Coupling Resonance $2Q_y - 2Q_x = 0$ Excited by Sextupolar Magnetic Fields

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Abstract

The Hamiltonian governing the on-momentum particle motion in accelerator magnetic field with arbitrary nonlinearity is derived in the frames of the second-order approximation of the averaging technique. As an example the effects of sextupolar field components on betatron oscillations are analyzed. The motion in the vicinity of the coupling resonance $2Q_y - 2Q_x = 0$ is investigated. The analytical results are shown to have a good agreement with tracking simulation in the UNK superconducting accelerator.

1 INTRODUCTION

To compensate chromaticity in a high energy superconducting accelerator magnetic fields having a higher sextupolar nonlinearity are required. The first-order approximation of the averaging method typically used to calculate similar systems proves to be insufficient for this purpose. Among the effects manifesting themselves in the second-order approximation, the most noticeable are the following ones [1]: the dependence of the betatron oscillation tunes $Q_{x,y}$ on amplitudes; the fourth-order resonances, one of them is the coupling resonance $2Q_y - 2Q_x = 0$. The presence of this resonance the work treats can bring about certain difficulties during slow extraction of the beam. Such effects may also occur due to a relatively large sextupolar nonlinearity in superconducting dipoles produced by persistent currents at low levels of magnetic field H. The above effects have to be taken into account when formulating the tolerable value of sextupolar nonlinearity in dipoles and choosing the structure of the chromaticity correction system [2].

2 EQUATIONS OF MOTION

In a superconducting high energy accelerator the magnet length is essentially larger than the inner aperture. Therefore, the magnetic field imperfections may be treated as plane, i.e. $\vec{h} = (h_x, h_y)$. In this case $h_{x,y}$ are expressed in terms of the longitudinal vector potential

$$A_s = \sum_{n=0}^{\infty} \left(\frac{\Delta H_n + i\Delta \widetilde{H}_n}{2(n+1)\, r^n} (x+iy)^{n+1} + \text{c.c.} \right),$$

where ΔH_n, $\Delta \widetilde{H}_n$ are normal end skew components of the n-th power nonlinearity, respectively; r is the reference radius accepted to be $35mm$ which coincides with the halfwidth of the vacuum chamber to be used in the UNK superconducting accelerator.

The equations of the on-momentum particle motion in the presence of the nonlinear magnetic field are

$$\zeta'' + g_\zeta \zeta = -\frac{R_o^2}{RH}\frac{\partial A_s}{\partial \zeta}, \qquad (1)$$

where ζ is used as general transverse coordinate for both x and y; R_o and R are, respectively, the average radius and the curvature radius of the reference orbit in the magnetic field H; 'prime' denotes the differentiation with respect to the generalized azimuth θ associated with the longitudinal coordinate s by the relation $\theta = s/R_o$. The solution of system 1 is sought in the form

$$\zeta = a_\zeta \varphi_\zeta + a_\zeta^* \varphi_\zeta^* \quad , \quad \zeta' = a_\zeta \varphi_\zeta' + a_\zeta^* \varphi_\zeta^{*\prime} \, , \qquad (2)$$

where $\varphi_\zeta = \sqrt{\beta_\zeta(\theta)/R_o}\exp(i\mu_\zeta(\theta))$ is Floquet's function with the normalization $\varphi_\zeta \varphi_\zeta^{*\prime} - \varphi_\zeta^* \varphi_\zeta' = -2i$, β_ζ is the betatron amplitude function and $\mu_\zeta = Q_\zeta \theta + \chi_\zeta$ is the phase of unperturbed betatron oscillation with periodic part $\chi_\zeta(\theta)$. Then from Eqs.1,2 it follows that

$$a_\zeta' = \frac{iR_o^2}{2RH}\frac{\partial A_s}{\partial \zeta}\varphi_\zeta^* \, . \qquad (3)$$

To describe the perturbed motion it is more convenient to use real variables $\widetilde{I}_\zeta, \widetilde{\eta}_\zeta$ instead of complex a_ζ and a_ζ^*

$$a_\zeta = \frac{r}{2}\sqrt{R_o/\beta_{max}}\; \widetilde{I}_\zeta^{\,1/2} \exp(i\widetilde{\eta}_\zeta) \, , \qquad (4)$$

where $\widetilde{I}_\zeta^{\,1/2}$ is the amplitude of oscillation referred to r at the point with $\beta_\zeta = \beta_{max}$. For the UNK it is taken to be $\beta_{max} = 152\,\mathrm{m}$ coinciding with the maximum of β_ζ in a regular cell [3]. From Eqs.3,4 we obtain the equations for $(\widetilde{I}_\zeta, \widetilde{\eta}_\zeta)$ in the canonical form

$$\widetilde{I}_\zeta{}' = -\frac{\partial D}{\partial \widetilde{\eta}_\zeta} \, , \qquad \widetilde{\eta}_\zeta{}' = \frac{\partial D}{\partial \widetilde{I}_\zeta} \qquad (5)$$

$$D = \frac{2\beta_{max} R_o}{r^2 RH} A_s(\zeta(\widetilde{I}, \widetilde{\eta}, \theta), \theta) \qquad (6)$$

