Theory of the Beta Function Shift Due to Linear Coupling*

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Abstract

This paper presents analytical perturbation theory results for β_1, β_2, the beta functions in the presence of linear coupling.

1 INTRODUCTION

This paper presents analytical perturbation theory results for β_1, β_2, the beta functions in the presence of linear coupling. It is a continuation of a previous paper[1] that gave analytical perturbation theory results for the tune ν_1, ν_2 in the presence of linear coupling. The results for β_1, β_2 hold when ν_x, ν_y are close to the resonance line $\nu_x - \nu_y = p$. The shift in beta functions is then linear in the skew quadrupole field given by $a_1(s)$. When ν_x, ν_y are far enough from the $\nu_x - \nu_y = p$ resonance, then the shift in the beta function becomes quadratic in the skew quadrupole field.

The analytical results show that the important harmonics in the skew quadrupole fields for producing large beta functions shifts are the harmonics near $\nu_x + \nu_y$. The harmonics near $\nu_x + \nu_y$ are also the important harmonics for the higher order tune (see Ref. 1). It is also shown that the beta function shift and the higher order tune shift have the same driving terms, thus, one may expect that an a_1 correction system that corrects the higher order tune shift will also correct the beta function shift.

2 LOWEST ORDER SOLUTION FOR β_1, β_2

The presence of the skew quadrupole fields will couple the x and y motions. New beta functions, β_1, β_2 can be defined[2] which are the beta functions of the normal modes and which are different from β_x, β_y, the beta functions of the unperturbed accelerator.

It will be shown below that β_1 and β_2 can be found from the solutions of the equations of motions, Eq. (2.1) in reference 1. These solutions were written there as

$$\begin{aligned}
&\eta_x = \zeta_x + \zeta_x^*, \quad \eta_y = \zeta_y + \zeta_y^* \\
&\zeta_x = A_s \exp(i\nu_{x,s}\theta_x) + \sum_{r\neq s} A_r \exp(i\nu_{x,r}\theta_x), \\
&\zeta_y = B_s \exp(i\nu_{y,s}\theta_x) + \sum_{r\neq s} B_r \exp(i\nu_{y,r}\theta_y), \\
&\nu_{x,s} - \nu_{y,s} = p
\end{aligned} \tag{2.1}$$

ζ_x^* is the complex conjugate of ζ_x.

The lowest order solution for the A_r, B_r are given by Eq. (2.7) Ref. 1, which can be put into Eq. (2.1) to find ζ_x, ζ_y. The first two equations in Eq. (2.7), Ref. 1 show that the two large coefficients A_s, B_s are related. For the ν_1 mode, where $\nu_1 \rightarrow \nu_x$ when $a_1 \rightarrow 0$, and using $\nu_{x,s} \simeq \nu_x$ one finds

$$B_s = \frac{-(\nu_1 - \nu_x)}{\Delta\nu(\nu_{x,s}, \nu_{y,s})} A_s . \tag{2.2a}$$

For the ν_2 mode, using $\nu_{y,s} \simeq \nu_y$, one finds

$$A_s = \frac{-(\nu_2 - \nu_y)}{\Delta\nu^*(\nu_{x,s}, \nu_{y,s})} B_s . \tag{2.2b}$$

$\Delta\nu(\nu_{x,s}, \nu_{y,s})$ is defined by Eq. (2.8), Ref. 1.

The last two equations of Eq. (2.7), Ref. 1, can be solved for A_r and B_r, which can then be put into Eq. (2.1) to find the Floquet solutions. Note that $A_r \neq 0$ only for $\nu_{x,y} = \nu_{y,s} + n$, $n \neq p$, and $B_r \neq 0$ only for $\nu_{y,r} = \nu_{x,s} + n$, $n \neq -p$. Assuming that ν_x, ν_y is close to the resonance line $\nu_{x,s} = \nu_{y,s} + p$, so that $\nu_{x,s} \simeq \nu_x$, and $\nu_{y,s} \simeq \nu_y$, then

$$\begin{aligned}
A_r &= \frac{-2\nu_x \, b_x(\nu_{x,r}, \nu_{y,s})}{(n + \nu_{y,s})^2 - \nu_x^2} B_s \\
A_r &= \frac{-2\nu_x \, b_x(\nu_{x,r}, \nu_{y,s})}{(n + \nu_x + \nu_y)(n - p)} B_s
\end{aligned} \tag{2.3}$$

where $n \neq p$, $\nu_{x,r} = \nu_{y,s} + n$ and b_x is defined by Eq. (2.6), Ref. 1.

