Synchrotron radiation in the presence of a perfect cylindrical mirror

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Abstract

Exact solutions are given for the Maxwell equations driven by a gyrating ultrarelativistic electron in the presence of a reflecting cylindrical boundary. The axis of the electron's circular trajectory and the axis of the surrounding cylinder are supposed to coincide. It is found that, at certain values of the ratio of the cylinder's and the trajectory's radii, the field amplitudes diverge as functions of time. The physical reason for this is the constructive interference between the radiation actually emitted and the radiation emitted earlier and (by reflection) getting back to the actual position of the electron.

1 INTRODUCTION

The cyclotron and synchrotron radiation in free space has long been thoroughly studied both theoretically and experimentally [1,2,3,4]. Recently in the context of cyclotron masers [5,6] and cavity electrodynamics, cavity effects have also been studied [7,8]. In the cyclotron masers the fundamental or low harmonics are generated, on the other hand, in the synchrotron radiation the very high harmonics are dominant. The question naturally emerges whether could one somehow feed back these high harmonics in order to have stimulated emission and manage perhaps lasing also. In the present paper we give a short study of one possible feedback mechanism, where the trajectories of the gyrating ultrarelativistic electrons are completely surrounded by a coaxial cylindrical mirror. We will show that, if the ratio of the cylinder's radius and the radius of the electron's trajectory satisfies certain geometrical resonance conditions, then pracically for all very high harmonics resonance can be achieved. We do not know yet whether our scheeme can be of relevance for a synchrotron laser, but we think that the problem discussed here is interesting in itself, too, so that it is worth presenting it.

2 EXCITATION OF CYLINDRICAL WAVE–GUIDE MODES BY AN ULTRARELATIVISTIC ELECTRON

In the present section we give an exact solution of Maxwell's equation driven by an ultrarelativistic electron gyrating inside a perfectly reflecting cylindrical wave guide.

The transverse position of the electron moving in the homogeneous magnetic field $\underline{e}_z B_0$ has the Cartesian components

$$x(t) = r_0 \cos(\omega_0 t + \varphi_0), \tag{1a}$$

$$y(t) = r_0 \sin(\omega_0 t + \varphi_0), \tag{1b}$$

where $r_0 = v/\omega_0$, $\omega_0 = \omega_c/\gamma$ with $\omega_c = |e|B_0/mc$ being the cyclotron frequency. v denotes the electron's velocity and $\gamma \equiv (1-\beta^2)^{-1/2}$, $\beta \equiv v/c$. We assume that the longitudinal component of the electron's velocity is zero, i.e. $v_z = 0$, and that the gyration takes place in the $z = 0$ plane.

The densities associated to the trajectory of one single electron with initial phase φ_0, given by eqs. (1a,b), can be conveniently expressed in cylindrical coordinates

$$\varrho = er^{-1}\delta(r - r_0)\delta(\varphi - (\omega_0 t + \varphi_0))\delta(z)u(t), \tag{2a}$$

$$\underline{j} = ev\underline{e}_\varphi r^{-1}\delta(r - r_0)\delta(\varphi - (\omega_0 t + \varphi_0))\delta(z)u(t), \tag{2b}$$

In eqs (2a,b) we have introduced the unit step function $u(t)$, being responsible for the switching of the interaction.

In order to obtain a physically meaningful solution of Maxwell's equations driven by the densities (2a,b) incide the cylinder, we have to take into account the boundary conditions $[\underline{n} \times \underline{E}]_C = 0$ and $[\underline{n} \cdot \underline{B}]_C = 0$. That is, the tangential component of the electric field, and the normal component of the magnetic induction vanish at the surface of the cylinder (at any vertical position z). Hence $\underline{E}$ and $\underline{B}$ are expanded into a superposition of the so-called cross-sectional vector-eigenfunctions.

$$\underline{E} = \sum_{m\ p} a_{mp}\underline{\nabla}_\perp \Phi_{mp} + \sum_{n\ s} b_{ns}\underline{e}_z \times \underline{\nabla}_\perp \Psi_{ns} + $$
$$+\underline{e}_z \sum_{m\ p} c_{mp}\Phi_{mp}. \tag{3a}$$

