A Novel Approach to the Nonlinear Longitudinal Dynamics in Particle Accelerators*

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Abstract

By using the recently proposed *thermal wave model for relativistic charged particle beam propagation* a new approach for studying some nonlinear effects in accelerator machines is developed. By taking into account the interaction of a relativistic charged particle bunch with both the RF and the self-induced wakes, and neglecting the synchrotron radiation emission,we show that the longitudinal dynamics is governed by a nonlinear Schrödinger equation for a complex wave function whose squared modulus is proportional to the longitudinal bunch density. This wave model, for which the diffraction parameter is represented here by the longitudinal emittance, is suitable to give a new description for the bunch instability and capable of reproducing both the well known coherent instability (stability) conditions and the well known longitudinal envelope equations (space charge effect included). Furthermore, we show that, in the case of RF *off*, soliton-like solution for the density profile are possible when the bunch propagates under the action of the self force. This model might deserve attention in understanding the bunch lengthening (shortening).

1 INTRODUCTION

Recently a *thermal wave model for relativistic charged particle beam propagation*, useful for a *quantum-like* description of the optics and the dynamics of charged particle beams, has been proposed in literature [1] and successfully applied to the transverse dynamics in both conventional accelerating machines and new plasma-based particle accelerator schemes [2]. In particular, this model seems suitable to describe both the spherical aberrations for the luminosity estimates at the interaction point when a quadrupolar-like lens with octupolar deviations is taken for the final focusing stage in linear colliders [3], and the self-consistent nonlinear interaction between the plasma wake field and the driving relativistic electron (positron) beam [2].

In this paper, in analogy to the transverse beam dynamics description given in the previous works [1]-[3], we suggest a novel approach to study the nonlinear longitudinal beam dynamics in particle accelerator. To this end, in Section 2 we propose a sort of nonlinear Schrödinger (NLS) equation for a complex wave function Ψ, the so called *beam wave function* (bwf), whose squared modulus gives the longitudinal density profile of the beam. In Section 3 an analysis of both coherent instability (stability) and soliton formation is developed within the contex of the thermal wave model which is capable of reproducing the well known coherent instability criterion [4], showing that the latter is fully similar to the well known criterion of *modulational instability* occuring when an e.m. bunch is travelling through a nonlinear medium. In Section 4, by using the thermal wave model, we are able of reproduce the well known envelope equation, which holds by taking into account both the RF field (potential well) and self interaction (wake fields). Section 5 concerns the comments and summarizes the conclusions.

2 WAVE EQUATION FOR THE LONGITUDINAL BEAM MOTION

It is well known that the longitudinal motion of a single particle within a stationary bunch travelling with longitudinal velocity βc in a circular accelerating machine is described, neglecting radiation damping, by the following motion equation [5]:

$$\frac{d^2x}{dt^2} + \omega_s^2 x = \eta \frac{F(x,t)}{m} \tag{1}$$

where x is the longitudinal particle coordinate with respect to the synchronous one, ω_s is the synchrotron angular frequency, $\eta = \alpha - \frac{1}{\gamma^2}$ is the common phase-slip factor (α is the momentum compaction and $\gamma = 1/\sqrt{1-\beta^2}$ is the usual relativistic gamma factor), F is the self force, and m is the particle rest mass. The term $-\omega_s^2 x$ accounts for the linear longitudinal force produced by the RF cavity (potential well) and $\frac{\eta}{m}F(x,t)$ is due to the presence of additional effects produced by the bunch itself (longitudinal wake field). Let us denote with $\frac{\Delta p}{p}$ the longitudinal momentum spread with respect to the synchronous particle and define $s = ct$, where c is the speed light. Thus, $\frac{dx}{ds}$ and $\frac{\Delta p}{p}$, are related by the following equation [6]:

$$\frac{dx}{ds} = -\eta \frac{\Delta p}{p} \tag{2}$$

Consequently, by introducing the wake potential $W(x,s)$ by:

$$\frac{F}{q} = -\frac{\partial W}{\partial x} \tag{3}$$

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with q the particle charge, and using the first integral of (1) (energy equation) and (2), we obtain the following dimensionless Hamiltonian:

$$H = \frac{\eta^2}{2}\left(\frac{\Delta p}{p}\right)^2 + \frac{1}{2}Kx^2 + \eta\frac{qW}{mc^2} \tag{4}$$

where $K \equiv \omega_s^2/c^2$ is the RF *focusing strength*. Note that here $\frac{1}{\eta^2}$ plays the role of an *effective mass*. In order to find an equation which describes the longitudinal evolution of the beam taking into account its thermal spreading (longitudinal emittance) while it interacts with the surrounding medium (potential well and wake fields), we put, according to the *thermal wave model for relativistic charged particle beam*, the following quantum-like correspondence rules:

$$\frac{\Delta p}{p} \rightarrow \mathcal{P} \equiv -i\epsilon\frac{\partial}{\partial x} \qquad H \rightarrow \mathcal{H} \equiv i\epsilon\frac{\partial}{\partial s} \tag{5}$$

where ϵ is the longitudinal beam emittance. So that, by considering (4) and (5), the following Schrödinger-like equation for the *beam wave function* (bwf) Ψ can be assumed

$$i\epsilon\frac{\partial\Psi}{\partial s} = \mathcal{H}\Psi, \quad \mathcal{H} = \frac{\mathcal{P}^2}{2(1/\eta^2)} + \frac{1}{2}Kx^2 + \eta\frac{qW}{mc^2} \tag{6}$$

namely:

$$i\epsilon\frac{\partial\Psi}{\partial s} = -\frac{\epsilon^2}{2(1/\eta^2)}\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{2}Kx^2\Psi + \eta\frac{qW}{mc^2}\Psi \tag{7}$$

