A NON CONFORMING FINITE ELEMENT METHOD FOR COMPUTING EIGENMODES OF RESONANT CAVITIES

F. Touze, G. Le Meur
Laboratoire de l'Accélérateur Linéaire
IN2P3-CNRS et Université de Paris-Sud
F-91405 ORSAY Cedex, France

Abstract

We present here a non conforming finite element in $\mathbb{R}^3$. This finite element, built on tetrahedrons, is particularly suited for computing eigenmodes. The main advantage of this element is that it preserves some structural properties of the space in which the solutions of the Maxwell's equations are to be found. Numerical results are presented for both two-dimensional and three-dimensional cases.

Introduction

Because of its flexibility, the finite element methods are now widely popular in the scientific and technical community and are a basis for a number of efficient finite element codes which provide the numerical solution of a large number of problems arising in practice. The purpose of this paper is to provide a description of a non conforming finite element in $\mathbb{R}^3$ for computing eigenmodes for resonant cavities. This element takes into account the continuity of the tangential components of the electric field along its edges and that permits the fluxes of the electric field to be continuous. This element has been proposed by Nedelec[1] and advocated by Bossavit[2] for solving eigenvalue problems. The main advantage of this finite element is the possibility of approximating Maxwell's equations in verifying Faraday's law. Moreover, this choice avoids the parasitic modes (modes which don't satisfy the zero divergence condition). These modes appear in finite-difference or in classical conforming Lagrange's finite element methods for instance.

A linear matrix eigenvalue problem is generated by applying a Galerkin type method. The numerical solution of this problem is used to provide an initial RF-eigenmode for the particle code PRIAM[3]. This application is at the origin of this study.

A mixed variational formulation in three dimensions

Consider for convenience, an half cavity Ω completely surrounded by conducting walls Γ_e. Let Γ_h be the plane of symmetry such that $\partial\Omega = \Gamma_e \cup \Gamma_h$.

The eigenmodes of this structure satisfy the time-harmonic Maxwell's equations :

$$\begin{cases} \mathbf{curl\ H} = i\omega \mathbf{D} & (1) \\ \mathbf{curl\ E} = -i\omega \mathbf{B} & (2) \end{cases}$$

and they are represented by the following expressions :

$$\mathbf{E}\ (x, y, z)\ e^{i\omega t}$$

$$\mathbf{H}\ (x, y, z)\ e^{i\omega t}\ \ldots$$

where $\mathbf{E}$, $\mathbf{H}$, $\mathbf{B}$ and $\mathbf{D}$ are three-dimensional fields and ω the pulsation.

Boundary conditions at the conducting surfaces Γ_e require the electric field $\mathbf{E}$ to be normal : we write

$$\mathbf{E} \wedge \mathbf{n} = 0 \text{ on conducting walls } \Gamma_e \qquad (3)$$

on a plane of symmetry Γ_h, the surface currents are symmetric : we have

$$\mathbf{H} \wedge \mathbf{n} = 0 \text{ on } \Gamma_h \qquad (4)$$

We briefly discuss the main properties and the relations between the linear differential operators **grad**, **curl** and div.

Let $L^2(\Omega)$ be the space of square integrable functions. Let us now define the following subspaces :

$$\text{dom}\ (\mathbf{grad}_h) = \left\{\varphi \in L^2(\Omega),\ \mathbf{grad}\ \varphi \in \left(L^2(\Omega)\right)^3,\ \varphi = 0 \text{ on } \Gamma_h\right\}$$

$$\text{dom}\ (\mathbf{curl}_h) = \left\{\mathbf{p} \in \left(L^2(\Omega)\right)^3,\ \int_\Omega |\mathbf{curl\ p}|^2 < \infty,\ \mathbf{p} \wedge \mathbf{n} = 0 \text{ on } \Gamma_h\right\}$$

$$\text{dom}\ (\text{div}_h) = \left\{\mathbf{p} \in \left(L^2(\Omega)\right)^3,\ \int_\Omega |\text{div}\ \mathbf{p}|^2 < \infty,\ \mathbf{p} \cdot \mathbf{n} = 0 \text{ on } \Gamma_h\right\}$$

We note that the following diagram :

$$L^2(\Omega) \xrightarrow[\mathbf{grad}_h]{} \left(L^2(\Omega)\right)^3 \xrightarrow[\mathbf{curl}_h]{} \left(L^2(\Omega)\right)^3 \xrightarrow[\text{div}_h]{} L^2(\Omega)$$

satisfies

$$\begin{cases} \text{Im}\ (\mathbf{grad}_h) \subset \text{dom}\ (\mathbf{curl}_h) & (5) \\ \text{Im}\ (\mathbf{curl}_h) \subset \text{dom}\ (\text{div}_h) & (6) \end{cases}$$