Similarly,

$$\underline{B} = \sum_{m\ p} \alpha_{mp}\underline{e}_z \times \underline{\nabla}_\perp \Phi_{mp} + \sum_{n\ s} \beta_{ns}\underline{\nabla}_\perp \Psi_{mp} + $$
$$+\underline{e}_z \sum_{n\ s} \gamma_{ns}\Psi_{ns}. \tag{3b}$$

In eqs (3a,b) Φ_{mp} and Ψ_{ns} are Dirichlet and Neumann eigenfunctions satisfying the scalar Helmholtz equation $(\nabla_\perp + k^2)f = 0$ with eigenvalues k_{mp} and k_{ns}, respectively. The unknown coefficients a_{mp}, b_{ns}, c_{mp} and α_{mp}, β_{ns}, γ_{ns} are to be determined as functions of time (t) and vertical position (z). According to the boundary condition $[\Phi_{mp}]_C = 0$, Φ_{mp} must have the form

$$\Phi_{mp} = J_m\left(x_{mp}\frac{r}{a}\right)\begin{Bmatrix}\sin(m\varphi)\\ \cos(m\varphi)\end{Bmatrix}, J_m(x_{mp}) = 0, \qquad (4a)$$

where x_{mp} is the p–th root of the Bessel function J_m. On the other hand, since $[\partial\Psi_{ns}/\partial r]_C = 0$, we have

$$\Psi_{ns} = J_n\left(y_{ns}\frac{r}{a}\right)\begin{Bmatrix}\sin(m\varphi)\\ \cos(m\varphi)\end{Bmatrix}, J_n'(y_{ns}) = 0 \qquad (4b),$$

where y_{ns} is the s–th root of the derivative of J_n. The eigenvalues of the corresponding wave numbers are $k_{mp} = \frac{x_{mp}}{a}$ and $k_{ns} = \frac{y_{ns}}{a}$, respectively, where a is the radius of the cylinder.

By taking into account the ortogonality property of the cross–sectional eigenfunctions, we can derive from the inhomogeneous Maxwell's equations two sets of coupled first order differential equations for the expansion coefficients. The set $\{a_{mp}, \alpha_{mp}, c_{mp}\}$ is responsible for the dynamics of the TM and longitudinal components of the electromagnetic field. On the other hand, the dynamics of the TE components are governed by the set $\{b_{ns}, \beta_{ns}, \gamma_{ns}\}$. It can be shown that $\alpha_{mp}(z = 0, t) = 0$ and $c_{mp}(z = 0, t) = 0$, moreover, in the case we shall discuss below a_{mp} vanishes to a good approximation at any vertical position. This meams that in the plane of the electron's gyration only the TE modes are excited, that is why henceforth we shall be dealing only with the TE modes.

The coupled system of equations for b_{ns},β_{ns} and γ_{ns} reads

$$\frac{\partial\beta_{ns}}{\partial z} - \frac{1}{c}\frac{\partial b_{ns}}{\partial t} - \gamma_{ns} = \frac{4\pi}{c}\frac{1}{N_{ns}^2}\int ds^2 \underline{j}\cdot(\underline{e}_z \times \underline{\nabla}_\perp \Psi_{ns}), \qquad (5a)$$

$$\frac{\partial b_{ns}}{\partial z} - \frac{1}{c}\frac{\partial \beta_{ns}}{\partial t} = 0, \qquad (5b)$$

$$b_{ns} - \frac{1}{ck_{ns}^2}\frac{\partial\gamma_{ns}}{\partial t} = 0, \qquad (5c)$$

where $N_{ns}^2 = (\pi/\epsilon_n)J_n^2(y_{ns})(y_{ns}^2 - n^2)$, and $\epsilon_0 = 1, \epsilon_n = 2$ for $n = 2, 3, \ldots$ The integration on the rhs of eq.(5a) is to be evaluated over the cross–section of the cylinder. Having eliminated the functions β_{ns} and γ_{ns} we can derive an inhomogeneous one–dimensional Klei–Gordon equation for $b_{ns}(z, t)$

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - k_{ns}^2\right) b_{ns} = B_{ns}\delta(z) f_n'(t)/c, \qquad (6)$$

where

$$B_{ns} \equiv 4e\beta\frac{(y_{ns}/a)J_n'(y_{ns}r_0/a)\epsilon_n}{J_n^2(y_{ns})(y_{ns}^2 - n^2)}, \qquad (6a)$$

and

$$f_n(t) \equiv \begin{Bmatrix}\sin[n(\omega_0 t + \varphi_0]\\ \cos[n(\omega_0 t + \varphi_0]\end{Bmatrix} u(t). \qquad (6b)$$

The Green's function of eq.(6) can be derived by using standard methods

$$g(t, z) = -\frac{c}{2}J_0(ck_{ns}\sqrt{t^2 - z^2/c^2})u(t - \frac{|z|}{c}), \qquad (7)$$

where $t \equiv t_1 - t_2$ and $z \equiv z_1 - z_2$. With the help of the Green's function (7) the solution of eq.(6) can be determined by simple integrations.

