Conversion of Power and Frequency*

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Abstract

This paper deals with a novel idea to excite electrons to radiate energy in the short (millimeter) wavelength range. A short electron bunch is made to travel along the axis of a waveguide where a TM electromagnetic wave is also traveling and causes the beam to perform transverse oscillations. The electrons radiate energy as a consequence of the oscillations. It is found that a convenient mode of operation is to drive the waveguide in proximity of the cut-off.

1 INTRODUCTION

It is well-known that there are no rf power sources of significant amount for frequencies larger than 3 GHz. Ordinary conceived methods to produce radiation with short wavelengths is to let a bunch of electrons to travel down a magnetic structure with alternating field direction, up and down, like in the *wigglers* or in the *undulators*, as it is done in circular synchrotron radiation facilities. Nevertheless these devices are ordinarly used to generate radiation of much shorter wavelength. We propose a new method to produce radiation of wavelength in the millimeter range, which still makes use of a short electron bunch, traveling this time down a waveguide which is at the same time excited by a lower TM propagating electromagnetic (EM) wave. Radiation can be obtained by *wiggling* the motion of the electrons in a direction perpendicular to the main one. The wiggling action can be induced by electromagnetic fields in a fashion similar to the one caused by wiggler magnets, as we shall show in this paper. We found that an interesting mode of operation is to drive the waveguide with an excitation of frequency very close to the cut off. Indeed for such excitation, the corresponding EM wave travels with a very large phase velocity which in turn has the effect of increasing the wiggling action on the electron bunch. Our method, to be effective, relies also on the *coherence* of the radiation from electron to electron; that is the bunch length is taken to be considerably shorter than the radiated wavelength. In this case, the total power radiated should be proportional to the square of the total number of the electrons in the bunch.

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2 EQUATIONS OF MOTION

2.1 Fields in the Waveguide

Consider a straight, square waveguide of perfectly conductive walls of width w. Define a cartesian coordinate system with z being the direction of the propagation of a TM traveling wave, along the waveguide axis. The scalar potential of one of the lowest modes is[1]

$$V = V_0 \sin\left(2\pi\frac{x}{w}\right)\cos\left(\pi\frac{y}{w}\right)\cos(kz-\omega t) \qquad (1)$$

which satisfies the condition that the z component of the electric field vanishes at the wall surfaces $x = \pm w/2$ and $y = \pm w/2$. The dispersion relation that relates the frequency ω to the wave number k is

$$k = \sqrt{\frac{\omega^2}{c^2} - 5\left(\frac{\pi}{w}\right)^2}. \qquad (2)$$

The propagation of the wave in the waveguide requires that ω is larger than the cutoff frequency ω_c with

$$\omega_c = \sqrt{5}\ \frac{\pi c}{w}.$$

Introduce the form factor $q = \omega_c/\omega$. q is related to the normalized phase velocity $\beta_w = \omega/kc$ by

$$\beta_w = \frac{1}{\sqrt{1-q^2}}. \qquad (3)$$

The range of value of q fulfilling the condition of propagation is

$$0 < q < 1 \text{ and } \beta_w > 1. \qquad (4)$$

Associated to V is the vector potential **A** which has only a longitudinal component $A = \beta_w V$. The electric **E** and magnetic **B** fields can be determined as

$$E_x = -2\pi\frac{V_0}{w}\cos\left(2\pi\frac{x}{w}\right)\cos\left(\pi\frac{y}{w}\right)\cos\phi \qquad (5)$$

$$E_y = \pi\frac{V_0}{w}\sin\left(2\pi\frac{x}{w}\right)\sin\left(\pi\frac{y}{w}\right)\cos\phi \qquad (6)$$

$$E_z = -k\left(\beta_w^2 - 1\right)V_0\sin\left(2\pi\frac{x}{w}\right)\cos\left(\pi\frac{y}{w}\right)\sin\phi \quad (7)$$

and

$$B_x = -\beta_w E_y \tag{8}$$

$$B_y = \beta_w E_x \tag{9}$$

$$B_z = 0 \tag{10}$$

where $\phi(z,t) = kz - \omega t$ is the phase of the electromagnetic wave.