Presenting each of variables $(\widetilde{I}_\zeta, \widetilde{\eta}_\zeta)$ as the sum of a part (I_ζ, η_ζ) varying slowly, and a small addition oscillating fast: $\widetilde{I}_\zeta = I_\zeta + \Delta I_\zeta$, $\widetilde{\eta}_\zeta = \eta_\zeta + \Delta \eta_\zeta$, we obtain from Eq.5 by means of the averaging method [1, 4]:

$$I_\zeta{}' = -\frac{\partial D^{(n)}}{\partial \eta_\zeta} \, , \qquad \eta_\zeta{}' = \frac{\partial D^{(n)}}{\partial I_\zeta} \qquad (7)$$

In the first and second approximations of this method the Hamiltonians $D^{(n)}$ have the form, respectively,

$$D^{(1)} = <D>, \tag{8}$$
$$D^{(2)} = <D> + \frac{1}{2}\sum_{x,y} < \frac{\partial D}{\partial \eta_\zeta}\frac{\partial \widehat{D}}{\partial I_\zeta} - \frac{\partial D}{\partial I_\zeta}\frac{\partial \widehat{D}}{\partial \eta_\zeta} >,$$

where in Eq.6 for D the variables $\widetilde{I}_\zeta$, $\widetilde{\eta}_\zeta$ are changed by I_ζ, η_ζ. Since $D(I_\zeta,\eta_\zeta,\theta) = \sum_j D_j(I_\zeta,\eta_\zeta)\exp(i\nu_j\theta)$, the operators $<D>$ and $\widehat{D}$ denote:

$$<D> = \sum_m D_m \exp(i\nu_m\theta), \quad |\nu_m| \ll 1 \; ; \tag{9}$$
$$\widehat{D}(I_\zeta,\eta_\zeta,\theta) = \sum_{j\neq m} \frac{D_j}{i\nu_j}\exp(i\nu_j\theta) \tag{10}$$
$$\equiv \int_{\vartheta\to\theta} (D(I_\zeta,\eta_\zeta,\vartheta) - <D(I_\zeta,\eta_\zeta,\vartheta)>)\,d\vartheta.$$

In the case of $<D>=0$ the expression for $D^{(2)}$ may be written as

$$D^{(2)} = \frac{1}{2}\sum_{\zeta=x,y} < \frac{\partial D(\theta)}{\partial \zeta} \int_{\vartheta\to\theta} \frac{\partial D(\vartheta)}{\partial \zeta} L_\zeta(\theta,\vartheta)d\vartheta >, \tag{11}$$
$$L_\zeta(\theta,\vartheta) = i\frac{\sqrt{\beta_\zeta(\theta)\beta_\zeta(\vartheta)}}{4\beta_{max}}\left(\exp(i\mu_\zeta(\theta) - i\mu_\zeta(\vartheta)) - \text{c.c.}\right).$$

We use the above procedure of obtaining the second-order approximation when there is only normal sextupolar nonlinearity in an accelerator.

3 INFLUENCE OF SEXTUPOLAR NONLINEARITY

According to Eqs.8,9 in the first approximation the nonlinearity ΔH_2 can excite first- and third-order resonances of betatron oscillations: $Q_x = k$, $2Q_y \pm Q_x = k$, $3Q_x = k$. For a sufficiently large distance of the working point (Q_x, Q_y) from these resonances (when one may neglect their influence on I_ζ, η_ζ) the equation $<D>=0$ will be satisfied [5].

As seen from Eq.11, the averaged over θ value contains, alongside with the constant constituent, also the terms with the frequencies ν_j which equal $4Q_{x,y} - k$, $2Q_y \pm 2Q_x - k$, $2Q_{x,y} - k$. This means that the sextupolar nonlinearity can excite the second- and fourth-order resonances whose strength is $\sim (\Delta H_2)^2$. In the UNK the sextupolar magnetic field of the chromaticity correction system and the systematic sextupolar nonlinearity in regular cells dipoles excite these resonances which are far from the working point. Therefore, one may neglect their effect, with the exception of the coupling resonance $2Q_y - 2Q_x = 0$ only near which the working point is located. In this case:

$$D^{(2)} = \frac{\alpha_{xx}}{2}I_x^2 + \alpha_{xy}I_xI_y + \frac{\alpha_{yy}}{2}I_y^2 + 2|\gamma|I_xI_y\cos(2w)\,, \tag{12}$$

where: $w = \eta_y - \eta_x + \arg(\gamma)/2$, $\quad \delta = Q_y - Q_x$

$$\alpha_{xx} = \int_o^\tau\int_o^\tau Y_1(\theta)Y_1(\vartheta)(K_{(3,0)} + 3K_{(1,0)})d\theta d\vartheta$$
$$\alpha_{yy} = \int_o^\tau\int_o^\tau Y_2(\theta)Y_2(\vartheta)(K_{(1,2)} + 4K_{(1,0)} - K_{(-1,2)})d\theta d\vartheta$$
$$\frac{\alpha_{xy}}{2} = \int_o^\tau\int_o^\tau Y_2(\theta)[Y_2(\vartheta)(K_{(1,2)} + K_{(-1,2)}) - 2Y_1(\vartheta)K_{(1,0)}]d\theta d\vartheta$$
$$\gamma = \int_o^\tau\int_o^\tau [Y_2(\theta)Y_2(\vartheta)(E_{(1,0)} + E_{(-1,2)}) - Y_1(\theta)Y_2(\vartheta)(E^*_{(1,2)} - E^*_{(-1,2)})/2 + Y_2(\theta)Y_1(\vartheta)(E_{(1,0)} - E_{(3,0)})/4]e^{if(\theta)}d\theta d\vartheta$$

$$E_{(n,m)} = -\frac{\exp[i(nF_x + mF_y)]}{\tau\sin[(n\mu_{x\tau} + m\mu_{y\tau})/2]}, \quad K_{(n,m)} = \frac{E_{(n,m)} + \text{c.c.}}{2}$$
$$Y_i = \frac{\Delta H_2 R_o}{HR}\frac{\beta_{max}}{8r}\left(\frac{\beta_x}{\beta_{max}}\right)^{5/2-i}\left(\frac{\beta_y}{\beta_{max}}\right)^{i-1},$$

$F_\zeta = \mu_\zeta(\theta) - \mu_\zeta(\vartheta) + \mu_{\zeta\tau}/2\cdot\text{sgn}(\vartheta - \theta)$; τ is the common period of the accelerator lattice and the sextupolar nonlinearity distribution along the accelerator; $\mu_{\zeta\tau}$ is the phase advance of oscillations per period τ; $\; f = 2(\chi_y - \chi_x)$.

In Eq.12 the parameter $|\gamma|$ characterizes the strength of the difference resonance, and the coefficients α define the nonlinear tune shifts. Since $D^{(2)}$ depends on $(\eta_y - \eta_x)$, then $I_y + I_x$ =const, the sum of the squared normalized amplitudes is conserved and, in the worst case, the energy exchange between two directions of betatron oscillations may only take place.

4 COUPLING RESONANCE

To study the resonance let us introduce new variables $I = I_y + I_x$, $v = \eta_y + \eta_x$, $J = I_y - I_x$, which, in accordance with Eqs.7,12, satisfy the equations

$$I' = -\partial G/\partial v\,, \qquad v' = \partial G/\partial I \tag{13}$$
$$J' = -\partial G/\partial w\,, \qquad w' = \partial G/\partial J$$
$$G = \lambda_0 I^2 + (\delta + \lambda_1 I)J + \lambda_2 J^2 + |\gamma|(I^2 - J^2)\cos(2w)$$
$$\lambda_i = [\alpha_{yy} + 2(1-i)\alpha_{xy} + (-1)^i\alpha_{xx}]/[3 + (-1)^i].$$

The Hamiltonian G which does not depend explicitly on θ and the quantity I are invariants of the motion. Therefore, for any fixed I one can calculate phase trajectories in the plane (J, w) and consequently define the motion completely in the frames of the approximation involved.

The existence of closed phase trajectories in the plane (J, w) and large modulation of amplitudes are due to the fixed points in this phase plane at which simultaneously $J' = 0$ and $w' = 0$. It is seen from Eqs.13 that the largest modulation occurs in the resonance centre when the fixed points lie on the line $J = 0$. The location of this centre in the betatron tune plane is defined by the equation $\delta = -\lambda_1 I$ and, consequently, will be different for

Table 1: Values of the lattice functions.

type of corrector	foc	def
$\sqrt{\beta_x}$ /m$^{1/2}$	11.828	5.912
$d\beta_x/ds \times 10^3$	-4.244	1.225
μ_x	0.746	0.114
$\sqrt{\beta_y}$ /m$^{1/2}$	5.911	11.826
$d\beta_y/ds \times 10^3$	1.224	-4.243
μ_y	0.834	0.025
ψ /m	2.332	1.293

particles with unequal I. With the fulfillment of the condition $|\delta + \lambda_1 I| \geq 2(|\lambda_2| + |\gamma|)I$, all the fixed points vanish.