Similarly, one finds for B_r

$$\begin{aligned}
B_r &= \frac{-2\nu_y \, b_y(\nu_{y,r}, \nu_{x,s})}{(n + \nu_{x,s})^2 - \nu_y^2} A_s \\
B_r &= \frac{-2\nu_y \, b_y(\nu_{y,r}, \nu_{x,s})}{(n + \nu_x + \nu_y)(n + p)} A_s
\end{aligned} \tag{2.4}$$

where $n \neq -p$, $\nu_{y,r} = \nu_{x,s} + n$

We can now find ζ_x for the ν_1 mode using Eqs. (2.3) and (2.2a) for A_r and putting these results into Eq. (2.1) for ζ_x,

$$\begin{aligned}
\zeta_x &= A_s e^{i\nu_1\theta_x} \left\{ 1 + \sum_{n\neq -p} f_n \right\} \\
f_n &= \frac{\nu_1 - \nu_x}{\Delta\nu(\nu_{x,s}, \nu_{y,s})} \frac{2\nu_x \, b_n \, \exp[-i(n+p)\theta_x]}{(n - \nu_x - \nu_y)(n+p)} \\
b_n &= \frac{1}{4\pi\rho} \int ds \, a_1 (\beta_x\beta_y)^{\frac{1}{2}} \exp[i((n - \nu_y)\theta_x + \nu_y\theta_y)]
\end{aligned} \tag{2.5}$$

*Work performed under the auspices of the U.S. Department of Energy.

A similar result can be found for ζ_y for the ν_2 mode

$$\zeta_y = B_s e^{i\nu_2\theta_y}\left\{1+\sum_{n\neq p} g_n\right\}$$

$$g_n = \frac{\nu_2-\nu_y}{\Delta\nu^*(\nu_{x,s},\nu_{y,s})}\frac{2\nu_y\, c_n\, \exp[-i(n-p)\theta_y]}{(n-\nu_x-\nu_y)(n-p)} \tag{2.6}$$

$$c_n = \frac{1}{4\pi\rho}\int ds\, a_1 (\beta_x\beta_y)^{\frac{1}{2}}\exp[i((n-\nu_x)\theta_y+\nu_x\theta_x)]$$

From the above Floquet solutions for ζ_x, ζ_y, one can find β_1 and β_2 the beta functions of the normal modes. This will be done below. It may be noted that b_n and c_n are just the integrals involved in computing the stopbands of $\nu_x+\nu_y = n$ sum resonance, but at certain choices of the ν-values on the resonance line. The b_n corresponds to the tune choice $n-\nu_y, \nu_y$, and c_n to $\nu_x, n-\nu_x$. The resonance denominator $1/(n-\nu_x-\nu_y)$ shows that the important n is near $\nu_x+\nu_y$.

The x motion given by $x = \beta_x^{1/2}\left(\zeta_x+\zeta_x^*\right)$ is the x-motion when only the ν_1 mode is excited. Similarly, $y = \beta_y^{1/2}\left(\zeta_y+\zeta_y^*\right)$ is the y motion when only the ν_2 mode is excited.

Results for β_1 and β_2

It was shown by Edwards and Teng[2] that one can transform from the x, x', y, y' coordinates to a new set of coordinates $v, v'u, u'$ which are uncoupled. The solutions of equations of motions for u and v can be written as[2]

$$\begin{aligned} v &= \sqrt{\beta_1\epsilon_1}\exp(i\psi_1) + \text{c.c.} \\ u &= \sqrt{\beta_2\epsilon_2}\exp(i\psi_2) + \text{c.c.} \end{aligned} \tag{2.7}$$

β_1 and β_2 are periodic functions and are the beta functions in the presence of linear coupling. If no solenoids are present, the ψ_1 and ψ_2 are related to β_1, β_2 by[2]

$$\begin{aligned} 1/\beta_1 &= d\psi_1/ds \\ 1/\beta_2 &= d\psi_2/ds \end{aligned} \tag{2.8}$$

ϵ_1 and ϵ_2 are two constants that turn out to be the emittances of each normal mode.