In general the explicit form of b_{ns} is a complicated expression, however, for large values of t a relatively simple expression can be derived (for short, we present only the upper component of b_{ns}):

$$b_{ns}(z, t) = -\frac{1}{2}B_{ns}\{J_0(\omega_{ns}\sqrt{t^2 - z^2/c^2})(\sin\varphi_0)u(t - \frac{|z|}{c} +$$

$$+\frac{\nu_{ns}u(\nu_{ns} - 1)}{\sqrt{\nu_{ns}^2 - 1}}\sin n\left[\omega_0(t - \frac{1}{c}\sqrt{\nu_{ns}^2 - 1}|z|) + \varphi_0\right] +$$

$$+\frac{\nu_{ns}u(1 - \nu_{ns})}{\sqrt{1 - \nu_{ns}^2}}(\cos n(\omega_0 t + \varphi_0))\exp\left[\frac{-n\omega_0}{c}\sqrt{1 - \nu_{ns}^2}|z|\right]\}, \qquad (8)$$

where $\nu_{ns} \equiv n\omega_0/\omega_{ns}$ with $\omega_{ns} = ck_{ns}$ being the TE eigenfrequencies. The first term on the rhs of eq.(8) represents a transient which vanishes as $1/\sqrt{t}$. the second and the third terms correspond to above–cutoff and below–cutoff waves, respectively. At exact resonance ($\nu_{ns} = 1$) eq.(8) loses its validity, and b_{ns} has a qualitatively different form (for short, we present here only the upper component of b_{ns} taken at $z = 0$):

$$b_{ns}(z = 0, t) = -\frac{1}{2}B_{ns}\{J_0(n\omega_0 t)(\sin n\varphi_0) +$$

$$+n\omega_0 t\left[J_0(n\omega_0 t)\cos n\varphi_0 - J_1(n\omega_0 t)\sin n\varphi_0\right]\}. \qquad (9)$$

It can be easily shown that for large t b_{ns} diverges as $\sqrt{t}\times$ (oscillatory function). Of course, since some sort of damping is always present in physical systems, such a divergence is not realistic. We have performed a similar analysis to the one presented above by introducing in addition a damping term $(-k_{ns}/cQ_{ns})(\partial b_{ns}/\partial t)$ on the lhs of eq.(6). In this case the structure of b_{ns} similar to that of eq.(8), but in this case tha oscillatory parts contain resonance denominators of the form $[(\nu_{ns}^2 - 1)^2 + \nu_{ns}^2/Q_{ns}^2]^{1/4}$. Hence, close to resonance the amplitudes are increased by a factor of $\sqrt{Q_{ns}}$. For the high harmonics we are interested in, Q_{ns} can be well approximated by a/δ_{ns}, where $\delta_{ns} = (2/\mu\sigma\omega_{ns})^{1/2}$ is the usual skin depth. For example, for silver $\delta_\omega \sim 6\times10^{-5}$ cm for $\omega/2\pi \sim 10^{10}$ Hz. In the optical region δ can be two orders of magnitudes smaller, thus Q can be very large if a is of order of meters, say.

3 RESONANCE CONDITIONS

In the present section we study the question of under what conditions simultaneous resonance can be reached for most of the higher harmonics of the synchrotron radiation in the cylindrical mirror.

The geometrical arrangement we are interested in is shown on Fig.1. On *geometrical resonance condition* we mean that the radii of the electron's trajectory and of the cylinder are adjusted such a way, that a signal emanating tangentially at point A, after reflection, gets back to the electron's trajectory at point B exactly at that time when the electron (possibly after N complete revolution) gets to that same point B. It is clear that, if once this condition is satisfied for the pair of points A,B, then it will be satisfied for the pair A',B' which can be obtained by rotating the pair A,B by an arbitrary angle. This way the electron, after a while, will continuously move in its own retarded radiation field which has been emitted earlier at different points on the trajectory. We think, that this arrangement would secure an effective feedback for obtaining stimulated emission.