The (7) describes the longitudinal beam dynamics with the following meaning of Ψ. If $\lambda(x,s)$ is the longitudinal beam density (number of particles per unit longitudinal length) and N is the total number of particles, thus:

$$\lambda(x,s) = N|\Psi(x,s)|^2 \tag{8}$$

so that the normalization condition is provided:

$$\int_{-\infty}^{\infty} |\Psi(x,s)|^2\, dx = 1. \tag{9}$$

This way $|\Psi|^2$ gives the longitudinal beam density profile. In general, the wake potential is function of $|\Psi|^2$, $W(x,s) = W(|\Psi(x,s)|^2)$, so this describes the longitudinal nonlinear beam dynamics in terms of an appropriate nonlinear Schrödinger (NLS) equation.

3 COLLECTIVE EFFECTS

In this section we develop, within the framework of the thermal wave model , an analysis of some collective effects occuring when the bunch interacts with the surrounding medium. To this end, we consider the special case of RF cavity *off* and take into account both the space charge effect and a purely inductive coupling impedance. Since in this hypothesis the self force is proportional to $\frac{\partial\lambda}{\partial x}$ [4], the wake potential is:

$$W(x,s) = -qNR\beta c\left[\frac{g_0}{R\beta c\gamma^2} - |Z'/n|\right]|\Psi(x,s)|^2 \tag{10}$$

where g_0 is the well known coupling coefficient, Z' is longitudinal coupling impedance per unit length divided by the mode number n, and R is the averaged orbit radius of the synchronous particle. Consequently, the (7) becomes:

$$i\epsilon\frac{\partial\Psi}{\partial s} = -\frac{\epsilon^2\eta^2}{2}\frac{\partial^2\Psi}{\partial x^2} - \eta\chi N r_b|\Psi|^2\Psi \tag{11}$$

where we have put (cgs units):
$\chi = R\beta c\left[\frac{g_0}{R\beta c\gamma^2} - |Z'/n|\right]$ and r_b is the classical radius of the particle. Note that (11) is formally identical to the cubic NLS equation which describes the propagation of an electromagnetic (e.m.) pulse through a nonlinear medium in paraxial approximation [7],[8]. In this analogy ϵ plays the role of the diffraction parameter (the inverse of the wavenumber), s corrisponds to the time and $-\eta\chi r_b|\Psi|^2$ corrisponds to a nonlinear refractive index. So that, an analysis of the bunch instability (stability) can be made in complete analogy to the electromagnetic one.

3.1 Stability criterion

For NLS equation of the form

$$i\frac{\partial\Phi}{\partial s} + P\frac{\partial^2\Phi}{\partial x^2} + Q|\Phi|^2\Phi = 0 \tag{12}$$

a small perturbation is stable (unstable) if the following condition (Lighthill criterion) is satisfied [7]:

$$PQ < 0 \qquad (PQ > 0). \tag{13}$$

Consequently, for the (11) we have coherent stability (instability) with respect to a small density perturbation of the bunch if

$$\chi\eta < 0 \qquad (\chi\eta > 0) \tag{14}$$

(here $P = \frac{\epsilon\eta^2}{2}$ and $Q = \frac{r_b N\eta\chi}{\epsilon}$). Thus, Eq. (14) immediately recovers the coherent stability (instability) criterion well known in the conventional theory [4]: (1) if the total longitudinal coupling impedance is capacitive ($\chi > 0$), the system is stable (unstable) only if it stays below (above) the transition energy, namely $\eta < 0$ ($\eta > 0$); (2) if the total longitudinal coupling impedance is capacitive ($\chi < 0$), the system is stable (unstable) only if it stays below (above) the transition energy, namely $\eta > 0$ ($\eta < 0$).

3.2 Solitons

A solitary solution of Eq. (11) is found by looking for a solution of a relativistic ($\beta_0 \approx 1$) envelope form:

$$\Psi(x,s) = G(x - \beta_0 s)e^{ik_0 x - i\omega_0 s} \tag{15}$$

with k_0, and ω_0 real numbers. Thus, according to the general theory of NLS equation [7], the following soliton-like solution for the beam density ($\lambda = NG^2$), which satisfies (9), is possible under the condition $\eta\chi > 0$:

$$\lambda(x,s) = \frac{N^2\chi r_b}{4\epsilon^2\eta} sech^2\left[\frac{N\chi r_b}{2\epsilon^2\eta}(x - \beta_0 s)\right] \tag{16}$$

where $k_0 = \beta_0/(\epsilon\eta^2)$ and $\omega_0 = (\epsilon\eta^2/2)k_0^2 - N^2\chi^2 r_b^2/(16\epsilon^3)$.