To control the obtained result the simplest case has been considered: there is $\Delta H_2 \neq 0$ only in the correctors of the chromaticity correction system. Such a case is possible at the highest energy of a superconducting accelerator when the sextupolar nonlinearity in dipoles is small. In the UNK sextupolar correctors are located in all of 160 regular cells near the focusing and defocusing quadrupoles. The values of the lattice functions in the centres of the correctors at $Q_x = 36.678$ and $Q_y = 36.696$ are presented in Table 1, where $\mu_{x,y}$ are measured off the beginning of a cell. The values $(\Delta H_2 l/HR)_{foc,def} = (1.376, -2.483) \times 10^{-5}$ in which l is the effective correctors length introduce the addition to the accelerator chromaticity $\Delta\xi_{x,y} = 70$. The values of parameters α–γ calculated for this case are presented in Table 2.

Table 2: Values of parameters α–γ.

$\alpha_{xx}10^3$	$\alpha_{yy}10^3$	$\alpha_{xy}10^3$	$\lvert\gamma\rvert 10^3$	$\arg(\gamma)$
-1.436	0.862	-9.495	3.890	2.922

Fig.1 shows the phase trajectories G =const in the plane $(J/I, w)$ at $\delta = 0$. These results have been verified by the tracking simulations according to the following algorithm: the motion between the correctors has been calculated via the standard matrix technique; the correctors were considered to be point-like, i.e. on passing a corrector a particle does not change ζ, while the quantity $d\zeta/d\theta$ gets a kick; slowly varying amplitudes and phases have been determined approximately according to Eqs.2,4 $I_\zeta \simeq \tilde{I}_\zeta(\zeta, \zeta', \theta)$, $\eta_\zeta \simeq \tilde{\eta}_\zeta(\zeta, \zeta', \theta)$. The results of these tracking simulations at $I = 0.33$, i.e. $rI^{1/2} = 20$ mm, and the same strengths of the correctors are presented by dots in Fig.1. The distance between the neighboring points corresponds to 10 turns in the accelerator. The phase trajectories obtained by the averaging method and the tracking simulations have a sufficiently good coincidence. Small deflections of the points from the theoretical trajectories do not bring about a systematic change in the total oscillation energy, i.e. in the value of I. This is illustrated in Fig.2 which shows the behaviour of $I_{x,y}$ calculated by means of the tracking simulations along the trajectories marked in Fig.1.

5 CONCLUSION

It follows from the form of Eq.12 for the Hamiltonian that a normal sextupolar nonlinearity in the second-order approximation of the averaging method produces the same effects on betatron oscillations as a normal octupolar nonlinearity does in the first-order one. This allows one to use a special octupolar correction system [6] for the compensation of the effects proportional to $(\Delta H_2/H)^2$, which may be appreciable.

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6 REFERENCES

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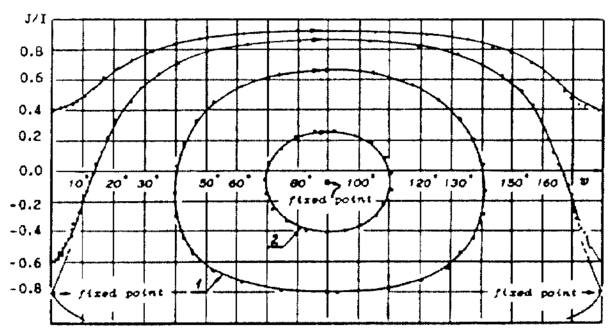

Fig.1 The phase trajectories at $Q_y = Q_x$ and I=0.33.

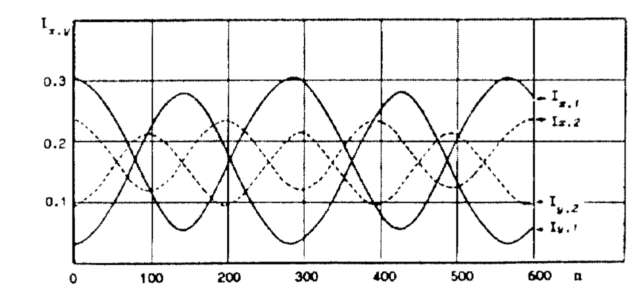

Fig.2 The squared amplitudes $I_{x,y}$ as functions of turn number n along the trajectories marked in Fig.1 (tracking result).