We guess that particular structure relative to Γ_h, can be carried on Γ_e :

$$L^2(\Omega) \xrightarrow[\mathbf{grad}_e]{} \left(L^2(\Omega)\right)^3 \xrightarrow[\mathbf{curl}_e]{} \left(L^2(\Omega)\right)^3 \xrightarrow[\text{div}_e]{} L^2(\Omega)$$

It is the obvious thing to bring closer these two structures as it is done in figure 1 :

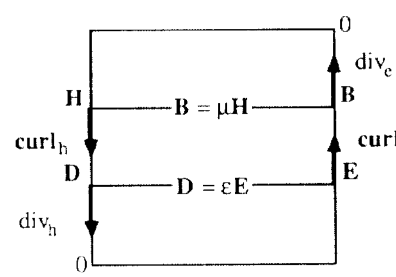

Fig. 1 : Tonti's diagram for the time-harmonic Maxwell's equations

Note that the Maxwell's equations are represented vertically and the constitutive relations appear horizontally.

Let us suppose Ω paved by tetrahedrons, and consider a finite element method of approximation. We have to construct finite dimensional subspaces $\mathcal{W}_h^0$ of dom $(\mathbf{grad}_h)$, $\mathcal{W}_h^1$ of dom $(\mathbf{curl}_h)$ and $\mathcal{W}_h^2$ of dom (div_h) (similarly for the $\mathcal{W}_e$ subspaces).

Let $\mathcal{W}_h^0$ be spaned by the functions w_n which are continuous, piecewise linear, equal to 1 at node n, to 0 at other nodes.

Let $\mathcal{W}_h^1$ (respectively $\mathcal{W}_h^2$) be spaned by the vector fields $\mathcal{W}_e$ (respectively $\mathcal{W}_f$) such that the circulation of $\mathcal{W}_e$ along edge e is 1 (and 0 along other edges), while the flux of $\mathcal{W}_f$ across face f is 1 (and 0 across other faces) : these elements are refereed in the literature as mixed elements (for more details see [1] and [2]).

In general, most of the properties of continuous problem don't carry over the approximate one. Nevertheless in a mixed finite element approximation the relations (5) and (6) mentioned above hold by construction. Hence it is important to notice that $\mathcal{W}_h^0 \subset$ dom $(\mathbf{grad}_h)$...

We have the same diagram as figure 1 :

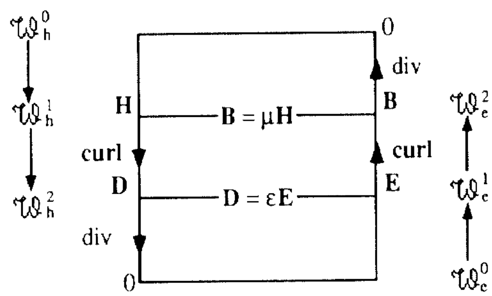

Fig. 2 : discrete structure

The approximate eigenmodes satisfy the Maxwell's equations (1) and (2). The error of approximation comes from the constitutive relations $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ which cannot be guaranteed because the discrete spaces $\mathcal{W}_\bullet^1$ and $\mathcal{W}_\bullet^2$ are distinct[2].

For our purpose we keep $\mathbf{E}$ in the space $\mathcal{W}_e^1$, therefore $\mathbf{B} = -\dfrac{\mathbf{curl\ E}}{i\omega}$ belongs to $\mathcal{W}_e^2$.

As $\mathbf{H} = \dfrac{1}{\mu}\mathbf{B}$, we can express the relation (2) in the following weak form :

$$\int_\Omega -\frac{1}{i\omega\mu}\mathbf{curl}\,(\mathbf{curl\ E}).\mathbf{p}\,dx - i\omega\int_\Omega \varepsilon\mathbf{E}.\mathbf{p}\,dx = 0 \text{ for } \mathbf{p} \in \mathcal{W}_e^1$$

Using Green's formula over Ω we obtain :

$$\int_\Omega \mathbf{curl\ E}.\mathbf{curl\ p}\,dx - \omega^2\varepsilon\mu\int_\Omega \mathbf{E}.\mathbf{p}\,dx = 0 \quad \forall\ \mathbf{p} \in \mathcal{W}_e^1$$