4 ENVELOPE EQUATIONS

In this Section we find an aberrationless solution of (7) with the potential (10) following the standard techniques of nonlinear e.m. wave optics [8]. This allows us to write an envelope equation for the longitudinal motion taking into account both the RF fields and the self interaction. To this end, we look for a solution of (7) in the form:

$$\Psi(x,s) \;=\; \frac{e^{-\frac{x^2}{4\sigma(s)}}}{\sqrt[4]{2\pi\sigma^2(s)}}\, e^{i\theta(x,s)} \qquad (17)$$

where the eikonal has been supposed as:

$$\theta(x,s) \;=\; \frac{x^2}{2\epsilon\rho(s)} \;+\; \phi(s). \qquad (18)$$

By substituting (17) and (18), separating the real part from the imaginary one, and expanding $|\Psi|^2$ up to the second-power of x (aberrationless approximation) we get a coupled equation system for the effective particle bunch width $\sigma(s)$, the curvature radius of the wavefront $\rho(s)$, and the phase $\phi(s)$. Therefore, by solving for σ we obtain the following envelope equation:

$$\frac{d^2\sigma}{ds^2} + \eta^2\left[K\sigma - \frac{\xi}{\sigma^2} - \frac{\epsilon^2\eta^2}{4\sigma^3}\right] = 0 \qquad (19)$$

where $\xi \equiv \frac{\eta\chi N r_b}{\sqrt{2\pi}}$. We first observe that in the limit of negligible self interaction ($\xi = 0$), (19) gives the envelope equation for the synchrotron motion (harmonic oscillator solutions for the bwf). In this case, by using the *quantum* uncertainty principle related to Eq. (7), the following relationship can be proved [1]: $\sigma_{p0}\sigma_0 = \epsilon/2 = constant$, where σ_0 is the bunch length and σ_{p0} is the the *quantum* expectation value of $\mathcal{P}$ at the equilibrium state. Therefore, by imposing $d^2\sigma/ds^2 = 0$ in Eq. (19), we easily recover the well known relationship between σ_0 and σ_{p0} [9]: $\sigma_0 = (|\eta|/\sqrt{K})\sigma_{p0} = (R|\eta|/h)\sigma_{p0}$, where h is the harmonic number.

Furthermore, by retaining the self interaction term ($\xi \neq 0$) in (19) we get, within the equilibrium condition, the following algebraic expression for σ: $\sigma^4 - \frac{\eta\chi N r_b c^2}{\omega_s^2}\sigma - \sigma_0^4 = 0$, which in the special case of parabolic density profile recovers the well known similar expression given in [10], used to try an explanation of the *potential well bunch lenghtening.*

5 CONCLUDING REMARKS

In this paper we have shown possible a novel approach to the nonlinear longitudinal dynamics of a relativistic particle bunch in circular accelerating machines within the contex of the recently proposed *thermal wave model for relativistic charged particle beam propagation* [1]. Neglecting the radiation damping, we have shown that the nonlinear interaction between the bunch and the surrondings (potential well and wake fields) is governed by an appropriated NLS equation (equation (7)), fully similar to that holds for the propagation of an e.m. bunch in a nonlinear medium in paraxial approximation [8]. By using this similarity we have recovered, when the RF is *off* the well known condition for the coherent instability (stability) [4]. We have pointed out that: *first*, in the e.m. analogy, these conditions correspond to the *Lighthill criterion* (modulational instability [7],[8]); *second*, the density can assume a soliton-like profile under the condition $\eta\chi > 0$. Physically, a sort of competition between the *diffractive energy* (i.e. *thermal energy*) and the self energy is established. We have instability when the self energy term overcomes the diffraction one. According to Section 3.2 condition $\eta\chi > 0$ suggests that soliton formation would be the natural evolution of the initial beam density modulation toward a self bunching which asimptotically gives a soliton-like envelope wave train. Furthermore, by taking into account both the RF and the self fields, the averaged-quantity description has allowed us to recover the well known envelope equation for the longitudinal motion [10].

Let us suppose that is possible to expand the wake potential $W(x,s)$ in powers of x up to an order $n > 2$, thus aberrations are introduced in terms of an anharmonic potential: the odd (even) powers are related to a *resistive* (*reactive*) contribution to the coupling impedance. If the initial bunch density profile is Gaussian, these anharmonic terms introduce a distortion which results in a modification of the space-distribution of the particles in such a way to produce, after many turns in the machine, a bunch length modification. In a forthcoming paper we discuss more carefully this effect by putting in (7) an expansion of $W(x,s)$ up to x^4. This way we try a realistic *wave* interpretation of the *anomalous bunch lengthening.*

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