# The Effect of Vacuum Chamber Wall Conductivity of an Accelerator on the Value of Coulomb Tune Shifts

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*Abstract*

The penetrating of a beam magnetic field into walls of a ring accelerator vacuum chamber provides an increase of Coulomb tune shifts from a head of a bunch, or bunch train to their tail. The effect can be practically important for accelerators with a low revolution frequency, mainly when there is a non-regular orbit filling by beam. The general formulae for incoherent tune shifts and the numerical results for the UNK accelerator are obtained.

## 1 INTRODUCTION

Common results for incoherent Coulomb tune shift in a bunched beam were obtained under an assumption of no penetration of an alternating magnetic field of the beam into vacuum-chamber walls [1, 2]. An independence of the tune shifts upon the position of a particle in a bunch is the most significant consequence of this assumption for very high energies, when the terms with factor $\gamma^{-2}$ can be neglected. Such shifts can be easily compensated by a tune correction system. As a result, only the limitations caused by a difference between coherent and incoherent tune shifts remain, which is not a serious problem. It is such a relatively favorable situation that has been expected for the UNK accelerator [2].

As a matter of fact, various harmonics of the magnetic field do penetrate into chamber wall up to a relevant skin-depth, the latter is comparable to the wall thickness in large accelerators (e.g., the skin-depth equals 3.5 mm for revolution frequency of 14.4 kHz in the UNK-1). It entails that the magnetic field would eventually penetrate into walls after passage of bunch. This paper shows an increase of Coulomb tune shift in direction from a head of a bunch, or bunch train, to their tail to be the most important consequence of this phenomenon. The effect can be practically important mainly in case of a non-regular orbit filling by beam. For example, an additional tune spread of 0.03 emerges in the UNK-1 accelerator, a total tolerable value of the spread being 0.05 [3].

## 2 CALCULATION OF TUNE SHIFTS

To describe the beam-induced field, use is made of a scalar potential $\mathcal{F}$, and of a vector potential with longitudinal component $\mathcal{A}$. Then, equation for horizontal betatron oscillations reads:

$$\frac{d^2x}{d\theta^2}+G(\theta)x=\frac{eR_0^2}{mc^2\beta^2\gamma}\left(-\frac{\partial\mathcal{F}}{\partial x}+\beta\frac{\partial\mathcal{A}}{\partial x}\right), \qquad (1)$$

with $G$ the coefficient of magnetic rigidity, the other notations being standard.

It is convenient to expand the beam charge density, as well as the field potentials, into a Fourier series

$$\rho=ef(x,z)\,\nu(\theta-\omega_s t)=ef(x,z)\sum_k \nu_k e^{ik(\theta-\omega_s t)}, \qquad (2)$$

$$\mathcal{F}=e\sum_k \nu_k\Phi_k e^{ik(\theta-\omega_s t)};\ \mathcal{A}=e\sum_k \nu_k A_k e^{ik(\theta-\omega_s t)}, \qquad (3)$$

where $\nu$ is the number of particles per unit of chamber length, $\omega_s$ is the revolution velocity, $f$ is a normalized function describing the distribution of particles in the beam cross-section. Then, for small oscillations, eq.(1) yields:

$$\begin{aligned}\frac{d^2x}{d\theta^2} \;+\; G(\theta)x \;&= \qquad (4)\\ &= \frac{xr_0R_0^2}{\beta^2\gamma}\sum_k\left(-\frac{\partial^2\Phi_k}{\partial x^2}+\beta^2\frac{\partial^2 A_k}{\partial x^2}\right)\nu_k\, e^{ik\vartheta},\end{aligned}$$

where $r_0$ is the electro-magnetic radius of a particle, $\vartheta$ is the distance from train's front to the particle; all the derivatives are taken at the origin of coordinates. Solution of this equation gives the following value of incohereht Coulomb tune shift:

$$\Delta Q_x=\frac{r_0R_0}{2\beta^2\gamma}\sum_k\left\langle\beta_x\left(\frac{\partial^2\Phi_k}{\partial x^2}-\beta^2\frac{\partial^2A_k}{\partial x^2}\right)\right\rangle\nu_k\, e^{ik\vartheta}, \qquad (5)$$

where $\beta_x$ is a horizontal $\beta$-function, $\langle\ldots\rangle$ denotes an averaging over a turn.

Fields $\Phi_k$ and $A_k$ obey the similar equations, e.g.:

$$\frac{\partial^2A_k}{\partial x^2}+\frac{\partial^2A_k}{\partial z^2}=-4\pi f(x,z). \qquad (6)$$

These include both the beam and wall-image fields. The first one is well-known, and is vanishingly small for high energy accelerators due to $\gamma^{-2}$-factor [4]. Therefore, only the image fields are investigated.