Now we consider the following approximate problem :

$$\left.\begin{array}{l} \text{Find a pair } (\mathbf{E}, k) \in \mathcal{W}_e^1 \times \mathbb{R} \text{ solution of} \\ \displaystyle\int_\Omega \mathbf{curl\ E}\,.\,\mathbf{curl\ p}\,dx = k^2\int_\Omega \mathbf{E}.\mathbf{p}\,dx \quad \forall\ \mathbf{p} \in \mathcal{W}_e^1 \end{array}\right| \quad (7)$$

where k is the wavenumber $\mathbf{k} = \omega/c \neq 0$ (c = light velocity)

Mixed finite element

The geometrical domain Ω over which the finite element method is applied is subdivided into tetrahedron. In this section, we present a mixed element, first introduced by Nedelec[1], which is conforming in the space $H(\mathbf{curl},\Omega) = \left\{\mathbf{p} \in (L^2(\Omega))^3, \int_\Omega |\mathbf{curl\ p}|^2 < \infty\right\}$ and exactly preserves the continuity of the tangential components of the electric field .These elements permit a first order interpolation of the field in the interior of the tetrahedrons in which they apply. Inside each tetrahedron T, the field $\mathbf{E}$ is interpolated by the function

$$\mathbf{E}_h = \begin{cases} \alpha_1 + \beta_2 z - \beta_3 y \\ \alpha_2 + \beta_3 x - \beta_1 z \\ \alpha_3 + \beta_1 y - \beta_2 x \end{cases}$$

The degrees of freedom are the values of the tangential component $\phi = \mathbf{E.t}$ at the midpoints of the sides of ∂T, where $\mathbf{t}$ is an unit tangent along the edge ∂T of T.

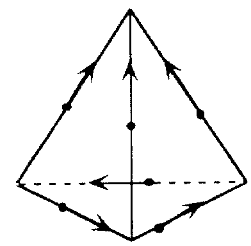

Fig. 3 : degrees of freedom in a tetrahedron

The electric field is expanded as $\mathbf{E}_h = \sum_{i=1}^{6} \phi_i \mathbf{N}_i\ (x, y, z)$ where N_i's are basis polynomials such that $\int_{\partial T_i} \mathbf{N}_i \,.\, \mathbf{t}_j = \delta_{ij}$ and ϕ_i are unknown expansion coefficients.

In the finite element method, this relation is properly combined to yield a global linear eigenvalue problem; the finer the mesh is, the better the approximation of vector field will be.

In each tetrahedron $\mathbf{curl\ E}_h = \sum_{i=1}^{6} \phi_i/V$ (where V is the volume of the element) is constant and div $\mathbf{E}_h = 0$.

Remark

Let us start from the eigenvalue problem in terms of $\mathbf{E}$:

$$\mathbf{curl}\ (\mathbf{curl\ E}) = k^2\mathbf{E}$$

if $k \neq 0$ then we have div $\mathbf{E} = 0$.

Presently, it is possible to construct a lot of equivalent formulations for solving our eigenvalue problem. For example, the classical conforming Lagrange's finite element method seems to be convenient tool.

However, the disadvantage of this formulation is in its limitation by parasitic modes : fields we obtain don't verify the zero divergence condition.

On the other hand, no problem exists with "mixed" element : consider $\mathbf{E} \in \mathcal{W}_e^1$ such that

$$(\mathbf{curl\ E}, \mathbf{curl\ p}) = k^2\,(\mathbf{E}, \mathbf{p}) \quad \forall\ \mathbf{p} \in \mathcal{W}_e^1$$

Let φ be a test function suitably chosen such that $\mathbf{p} = \mathbf{grad}\ \varphi \in \mathcal{W}_e^1$ (this exists because the operator $\mathbf{grad}$ maps the finite-dimensional subspace $\mathcal{W}_e^0$ onto $\mathcal{W}_e^1$). We obtain $(\mathbf{E}, \mathbf{grad}\ \varphi) = -(\text{div}\ \mathbf{E}, \varphi) = 0 \ \ \forall\ \varphi$ i.e. the equation div $\mathbf{E} = 0$ is preserved.

Results for a two-dimensional configuration

In order to investigate the usefulness of our element, we have written a code for the electric field in an axisymmetrical configuration. Only TM eigenmodes are computed in which (E_r, E_z, B_φ) occur.

The method was applied in order to solve the eigenvalue problem $\mathbf{curl}$ curl $\mathbf{E} = k^2\mathbf{E}$ in a homogeneous medium. On the axis, we have $B_\varphi = 0$. As in our case Maxwell's equations yield to curl $\mathbf{E} = -i\omega B_\varphi$ we finally obtain the boundary condition on the axe : curl $\mathbf{E} = 0$.