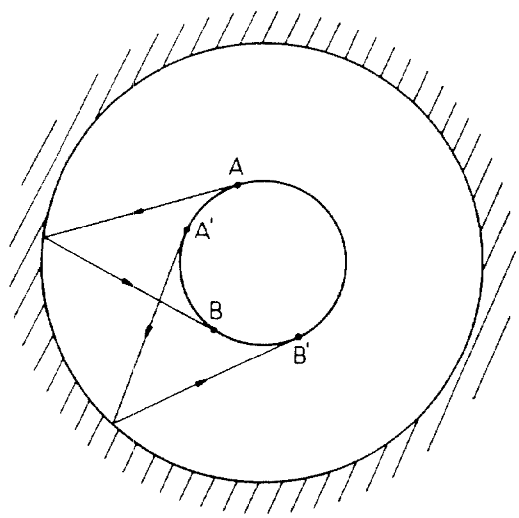

Fig.1. A ray of radiation emanating from the electron at point A gets reflected on the cylindrical mirror, and arrives at point B exactly at the instant when the electron arrives there. For further explanation see section 3.

By simple kinematic considerations, it can be shown that, if the geometrical resonance condition holds, then the ratio a/r_0 satisfies the following transcendental equation

$$\beta\sqrt{x^2-1} - \arccos(1/x) = N\pi, \qquad x \equiv a/r_0 \qquad (10)$$

where N is the number of complete revolutions of the electron before the first encounter with its own radiated field after one reflection. For $N = 1$, we obtain approximately $a/r_0 \simeq 3\pi/2$, for $N = 0$, $a/r_0 \simeq 1 + 3/4\gamma^2$. Henceforth we shall study the case $N = 1$.

The *wave resonance condition* $1 = \nu_{ns} \equiv n\omega_0/\omega_{ns} = (n/y_{ns})(\beta a/r_0)$ of the previous section can be considered by using the asymptotic form of the roots y_{ns} of the derivative of the Bessel function J_n. Here we restrict ourselves to the case when not only n but also s are large. (It can be shown that if s considerably differs from n then the resonance cannot be reached.) For large n we have

$$\frac{a}{r_0} \simeq \beta\frac{a}{r_0} = \frac{y_{ns}}{n} = z(\zeta) + O(\frac{1}{n^2}), \qquad (11a)$$

where $z(\zeta)$ is the inverse function defined by the relation

$$\sqrt{z^2-1} - \arccos(1/z) = \frac{2}{3}(-\zeta)^{3/2}. \qquad (11b)$$

$\zeta = n^{-3/2}a'_s$, with a'_s are the s-th negative zeros of the derivative of the Airy function. Since n is very large, $z = x$, and because β is practically unity, the left-hand sides of eqs.(10) and (11b) coincide. Now, for large s $-\zeta = (3\pi s/2n)^{2/3}$ holds to a good approximation. Having taken this relation into account, we can easily check that, if the geometrical resonance condition is satisfied then the wave resonance condition (11a) becomes an identity for $s = n$. As a consequence, the wave resonance condition is independent of the n values, if these are large enough. This means that (if Q_{nn} is a smooth function of n) there exists a broad band in the highfrequency part of the spectrum which is almost uniformly "lifted up". This accumulation process can be interpreted as a result of the constructive interference between the radiation actually emiited and the radiation emitted earlier and (by reflection) getting back to the actual position of the electron.

4 SUMMARY

In the present paper we have discussed some of the characteritics of the synchrotron radiation emitted by an ultrarelativistic electron in the interior of a cylindrical mirror. We have shown that near the plane of the electron's gyration only the TE modes are excited, and we have briefly dicussed the role of the damping close to resonance. In section 3 we have introduced the *geometrical resonance condition*, and we have shown that if this condition is satisfied, then the *wave resonance condition* does not depend on the excitation index of the very high harmonics. At such a resonance there is an accumulation process taking place due to which the intensity of the emitted radiation can be increased by many orders of magnitude (depending on the value of Q). We have also given a simple physical interpretation for this enhancement.

5 REFERENCES

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