The total power flow P along the waveguide is obtained by integrating the axial component of the Poynting vector over the cross-section of the waveguide

$$P = \frac{c}{8\pi}\int_{-\frac{w}{2}}^{+\frac{w}{2}}\int_{-\frac{w}{2}}^{+\frac{w}{2}} (E_x B_y - B_x E_y)\, dx\, dy = \frac{5\pi c}{64}\, \beta_w\, V_0^2. \tag{11}$$

2.2 Motion in the Waveguide

Consider an electron of rest mass m and electric charge e moving along the waveguide. The equations of motion including the force $\mathbf{F}_{\rm rad}$ of radiation reaction is

$$m\frac{d(\gamma \mathbf{v})}{dt} = e\mathbf{E} + e\frac{\mathbf{v}}{c} \times \mathbf{B} + \mathbf{F}_{\rm rad} \tag{12}$$

where $\mathbf{v} \equiv (\dot{x}, \dot{y}, \dot{z}) = \vec{\beta} c$ is the velocity vector of the electron. Assume that the amplitude of the transverse motion of the electron is small compared with the transverse dimension w. With the chosen mode, the vertical displacement is small compared with the horizontal one. The transverse components of Eq. (12) can be approximated by

$$m\,\frac{d(\gamma v_x)}{dt} = 2\pi e\,(\beta_z \beta_w - 1)\,\frac{V_0}{w}\cos\left(2\pi\frac{x}{w}\right)\cos\phi \tag{13}$$

$$m\,\frac{d(\gamma v_y)}{dt} = 0. \tag{14}$$

The longitudinal component of Eq. (12) can be replaced by the one describing the energy change of the electron,

$$mc^2\frac{d\gamma}{dt} = e\mathbf{v}\cdot\mathbf{E} - P_{\rm rad}$$

where

$$P_{\rm rad} = \frac{2e^2}{3c}\gamma^6\left[\dot{\vec{\beta}}^2 - \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2\right] \tag{15}$$

is the power loss due to the radiation.

For relativistic electrons, $\beta_{x,y} \ll \beta_z \le 1$. Under the condition that $|\beta_z\beta_w - 1| \gg \beta_x/\beta_z$, the horizontal equation of motion becomes for small displacement $x \ll w$

$$\ddot{x} = \Omega^2 w \cos\phi \tag{16}$$

where

$$\Omega^2 = 2eV_0\frac{\pi}{w^2}\frac{\beta\beta_w - 1}{m_0\gamma}. \tag{17}$$

The solution to Eq. (16) is

$$x = -w\,\nu^2\,\cos\,\phi, \quad \nu = \Omega/\Omega_0 \tag{18}$$

which is an oscillation at the frequency of the phase slippage

$$\Omega_0 = -\dot{\phi} = -ck\,(\beta_z - \beta_w)\,. \tag{19}$$

In the case that $\beta_z \approx 1$, the condition for small-amplitude oscillation $\nu^2 \ll 1$ becomes

$$\frac{eV_0}{mc^2\gamma} \ll \frac{5\pi}{2}\frac{1}{\beta_w + 1} \tag{20}$$

which limits the value of V_0. Based on Eq. (16), the energy change becomes

$$\frac{d\gamma}{dt} = \left(\frac{eV_0}{mc^2}\right)^2\left[\frac{4\omega\,(\beta_w^2 - 1)\,(\beta_w + 2)}{5\gamma\beta_w}\sin\phi\cos\phi - \frac{8\pi^2}{3}\frac{r_0\gamma^2 c\,(\beta_w - 1)^2}{w^2}\cos^2\phi\right] \tag{21}$$

where $r_0 = e^2/mc^2$ is the classical electron radius. The last term in Eq. (21) corresponds to the radiation loss.