The x, x', y, y' and the v, v', u, u' coordinates are related by[2]

$$x = Rv \tag{2.9}$$

where R is a 4×4 matrix given by

$$R = \begin{pmatrix} I\cos\varphi & \overline{D}\sin\varphi \\ -D\sin\varphi & I\cos\varphi \end{pmatrix} \tag{2.10}$$

D and $\overline{D}$ are 2×2 matrices, and $\overline{D} = D^{-1}$. I is the 2×2 identity matrix. D and φ can be computed from the one turn transfer matrix.[2]

Let v, v' be the coordinates that have the tune ν_1 where $\nu_1 \to \nu_x$ when $a_1 \to 0$. Then if only this mode is present then x is given by

$$x = \cos\varphi\; v \tag{2.11}$$

From Eq. (2.11) one finds

$$\cos\varphi\sqrt{\beta_1\epsilon_1}\exp(i\psi_1) = \sqrt{\beta_x}\zeta_x \tag{2.12}$$

where ζ_x is given by Eq. (2.5). It follows that

$$\begin{aligned} \beta_1 &= d\psi/ds \\ \zeta_x &= |\zeta_x|\exp(i\psi) \ . \end{aligned} \tag{2.13}$$

and ψ can be found from Eq. (2.5).

$$\zeta_x = A_s\exp(i\nu_1\theta_x)\left(1+\sum_{n\neq -p} f_n\right)$$

$$\zeta_x = A_s\left(1+\frac{1}{2}\sum_{n\neq -p}(f_n+f_n^*)\right)$$

$$\exp\left[i\left(\nu_1\theta_x+\frac{1}{2i}\sum_{n\neq -p}(f_n-f_n^*)\right)\right] \tag{2.14}$$

$$\psi = \nu_1\theta_x + \frac{1}{2i}\sum_{n\neq -p}(f_n-f_n^*)$$

$$\frac{1}{\beta_1} = \frac{d\psi}{ds} = \frac{\nu_1}{\nu_x\beta_x} + \frac{1}{2\nu_x\beta_x}\sum_{n\neq -p}(-n-p)(f_n+f_n^*) \ .$$

Using $1/\beta_1 - 1/\beta_x \simeq -(\beta_1-\beta_x)/\beta_x^2$ one finds

$$\frac{\beta_1-\beta_x}{\beta_x} = -\frac{\nu_1-\nu_x}{\nu_x} - \sum_{n\neq -p}\frac{(-n-p)}{2\nu_x}(f_n+f_n^*) \tag{2.15}$$

$$\frac{\beta_1-\beta_x}{\beta_x} = -\sum_{\text{all } n}\left\{\frac{\nu_1-\nu_x}{\Delta\nu(\nu_1,\nu_1-p)}\frac{b_n}{n-\nu_x-\nu_y}\right.$$

$$\left.\exp[-i(n+p)\theta_x] + \text{c.c.}\right\} \ . \tag{2.16}$$

In a similar way, one also finds

$$\frac{\beta_2-\beta_y}{\beta_y} = -\sum_{\text{all } n}\left\{\frac{\nu_2-\nu_y}{\Delta\nu^*(\nu_2+p,\nu_2)}\frac{c_n}{n-\nu_x-\nu_y}\right.$$

$$\left.\exp[-i(n-p)\theta_y] + \text{c.c.}\right\} \ . \tag{2.17}$$

Eq. (2.16) can be written in an integral form by using the result

$$\sum_{\text{all } n}\frac{\exp[in(\theta-\theta')]}{n-\nu} = -\pi\frac{\exp(\mp i\pi\nu)}{\sin\pi\nu}e^{i\nu(\theta-\theta')} \tag{2.18}$$

where the top sign is used for $\theta > \theta'$, and the bottom sign for $\theta < \theta'$. Replacing b_n using Eq. (2.5) one finds

$$\frac{\beta_1 - \beta_x}{\beta_x} = -\frac{(\nu_1 - \nu_x)}{|\Delta\nu(\nu_1, \nu_1 - p)|} \frac{1}{2\rho \sin\pi(\nu_x + \nu_y)} \times \int ds'\, a_1(s') \left(\beta_x(s')\beta_y(s')\right)^{\frac{1}{2}} \cos\left[\pm\pi(\nu_x + \nu_y) - (\nu_x + \nu_y)(\theta_x - \theta_x') + \nu_y(\theta_y' - \theta_x') - p\theta_x - \delta_1\right] . \quad (2.19a)$$

$\delta_1 = \text{phase}\,[\Delta\nu(\nu_1, \nu_1 - p)]$, and in the $\pm$ sign, the $+$ sign is used for $\theta > \theta'$, and the $-$ sign for $\theta < \theta'$.

In a similar way one can find $(\beta_2 - \beta_y)/\beta_y$ as

$$\frac{\beta_2 - \beta_y}{\beta_y} = -\frac{(\nu_2 - \nu_y)}{|\Delta\nu(\nu_2 + p, \nu_2)|} \frac{1}{2\rho \sin\pi(\nu_x + \nu_y)} \times \int ds'\, a_1(s') \left(\beta_x(s')\beta_y(s')\right)^{\frac{1}{2}} \cos\left[\pm\pi(\nu_x + \nu_y) - (\nu_x + \nu_y)(\theta_y - \theta_y') + \nu_x(\theta_x' - \theta_y') + p\theta_y + \delta_2\right] . \quad (2.19b)$$

$\delta_2 = \text{phase}\,[\Delta\nu(\nu_2 + p, \nu_2)]$.