Functions $\Phi_k$ satisfy homogeneous boundary conditions of the first kind at the vacuum chamber wall, while $A_0$ obeys these of the second kind at the yoke surface. Following [2], their derivatives can be written as

$$\frac{\partial^2\Phi_k}{\partial x^2}=\varepsilon\left(\frac{1}{b_z^2}-\frac{1}{b_x^2}\right),\quad -\frac{\partial^2A_0}{\partial x^2}=\mu\left(\frac{1}{g_z^2}-\frac{1}{g_x^2}\right), \qquad (7)$$

where $b_{x,z}$ are semi-axes of the vacuum chamber, $g_{x,z}$ are these for the yoke which is as well supposed to be elliptical.

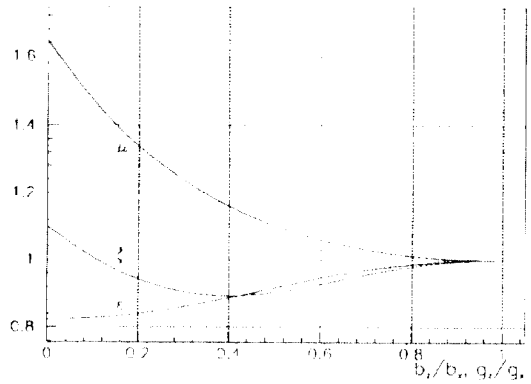

Figure 1: Form-factors for elliptical chamber.

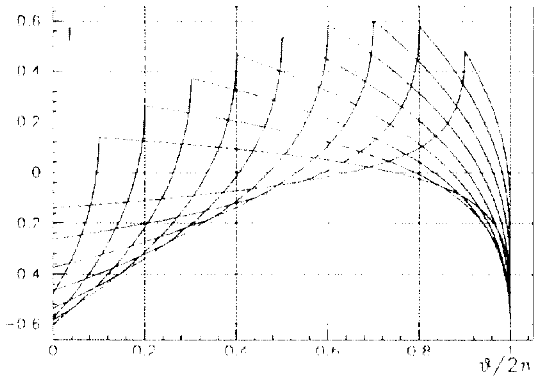

Figure 2: Function $F(\vartheta)$ for rectangular train.

Positive quantities $\varepsilon$ and $\mu$ depend on ratio of semi-axes, and are shown in Fig.1 (see also Appendix). Dimensional multipliers in eq.(7) show straight-forwardly that the boundaries do not affect the incoherent tune shift in an axially-symmetrical system.

The boundary conditions, as well as the solutions for harmonics $A_{k\neq 0}$, are discussed in Appendix. Use these results to put down:

$$\begin{aligned} \frac{\partial^2 A_k}{\partial x^2} &= \frac{\partial^2 \Phi_k}{\partial x^2} - \frac{3\zeta}{4}\left(\frac{1}{b_z^2}-\frac{1}{b_x^2}\right) \\ &\times \left(\frac{1}{b_z}+\frac{1}{b_x}\right)\left(1+i\frac{|k|}{k}\right)\delta\ |k|^{-1/2}, \end{aligned} \tag{8}$$

where $\delta$ is skin-depth at frequency $\omega_s$, and a positive coefficient $\zeta$ is close to 1 (Fig.1).

Substitute eqs.(7),(8) into eq.(5) to get:

$$\begin{aligned} \Delta Q_x &= \frac{N r_0}{4\pi\gamma}\left\{\frac{3\pi\delta F(\vartheta)}{\Delta\vartheta}\left\langle\beta_x\zeta\left(\frac{1}{b_z^2}-\frac{1}{b_x^2}\right)\left(\frac{1}{b_z}+\frac{1}{b_x}\right)\right\rangle\right. \\ &+ \left(1+\frac{\nu(\vartheta)}{\beta^2\gamma^2\nu_0}\right)\left\langle\beta_x\varepsilon\left(\frac{1}{b_z^2}-\frac{1}{b_x^2}\right)\right\rangle \\ &+ \left.\left\langle\beta_x\mu\left(\frac{1}{g_z^2}-\frac{1}{g_x^2}\right)\right\rangle\right\}, \end{aligned} \tag{9}$$

$$F(\vartheta) = \frac{\Delta\vartheta}{2\pi}\mathrm{Re}\ (1+i)\sum_{k=1}^{\infty}\frac{\nu_k}{\nu_0}\ |k|^{-1/2}e^{ik\vartheta}, \tag{10}$$

where $\Delta\vartheta$ is the train's length.