2.3 Radiation

In the frame with the reference electron at rest, the EM wave in the waveguide is incident on the electrons and results in Thomson scattering. Using Lorentz transformations, the frequency of the wave emitted from the scattering observed in the laboratory frame, is

$$\omega_r = \frac{\omega}{1-\beta_z}\frac{\beta_w - \beta_z}{\beta_w}. \tag{22}$$

The power radiated by a single electron is given by Eq. (15). In the case that the number of oscillations of the electron in the waveguide is very large, this power is mostly emitted at the frequency ω_r.

With a bunch of N electrons, the maximum amount of power can be achieved if the electrons radiate coherently. When the longitudinal bunch length is considerably smaller than the radiated wavelength, and when the transverse bunch width is smaller than the oscillation amplitude, the power is

$$P_{\rm rad} = \frac{N^2 e^2\gamma^4 w^2\Omega^4}{3c^3}. \tag{23}$$

For a monochromatic radiation, the relative variation of the electron energy within an oscillation period must be small, i.e. according to Eq. (21)

$$\frac{1}{\Omega_0\gamma}\frac{d\gamma}{dt} = 5\pi^2\nu^4\frac{(\beta_w + 2)}{(\beta_w + 1)} \ll 1. \tag{24}$$

3 APPLICATIONS

3.1 Frequency Transformer and Power Amplifier

The device that consists of the wiggling electrons and the waveguide with propagating TM wave, can be characterized by two parameters R_ω and R_P describing the frequency and power amplification, respectively. Since $\beta_w > 1$,

$$R_\omega = \omega_r/\omega > 1 \quad (25)$$

indicating that the radiated frequency is always larger than that of the waveguide input.

Under the condition of coherent radiation, the ratio of power amplification is

$$R_P = \frac{256\pi N^2\gamma^2 r_0^2}{15w^2}\frac{(\beta\beta_w - 1)^2}{\beta_w} \quad (26)$$

which does not depend on the voltage amplitude V_0 in the waveguide.

3.2 Modes of Operation

In the case $R_\omega \sim 1$ and $R_P \gg 1$, the device is a power amplifier. It is thus required that the electron speed is low, and the waveguide operates well above the cutoff frequency. In the case $R_\omega \gg 1$, the device is a frequency transformer. Eq. (25) implies that the electron energy should be high, $\gamma \gg 1$. Eq. (26) further indicates that higher power can be achieved if the waveguide is operated near the cutoff frequency, $\beta_w \gg 1$.

Operating near the cutoff frequency, however, imposes a stringent condition on the beam quality. From Eq. (21), the oscillation amplitude must be small enough to satisfy

$$\nu^4 \ll \frac{1}{5\pi^2}. \quad (27)$$

An example of this mode of operation is shown in Table 1.

Table 1: An Example of Frequency Transformer

Kinetic Energy of Electrons	4 MeV
Number of Electrons	4×10^{11}
Bunch Length	1 mm
Input Frequency	1.3 GHz
Radiated Frequency	190 GHz
Frequency Transform Ratio	146
Power Amplification Factor	1.06
Phase Velocity, β_w	15.5
Cutoff Frequency	1.2973 GHz
Waveguide Dimension, w	25.8 cm
Period of Oscillations	24.5 cm

4 CONCLUSION

A practical demonstration of the method discussed in this paper would be valuable even, for instance, with low initial power amplification and modest electron beam intensity. There are nevertheless still a number of technical and design issues to be solved. For instance, an experimental evidence of *coherent* radiation is crucial in our model. We plan to continue by examining closely the interaction of a short electron bunch with an electromagnetic wave of general propagating mode traveling in a waveguide. We would also like to consider more realistic, less conductive wall material which could have the effect of slowing down the phase velocity.

5 REFERENCES

1. J.D. Jackson, Classical Electrodynamics (John Wiley & Sons, New York, 1975).