Diffusion Rate for the Emittance Growth due to Periodic Crossings of Nonlinear Coupled Resonances

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Abstract

Assuming that many betatron oscillations occur between crossings so that the betatron phase is uncorrelated from one crossing to the next, we estimate the diffusion rate for the emittance growth due to periodic crossing of coupled nonlinear resonances. It was shown that the diffusion rate is more or less independent of the frequency, but it is inversely proportional to the modulation amplitude.

Due to tune modulation the transverse tunes periodically cross high order resonances. It is commonly believed that these repeated crossings of high order resonances can lead to beam size growth. In previous work, Chasman et al[1], Evens and Gareyte[2], and Bruck[3] considered the repeated crossings of a single resonance with a single multipole term. As the working point is near the diagonal in tune space, however, the tunes actually cross many coupled resonances of a given order since, for each high order resonance, there is a group of dense resonance lines near the diagonal. Furthermore, as it will be shown in Appendix, the periodic crossings of a single resonance with a single multipole term could lead to an infinite growth within a finite time. In this paper we study periodic crossings of high order coupled resonances. We consider the case in which the tunes are near the diagonal and modulated at the synchrotron frequency due to momentum oscillation with finite chromaticity. The modulation amplitude is assumed to be large enough for the working point to cross all resonances of a given order repeatedly. It is, however, small enough for these coupled resonances to be treated as an isolated group of resonances. By assuming that the betatron phase at the crossing is uncorrelated, we find that the rms growth rate of emittances is nearly independent of the modulation frequency and inversely proportional to the modulation amplitude.

In terms of action-angle variables $(I_{x,y},\ \phi_{x,y})$, the Hamiltonian for betatron oscillations can be written as

$$H(I_x, I_y, \phi_x, \phi_y, \theta) = \nu_x I_x + \nu_y I_y + U(\sqrt{2I_x\beta_x}\cos\psi_x, \sqrt{2I_y\beta_y}\cos\psi_y, \theta) \quad (1)$$

where

$$\psi_{x,y} = \phi_{x,y} - \nu_{x,y}\theta + \frac{C}{2\pi}\int_0^\theta \frac{1}{\beta_{x,y}}d\theta = \phi_{x,y} + \alpha_{x,y}\ . \quad (2)$$

θ and C are the azimuth and circumference, respectively. The nonlinear perturbation U is the beam-beam interaction or magnetic field imperfections. In general, U can be expanded as

$$U = \sum_{klh} \mu_{klh}(2I_x\beta_x)^{k/2}(2I_y\beta_y)^{l/2} \sum_m^k \sum_n^l C_{km}C_{ln} \times \cos(h\theta)\cos[m(\phi_x + \alpha_x)]\cos[n(\phi_y + \alpha_y)] \quad (3)$$

where

$$C_{k0} = \begin{cases} \frac{k!}{2^k(\frac{k}{2}!)^2} & k \text{ even} \\ 0 & k \text{ odd} \end{cases} ; \quad (4)$$

$$C_{km} = \frac{k!}{2^{k-1}(\frac{k+m}{2})!(\frac{k-m}{2})!} \quad \text{for } m \neq 0\ . \quad (5)$$

If we define

$$A^\pm(\text{index})e^{i\eta^\pm(\text{index})} = \frac{1}{2\pi}\int_0^{2\pi} \beta_x^{k/2}\beta_y^{l/2} e^{i(m\alpha_x \pm n\alpha_y)}\cos(h\theta)e^{iq\theta}d\theta \quad (6)$$

where index $= (k, l, h, m, n, q)$,

$$U = \sum_{klh} \frac{1}{4}\mu_{klh}(2I_x)^{k/2}(2I_y)^{l/2} \sum_m^k \sum_n^l C_{km}C_{ln} \times \sum_{q=-\infty}^{\infty} \left\{A^+\cos\left(m\phi_x + n\phi_y - q\theta + \eta^+\right) + A^-\cos\left(m\phi_x - n\phi_y - q\theta + \eta^-\right)\right\}\ . \quad (7)$$

Eq. (7) shows that the betatron resonances are excited when tunes satisfy the relation

$$s\nu_x + t\nu_y = q + \delta q \quad \text{with} \quad \delta q << 1\ . \quad (8)$$