Suppose all bunches of the train to be identical and equidistantly located. Then

$$\frac{\nu_k}{\nu_0} = \frac{\sin(k\Delta\vartheta/2)}{k\Delta\vartheta/2}e^{ik(\vartheta_0-\Delta\vartheta/2)}, \tag{11}$$

where $\vartheta_0$ is the train's front position. Plots of functions $F(\vartheta)$ for this case are shown in Fig.2 where $\vartheta_0 = 2\pi$, $\Delta\vartheta/2\pi = 0.1,\ 0.2,\ldots,1$. The trains are moving from left to right, and a position of train's tail coincides with the break-point of a curve.

The similar formula for $\Delta Q_z$ can be got from eq.(9) by a mere permutation of indices x and z. Coefficients $\varepsilon$, $\mu$, $\zeta$ are the same because the image fields inside a chamber satisfy a Laplace equation.

## 3 EXAMPLE

As an example, consider the first stage of the UNK-accelerator which is to be arranged of 2 types of dipole magnets:

Type A: $b_z = 23$ mm, $b_x = 45$ mm, $g_z = 24$ mm, $\beta_x = 109$ m, $\beta_z = 49$ m, $\varepsilon = 0.92$, $\zeta = 0.91$;

Type B: $b_z = 31$ mm, $b_x = 35$ mm, $g_z = 32$ mm, $\beta_x = 49$ m, $\beta_z = 109$ m, $\varepsilon = 1$, $\zeta = 0.99$,

where the average value of $\beta$-function for each dipole type is given. It is possible to treat the width of dipole poles as an infinite one, in which case take $\mu = \pi^2/6$ and multiply the relevant part of the formula by factor 0.606. The latter is the ratio of net length of dipoles to the orbit length. On putting into eq.(11) values $\delta = 3.5$ mm and $\gamma = 69.3$ (injection) one obtains:

$$\Delta Q_x = 3.4\cdot 10^{-16}N + 4.5\cdot 10^{-17}\frac{2\pi N}{\Delta\vartheta}F(\vartheta), \tag{12}$$

$$-\Delta Q_z = 2.5\cdot 10^{-16}N + 2.6\cdot 10^{-17}\frac{2\pi N}{\Delta\vartheta}F(\vartheta). \tag{13}$$

The accelerator is to be filled by 12 injection pulses. Each of them contains $5\cdot 10^{13}$ particles and has azimuthal length $\Delta\vartheta = 0.078$. In which case eqs.(12), (13) give:

$$\Delta Q_x = 0.017\ m + 0.029\ F(\vartheta), \tag{14}$$

$$-\Delta Q_z = 0.0125 m + 0.0165 F(\vartheta). \tag{15}$$

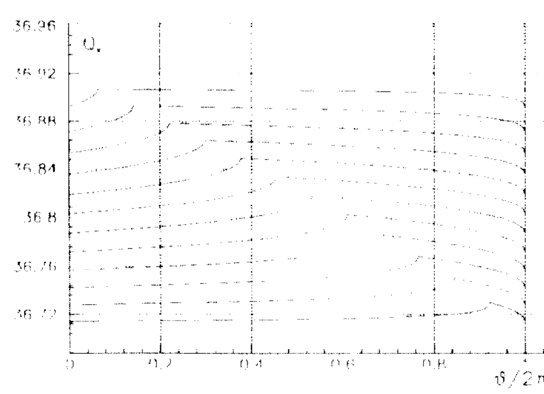

Figure 3: Distribution of $Q_x$ along train.

where $m$ is a number of injection pulses. Fig.3 shows the distribution of horizontal betatron frequencies along the ring after each injection pulse. It is possible to adjust a correction regime so as to provide average betatron frequencies in the train to be constant. Then one gets a picture similar to that shown in Fig.2, but with a maximum spread of $\Delta Q_x = 0.032$. Distribution of $\Delta Q_z$ has the same pattern, but with an extra coefficient of $-0.57$.

## 4 REFERENCES


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## 5 APPENDIX. CALCULATION OF ELECTROMAGNETIC FIELD

Solve eq.(6) by using an elliptical coordinate system:

$$x = b\cosh\xi\,\cos\eta, \quad z = b\sinh\xi\,\sin\eta, \tag{16}$$

where $b^2 = b_x^2 - b_z^2$. The chamber wall has a coordinate $\xi = \xi_0 = \arctan(b_z/b_x)$. A beam is supposed to be placed at the chamber axis. Only the image fields being of interest, neglect the beam thickness.

