{q}$ and s. The components of the electric field $\boldsymbol{E}$ is given by

$$\boldsymbol{E} \cdot \boldsymbol{e} = -\frac{\partial \phi}{\partial q_e},$$

$$\boldsymbol{E} \cdot \boldsymbol{n} = -\frac{\partial \phi}{\partial q_n}$$

and

$$\boldsymbol{E} \cdot \boldsymbol{t} \equiv -\partial_s \phi = (\boldsymbol{e}' \cdot \boldsymbol{n})\left(q_e \frac{\partial \phi}{\partial q_n} - q_n \frac{\partial \phi}{\partial q_e}\right) + \frac{\partial \phi}{\partial s}.$$

The Laplace equation has a form

$$\Delta\phi = \frac{1}{\lambda}\left\{\frac{\partial}{\partial q_e}\left(\lambda\frac{\partial \phi}{\partial q_e}\right) + \frac{\partial}{\partial q_n}\left(\lambda\frac{\partial \phi}{\partial q_n}\right)\right\} - \frac{1}{\lambda}\partial_s\left(\frac{1}{\lambda}\partial_s\phi\right) = 0.$$

The solution can be expanded into a power series of $\boldsymbol{q}$ in the plane containing $\boldsymbol{q}$. Taking the polar coordinates, q and θ, we have

$$\phi = \sum_{n \geq 0} \phi_n,$$

where

$$\phi_n = Real\,\{a_n(s)q^n e^{in\theta} + \sum_{\substack{k>n \\ m \leq k-2}} c_{nkm}(s) q^k e^{im\theta}\},$$

a_n and c_{nkm} are complex functions of s, and m is an odd or even non-negative integer. Note that the factor $q^n e^{in\theta}$ is a solution of the Laplace equation when the reference curve is a straight line. The coefficient a_0 must be independent of s since the electric field is perpendicular to C and a_1 is determined by K and F. The other coefficients a_n introducing the $2n$-pole electric potential are free parameters for the shape of the inflector electrodes. For example, the electric focusing by quadratic electrode surfaces and the effects of disturbance of the electric field at the edges of the electrodes can be studied by adding appropriate higher order terms through a_n.

The effects of twist of the reference curve appear in the correction terms containing c_{nkm} which can be uniquely determined from a_n. The magnitude of c_{nkm} is of the order of the k-th power of the curvature of the reference curve. The series converges rapidly if q is much smaller than the radius of curvature. A computer program has been written to calculate c_{nkm} symbolically starting from a given a_n. The truncation error in this expansion can be estimated as the space charge appearing in the corresponding Poisson equation.

Conversely the space charge effects due to the beam can be simulated by adding a potential which vanishes on the electrode surfaces, when the beam intensity through the inflector becomes large.

4. NUMERICAL EXAMPLES

As numerical examples, we have studied an inflector for the SF cyclotron at the Institute for Nuclear Study, University of Tokyo, whose extraction radius is 73 cm. If we use a constant orbit with 250 turns for acceleration, all parameters except the ratio of the injection voltage of the ions to the dee voltage are fixed for centering the cyclotron orbit. Since the dee voltage is 20~60 kV in our case, we took the ratio to be 0.3. For simplicity and comparison, we employed the electrodes 10 mm wide with and without tilt. The magnetic and electric radii of curvature are 17.9 mm and 41.3 mm, respectively and $K_0 = 0.2K_e$ for the tilt.

It should be noted that the electrode surfaces are no more flat due to the twist of the reference curve. At the edges, the shift of the surfaces introduced by the twist amounts to about 2 mm, which is not negligible compared with the gap about 5 mm. The error introduced by truncating the terms higher than the fifth order in the expansion was equivalent to the space charge due to a beam of about 100 μA of 10-keV Ar^{10+}.

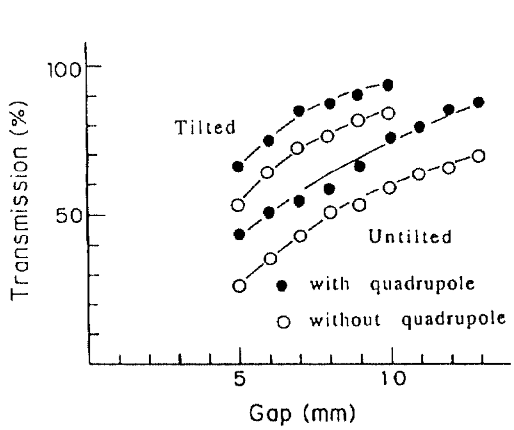

Figure 1. Transmission efficiency vs the electrode gap.

Figure 1 shows the transmission efficiency of a beam as a function of the gap between the electrodes. The top two curves are for the tilted cases and the bottom two, for the untilted (open circles: without quadrupole field, closed circles: with quadrupole field). The beam was assumed to have a waist with a diameter 5 mm at the entrance and an emittance of 200 mm.mrad. The position and angle of the ions in the initial beam were assumed not to be correlated by the longitudinal magnetic field of the cyclotron. If a correlated beam with the same size was assumed, the efficiency was found to be much higher. The transmission efficiency turned out to be much lower, about half, than that calculated with a potential without the correction for the twist. It is natural that the correction terms strongly affect the transmission since the magnetic radius of curvature is not large enough compared with the deviation of the ions from the reference curve.

The higher efficiency for the tilted case is considered to arise from the shorter path length, 46.1 mm, through the inflector compared to that of the untilted case, 64.1 mm. In both cases, addition of a quadrupole electric potential of a form $a(x^2-y^2) + bxy$ with a set of parameters, $a = b = 0.2\ K_e^2$ results in better transmission by about 20 %.

5. DISCUSSIONS

The simple analytical solution of the central orbit of a spiral inflector has been shown to be useful and have a possibility not studied so far. The solution is applicable for the spiral inflectors for the cyclotrons with various sizes. It offers a transparent method for design.

We have formulated a method to analyze the ion-optical properties of a spiral inflector by using an electric potential. The previous works [1-3] have devised to expand the electric field around the central trajectory instead of the potential. Using a potential instead of the electric field brings about an advantage that the electric field thus obtained can be expanded around the central trajectory together with the method of expansion described in Sec. 3. This method makes it possible to obtain the electric potential with enough accuracy without using a 3-dimensional computer code for calculating the electric field.

With advent of numerically controlled machining of the electrode surfaces, there arise possibilities to use more complicated surfaces to get better optical properties. As is seen from the formulae presented in this paper, there still remains much freedom to choose the electric potential around the central trajectory. The formalism developed here will be useful in those cases.

6. REFERENCES

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