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Consider the effects of the highest order betatron resonances due to the kth order multipoles, i.e.,

$$s\nu_x^0 + t\nu_y^0 = q \qquad \text{with } s + t = k \quad . \tag{9}$$

By averaging over all rapidly varying oscillations except for the resonances of Eq. (9), we reduce the multipole perturbation U in Eq. (7) to

$$U = U_0 + U_1 \tag{10}$$

with U_0 the average value of U with respect to $\phi_{x,y}$ and θ. This leads to the amplitude dependences of the tunes,

$$U_0 = \sum_h \sum_{m=0}^{k} \sum_{l=0}^{k-m} E_{hlm}(2I_x)^{m/2}(2I_y)^{l/2} \quad , \tag{11}$$

where

$$E_{hlm} = \frac{\mu_{mlh}\, m!l!}{2^{m+l+1}\pi[(\frac{m}{2})!(\frac{l}{2})!]^2} \int_0^{2\pi} \beta_x^{m/2}\beta_y^{l/2}\cos(h\theta)d\theta \tag{12}$$

for m and l even; and $E_{hlm} = 0$, for m or l odd. U_1 includes all of the resonance terms

$$U_1 = \sum_h \sum_{l=0}^{k} D_{hl}(2I_x)^{(k-l)/2}(2I_y)^{l/2}\cos(\chi_{hl}) \tag{13}$$

where $\chi_{hl} = (k-l)\phi_x + l\phi_y - q\theta + \eta_{hl}$ and $D_{hl} = 2^{4-k}\mu_{k-llh}A_{hl}$ with

$$A_{hl}e^{i\eta_{hl}} = \frac{1}{2\pi}\int_0^{2\pi} \beta_x^{(k-l)/2}\beta_y^{l/2} \times \exp\{i[(k-l)\alpha_x + l\alpha_y + q\theta]\}\cos(h\theta)d\theta \quad . \tag{14}$$

For k large, and any even m and l with $m + l \le k$,

$$\left|\frac{E_{hlm}}{2D_{hl}}\right| \ge \left|\frac{E_{hl(k-l)}}{2D_{hl}}\right| \ge \frac{8(k-l)!l!}{[(\frac{k-l}{2})!(\frac{l}{2})!]^2} >> 1 \quad . \tag{15}$$

Thus, for high order resonances, $|U_0| >> |U_1|$ and the amplitude dependence of the tunes is important. In the rotating frame

$$b_{x,y} = \phi_{x,y} - \nu_{x,y}^0\theta \quad , \tag{16}$$

U_1 becomes independent of θ and the new Hamiltonian is

$$\begin{aligned} H(I_x, I_y, b_x, b_y) &= (\nu_x - \nu_x^0)I_x + (\nu_y - \nu_y^0)I_y \\ &+ U_0(I_x, I_y) + U_1(I_x, I_y, b_x, b_y) \end{aligned} \tag{17}$$

where U_0 is given in Eq. (11) and

$$U_1 = \sum_h \sum_{l=0}^{k} D_{hl}(2I_x)^{(k-l)/2}(2I_y)^{l/2}\cos(\chi_{hl}) \tag{18}$$

with $\chi_{hl} = (k-l)b_x + lb_y + \eta_{hl}$. The equations of motion are

$$I_x' = \sum_h \sum_{l=0}^{k} (k-l)D_{hl}(2I_x)^{(k-l)/2}(2I_y)^{l/2}\sin(\chi_{hl}) \tag{19}$$

$$I_y' = \sum_h \sum_{l=0}^{k} lD_{hl}(2I_x)^{(k-l)/2}(2I_y)^{l/2}\sin(\chi_{hl}) \tag{20}$$

$$b_x' = \nu_x - \nu_x^0 + \frac{\partial U_0}{\partial I_x} + \frac{\partial U_1}{\partial I_x} \tag{21}$$

$$b_y' = \nu_y - \nu_y^0 + \frac{\partial U_0}{\partial I_y} + \frac{\partial U_1}{\partial I_y} \tag{22}$$

where prime denotes $d/d\theta$.

Consider the tunes $\nu_{x,y}$ modulated at the synchrotron frequency ν_s,

$$\nu_{x,y} = \tilde{\nu}_{x,y} + (\Delta\nu)\cos(\nu_s\theta) \quad . \tag{23}$$