General solution of eq.(6) is

$$\begin{aligned} A_k = -2\xi \quad &+ \quad 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\,e^{-2n\xi}\cos 2n\eta + \\ &+ \quad \sum_{n=0}^{\infty} a_n\cosh 2n\eta. \end{aligned} \tag{17}$$

Here the first line gives the beam field in an empty space (to be neglected), while the last sum is the image field we are interested in. Take account of the latter to write down the derivatives for eq.(5) as

$$\begin{aligned} \frac{\partial^2 A_k}{\partial x^2} &= \frac{1}{b^2}\frac{\partial^2 A_k}{\partial \eta^2}(\xi = 0, \eta = \pi/2) = \\ &= -\frac{4}{b^2}\sum_{n=1}^{\infty}(-1)^n n^2 a_n. \end{aligned} \tag{18}$$

Coefficients $a_n$ can be found by boundary conditions. Scalar potential is zero at the chamber wall. Hence,

$$a_0 = 2\xi_0; \quad a_{n\neq 0} = -\frac{2(-1)^n\,e^{-2n\xi_0}}{n\cosh 2n\xi_0}. \tag{19}$$

Therefore, definition of eq.(7) gives:

$$\varepsilon = 2\sinh^2\xi_0\sum_{n=1}^{\infty}\frac{n\,e^{-2n\xi_0}}{\cosh 2n\xi_0}. \tag{20}$$

Harmonic $A_0$ obeys boundary condition $\partial A_0/\partial\xi = 0$ at the yoke surface, i.e.

$$\mu = 2\sinh^2\xi_0\sum_{n=1}^{\infty}\frac{n\,e^{-2n\xi_0}}{\sinh 2n\xi_0}, \tag{21}$$

where, of course, $\xi_0 = \arctan(g_z/g_x)$.

Harmonics $A_{k\neq 0}$ inside a chamber wall follow the equation:

$$\frac{\partial^2 A_k}{\partial x^2} + \frac{\partial^2 A_k}{\partial z^2} = -2i\,\frac{|k|}{k}\,\frac{b^2}{\delta_k^2}\,(\cosh^2\xi - \cos^2\eta)A_k, \tag{22}$$

where $\delta_k$ is a skin-depth at frequency $k\omega_s$. Solution in a thick-wall approximation is

$$A_k \propto \exp\left\{\frac{b\xi}{\delta_k}\left(-1 + i\frac{|k|}{k}\right)\sqrt{\cosh^2\xi_0 - \cos^2\eta}\right\}, \tag{23}$$

which gives the boundary condition at the inner surface of the chamber wall:

$$\begin{aligned} \frac{\partial A_k}{\partial\xi}(\xi_0,\eta) \quad &= \quad \frac{b}{\delta_k}\left(-1 + i\frac{|k|}{k}\right)\times \\ &\times \quad \sqrt{\cosh^2\xi_0 - \cos^2\eta}\quad A_k(\xi_0,\eta). \end{aligned} \tag{24}$$

Whereof, the coefficients $a_n$ are to satisfy the set of eqs.:

$$\begin{aligned} 2 \quad + \quad & 4\sum_{n=1}^{\infty}(-1)^n e^{-2n\xi_0}\cos 2n\eta - \\ - \quad & 2\sum_{n=1}^{\infty} n a_n\sinh 2n\xi_0\cos 2n\eta = \\ = \quad & \left(-1 + i\frac{|k|}{k}\right)\frac{b}{\delta_k}\sqrt{\cosh^2\xi_0 - \cos^2\eta}\;\times \\ \times \quad & \left(2\xi_0 - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-2n\xi_0}\cos 2n\eta - \right. \\ - \quad & \left.\sum_{n=0}^{\infty} a_n\cosh 2n\xi_0\cos 2n\eta\right). \end{aligned} \tag{25}$$

Its solution, up to the first approximation in $\delta_k/b$, is:

$$\begin{aligned} a_{n\neq 0} \quad &= \quad -\frac{2(-1)^n e^{-2n\xi_0}}{n\cosh 2n\xi_0} + \frac{\delta_k(1 + i|k|/k)}{\pi b\cosh 2n\xi_0}\times \\ &\times \quad \sum_{m=-\infty}^{\infty}\frac{(-1)^m}{\cosh 2m\xi_0}\int_{-\pi}^{\pi}\frac{\cos 2n\eta\cos 2m\eta d\eta}{\sqrt{\cosh^2\xi_0 - \cos^2\eta}}. \end{aligned} \tag{26}$$

Use definition of eq.(8) and take account of eq.(19) to obtain:

$$\begin{aligned} \zeta \quad &= \quad \frac{2\sinh^3 2\xi_0}{3\pi e^{\xi_0}}\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{\cosh 2n\xi_0}\times \\ &\times \quad \sum_{m=-\infty}^{\infty}\frac{(-1)^m}{\cosh 2m\xi_0}\int_{-\pi}^{\pi}\frac{\cos 2n\eta\cos 2m\eta d\eta}{\sqrt{\cosh^2\xi_0 - \cos^2\eta}}. \end{aligned} \tag{27}$$