The eigenvalue equation was solved by using a subspaces iteration method (Fig. 1, 2 and 3).

Results for a three-dimensional configuration

At a first test of our three-dimensional code, we have applied it to the case of a circular cylindrical resonant cavity. The boundary condition $\mathbf{E} \wedge \mathbf{n} = 0$ was used on conducting walls (Fig. 4, 5, 6 and 7).

Conclusion

We have illustrated that the use of the mixed finite element method for computing electromagnetic fields is very well suited. The advantage in the case of Maxwell's equations is that mixed finite element method can be interpreted by means of physical laws.

Our developments are achieved by using the Modulef facilities[4] and all computations have been carried out on a Vax 8600 computer.

References

[1] J.C. Nedelec, "Mixed Finite Elements in $\mathbb{R}^3$", Numerische Mathematik, vol. 35, pp. 315-341, 1980.

[2] A. Bossavit, "Un nouveau point de vue sur les éléments mixtes", Revue Matapli, vol. 20, pp. 23-25, October 1989.

[3] G. Le Meur, F. Touze, "PRIAM : A self consistent finite element code for particle simulation in electromagnetic fields", (these proceedings).

[4] M. Bernadou et al., MODULEF, a modular finite element library. Rocquencourt : INRIA, 1986.

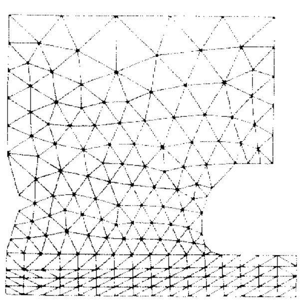

Fig. 1 : Finite element mesh of an axisymmetrical resonant cavity (First cell of LAL RF-Gun)

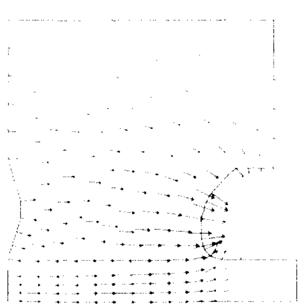

Fig. 2 : eigenmode of the RF cavity at 3 GHz frequency : the electric field vector

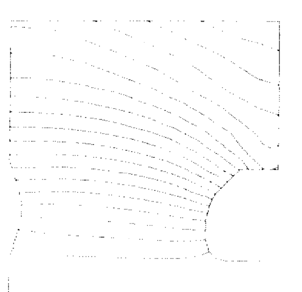

Fig. 3 : iso-B_φ lines multiplied by the radius

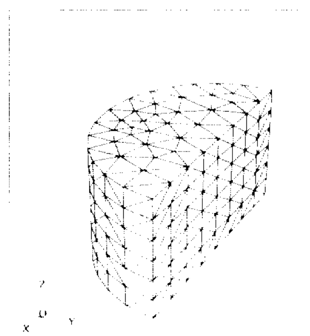

Fig. 4 : Half cylindrical cavity finite element mesh

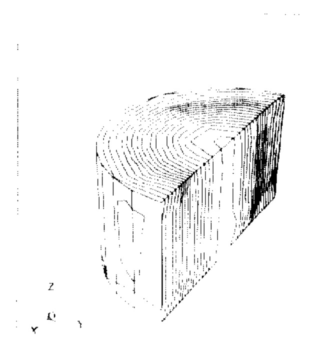

Fig. 5 : iso-E_z lines for TM_{010} mode

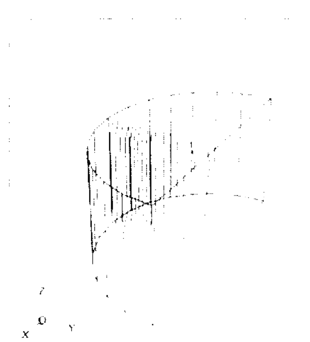

Fig. 6 : Cross section of the iso-E_z lines for TM_{010} mode with respect to the x = 0 plane

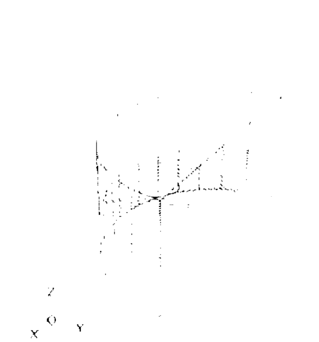

Fig. 7 : Cross section of the iso-E_z lines for TM_{210} mode with respect to the z = 0.5 plane