We assume that within each synchrotron oscillation all betatron resonances of Eq. (9) are crossed forward and backward once. The condition for this assumption is $\Delta\nu > |\nu_x^0 - \nu_y^0|$. Since $I_{x,y}$ change very slowly compared with changes in $b_{x,y}$, they are adiabatic invariants. To estimate the changes of $I_{x,y}$, one can approximate the variation in phase as

$$b_{x,y} \simeq \delta_{x,y}\theta + \frac{\Delta\nu}{\nu_s}\sin(\nu_s\theta) + b_{x,y}(0) \tag{24}$$

where $\delta_{x,y} = \tilde{\nu}_{x,y} - \nu_{x,y}^0 + \partial U_0/\partial I_x$, and

$$\chi_{hl} \simeq [(k-l)\delta_x + l\delta_y]\theta + k\frac{\Delta\nu}{\nu_s}\sin(\nu_s\theta) + \overline{\eta}_{hl} \tag{25}$$

where $\overline{\eta}_{hl} = (k-l)b_x(0) + lb_y(0) + \eta_{hl}$. Then the changes of $I_{x,y}$ during each crossing can be estimated by substituting Eq. (25) to Eqs. (19) and (20), and integrating over θ in a half period of synchrotron oscillation. Since

$$sin\left([(k-l)\delta_x + l\delta_y]\theta + k\frac{\Delta\nu}{\nu_s}\sin(\nu_s\theta) + \overline{\eta}_{hl}\right) = \sum_{n=-\infty}^{\infty} J_n\left(k\frac{\Delta\nu}{\nu_s}\right) sin\left[((k-l)\delta_x + l\delta_y - n\nu_s)\theta + \overline{\eta}_{hl}\right] \tag{26}$$

where J_n is the nth order Bessel function, the change of I_x during each crossing is approximately

$$\Delta I_x = \sum_{hn}\sum_{l=0}^{k}(k-l)D_{hl}(2I_x)^{(k-l)/2}(2I_y)^{l/2}J_n\left(k\frac{\Delta\nu}{\nu_s}\right) \times \frac{\cos(\overline{\eta}_{hl}) - \cos\left[((k-l)\delta_x + l\delta_y - n\nu_s)\frac{\pi}{\nu_s} + \overline{\eta}_{hl}\right]}{(k-l)\delta_x + l\delta_y - n\nu_s} \quad . \tag{27}$$

ΔI_y can be obtained by exchanging x and y in Eq. (27). For large k, the dominant contributions in the summations of Eq. (27) are from the terms with

$$(k-l)\delta_x + l\delta_y \simeq n_s\nu_s \quad . \tag{28}$$

These are the synchro-betatron sidebands. Therefore most of the emittance growth during the crossings of betatron resonances occur near the sidebands. Keeping only the dominant terms, the changes of $I_{x,y}$ per half synchrotron oscillation is

$$\Delta I_x = \frac{\pi}{\nu_s}\sum_h\sum_{l=0}^{k}(k-l)D_{hl}(2I_x)^{(k-l)/2}(2I_y)^{l/2} \times J_{n_s}\left(k\frac{\Delta\nu}{\nu_s}\right)\sin(\overline{\eta}_{hl}) \tag{29}$$

where n_s is the order of the synchrotron-betatron resonance satisfying Eq. (28). Since

$$J_{n_s}\left(k\frac{\Delta\nu}{\nu_s}\right) \simeq \sqrt{\frac{2\nu_s}{\pi k\Delta\nu}}\cos\left(k\frac{\Delta\nu}{\nu_s} - \left(\frac{n_s}{2}+\frac{1}{4}\right)\pi\right) , \quad (30)$$

$$\Delta I_x = \sqrt{\frac{2\pi}{k\nu_s\Delta\nu}}\sum_h\sum_{l=0}^{k}(k-l)D_{hl}(2I_x)^{(k-l)/2}(2I_y)^{l/2} \times\cos\left(k\frac{\Delta\nu}{\nu_s} - \left(\frac{n_s}{2}+\frac{1}{4}\right)\pi\right)\sin(\overline{\eta}_{hl}) . \quad (31)$$

If we assume many betatron oscillations between resonance crossings, it is reasonable to assume that the betatron phase at the crossing is uncorrelated from one crossing to the next. In this case the repeated crossing can be treated as a random walk process and the rms growth of $I_{x,y}$ during a half period of the synchrotron oscillation can be obtained by averaging $(\Delta I_{x,y})^2$ over $\overline{\eta_{hl}}$, i.e.,

$$\Delta I_x^2 = \frac{\pi}{k\nu_s\Delta\nu}\sum_h\sum_{l=0}^{k}(k-l)^2D_{hl}^2(2I_x)^{(k-l)}(2I_y)^l \times\cos^2\left(k\frac{\Delta\nu}{\nu_s} - \left(\frac{n_s}{2}+\frac{1}{4}\right)\pi\right) . \quad (32)$$

The diffusion rate of the emittance growth is then

$$K_x = \frac{\Delta I_x^2}{\frac{\pi}{\nu_s}} = \frac{1}{k\Delta\nu}\sum_h\sum_{l=0}^{k}(k-l)^2D_{hl}^2 \times(2I_x)^{(k-l)}(2I_y)^l\cos^2\left(k\frac{\Delta\nu}{\nu_s} - \left(\frac{n_s}{2}+\frac{1}{4}\right)\pi\right) , \quad (33)$$

and K_y can be obtained by exchanging x and y in Eq. (33).