Eq. (2.16) shows that the important harmonics in a_1 are the harmonics near $\nu_x + \nu_y$. However, Eq. (2.16) shows that the dominant harmonic excited in β_1 due to the a_1 field is the $2\nu_x$ harmonic, and in β_2 the $2\nu_y$ harmonic.

One may note the factor $(\nu_1 - \nu_x)/\Delta\nu$. Close to the resonance line $\nu_x = \nu_y + p$ where $|\Delta\nu| >> |\nu_x - \nu_y - \nu_p|$, then this factor approaches 1. This may be seen from Eq. (2.10) in Ref. 1 for ν_1 and ν_2. According to Eq. (2.10), Ref. 1, $(\nu_1 - \nu_x)/|\Delta\nu| \to 1$ for large $\Delta\nu$, and $(\nu_1 - \nu_x)/|\Delta\nu| \sim 2|\Delta\nu|/|\nu_x - \nu_y|$ for small enough $\Delta\nu$. Thus $(\beta_1 - \beta_x)/\beta_x$ is linear in a_1 for large enough $\Delta\nu$, ν_x, ν_y close enough to the resonance line, and quadratic in a_1 for small enough $\Delta\nu$, far enough from the resonance line. For small enough $\Delta\nu$ where $(\beta_1 - \beta_x)/\beta_x$ becomes quadratic in a_1, then Eq. (2.16) is no longer correct because of the neglect of a_1^2 terms in deriving it.

A result for the rms value of $(\beta_1 - \beta_x)/\beta_x$ due to a random distribution of a_1 errors may be obtained from the integral form Eq. (2.19), for the case when $|\Delta\nu| >> |\nu_x - \nu_y - p|$. In this case $|\nu_1 - \nu_x|/|\Delta\nu| \simeq 1$ and

$$\left(\frac{\beta_1 - \beta_x}{\beta_x}\right)^2_{\rm rms} = \sum_k \left(\frac{\beta_1 - \beta_x}{\beta_x}\right)^2_{k,\rm rms}$$

$$\left(\frac{\beta_1 - \beta_x}{\beta_x}\right)_{k,\rm rms} = N_k^{1/2} \frac{\left((\beta_x\beta_y)^{1/2} a_{1,\rm rms}\right)_k}{2.8\rho \sin\pi(\nu_x + \nu_y)} \quad (2.20)$$

where the index k indicates the different types of magnets. N_k is the number of magnets of a certain type. Eq. (2.19) also gives the result for $((\beta_2 - \beta_y)/\beta_y)_{rms}$. One also sees that

$$((\beta_1 - \beta_x)/\beta_x)_{\rm rms} = [4\pi/(2.8 \sin\pi(\nu_x + \nu_y))]\,\Delta\nu_{\rm rms} \quad (2.21)$$

where $\Delta\nu_{\rm rms}$ is the rms value of $\Delta\nu$.

3 CORRECTION OF β_1, β_2

The above analytical results for the beta function shifts show that when the higher order tune shifts $\nu_1 - \nu_x$ and $\nu_2 - \nu_y$ are corrected, then the beta function shifts are also corrected. This can be seen by comparing Eq. (2.5) for the beta function shift with Eq. (3.2 and 3.3) in Ref. 1, for the higher order tune shift. Both these effects have the same driving terms b_n and c_n, and for both effects the important b_n, c_n are those for which n is close to $\nu_x + \nu_y$.

This result has been observed in numerical computations[3] for the RHIC accelerator, where an a_1 correction system has been provided to correct the higher order tune shift.[4,5] In order to correct the shift in the beta functions it is important that in correcting the higher order tune shift, that one correct not only the tune splitting $|\nu_1 - \nu_2|$ but also the shift in the average tune $(\nu_1 + \nu_2)/2$. The harmonic closest to $\nu_x + \nu_y$ do not have much effect on $|\nu_1 - \nu_2|$ but are most important for the average tune $(\nu_1 + \nu_2)/2$, and also for the beta function shift. One might be able to correct the average tune $(\nu_1 + \nu_2)/2$ using the normal tune adjusting quadrupoles instead of the a_1 correctors, but this would not help to correct the beta function shift.

4 REFERENCES

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