As can be seen from Eq. (33), the diffusion rate is more or less independent of ν_s. This is because each change in $I_{x,y}$ is larger for smaller ν_s, but the number of crossings per turn is correspondingly smaller for smaller ν_s. Eq. (33) also shows that the diffusion rate increases as $\Delta\nu$ decreases. To justify this result we should point out that $\Delta\nu$ here was assumed to be much larger than the total resonance width and these resonances are isolated. Therefore a "small" change of $\Delta\nu$ will not change the crossing process. However, for a given modulation frequency, the smaller the $\Delta\nu$, the longer the time spent in each individual resonance. Consequentially the diffusion will be enhanced.

The validity of the random walk approximation for the emittance growth is based on the assumption that the betatron phase is uncorrelated between successive crossings. One possibility of this betatron phase randomness is chaotic motion near the synchro-betatron resonances. By using the resonance overlap criterion,[4] the sufficient condition for the chaotic motion on the synchro-betatron resonances is estimated as

$$D_{hl}(2I_x^*)^{(k-l)/2}(2I_y^*)^{l/2}\left[(k-l)^2\frac{\partial^2U_0}{\partial I_x^{*2}} + 2l(k-l)\frac{\partial^2U_0}{\partial I_x^*\partial I_y^*} + l^2\frac{\partial^2U_0}{\partial I_y^{*2}}\right] \geq \frac{1}{32}\sqrt{2\pi k\Delta\nu\nu_s^3} \quad (34)$$

where $I_{x,y}^*$ are the betatron actions corresponding to the synchro-betatron resonance. In most cases the chaotic threshold of Eq. (34) is overestimated due to many simplifications in the model for the transition to chaos. As a matter of fact, the chaotic layers generically exist near the separatrices of resonances, and also there are many "external" noise in real machine. As many betatron oscillations occur between resonance crossings, the condition for the betatron phase to be uncorrelated between successive crossings should not be as stringent as Eq. (34).

APPENDIX

If we assume that a single multipole term is important, the Hamiltonian (17) can be written as

$$K(I_-,I_+,b_-,b_+) = [(\nu_x-\nu_x^0)(k-l) - (\nu_x-\nu_x^0)l]I_- + [(\nu_x-\nu_x^0)(k-l) + (\nu_x-\nu_x^0)l]I_+ + U_0(I_x,I_y) + (2I_x)^{(k-l)/2}(2I_y)^{l/2}\sum_h D_{hl}\cos(b_+ + \eta_{hl}) \quad (35)$$

where $I_\pm = I_x/(k-l) \pm I_y/l$ and $b_\pm = (k-l)b_x \pm lb_y$. Since $\partial K/\partial b_- = 0$,

$$I_- = \frac{1}{k-l}I_x - \frac{1}{l}I_y = \text{constant} . \quad (36)$$

Assuming crossings with uncorrelated betatron phase, we obtain the change of I_+^2 during a half period of the synchrotron oscillation

$$\Delta I_+^2 = \frac{\pi B}{k\nu_s\Delta\nu}(k-l)^2(2I_x)^{(k-l)}(2I_y)^l \quad (37)$$

where

$$B = \left(\sum_h D_{hl}^2\right)\cos^2\left(k\frac{\Delta\nu}{\nu_s} - \left(\frac{n_s}{2}+\frac{1}{4}\right)\pi\right) . \quad (38)$$

Then the rms growth of I_+ can be expressed by

$$\frac{dI_+^2}{d\theta} = \frac{\Delta I_+^2}{\frac{\pi}{\nu_s}} = \frac{(k-l)^2B}{k\Delta\nu}(2I_x)^{(k-l)}(2I_y)^l . \quad (39)$$

With Eq. (36), this equation can be integrated as

$$\int_{I_+(0)}^{I_+}\frac{I_+dI_+}{(I_++I_-)^{k-l}(I_+-I_-)^l} = \frac{B(k-l)^{k-l+2}l^l}{2^{1-k}k\Delta\nu}\theta . \quad (40)$$

For $k \geq 3$, the left side of Eq. (40) remains finite even if $I_+ \to \infty$, implying faster than exponential buildup, including infinite I_+ even for finite θ. It is shown that a single multipole term is no longer valid and the detuning effect due to the amplitude dependence of tunes must be considered.

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