NON-MATRIX ANALYSIS OF AG PROBLEMS WITH SPACE CHARGE

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We describe an integration method for AG problems with space charge giving simple analytic results for transportable current, beam ripple, and phase advances. Space charge does not need to be modeled by discrete impulses nor the focusing force represented by constant steps. Our approximate results are easy to use and sufficiently accurate for most design purposes.

Introduction

Thirty-two years ago Courant and Snyder published their famous paper on alternating-gradient (AG) focusing of particle beams without space charge [1]. Their powerful matrix approach, however, requires the focusing forces to be represented by step functions and is not easy to apply to space charge dominated problems.

A different way to treat regularly fluctuating forces was discussed briefly in [1] and [2]. It was originally introduced by Kapitza [3], developed mathematically by Bogolyubov [4], and applied to plasma physics problems by Morozov and Solov'ev [5]. This method treats multiple time scales by analyzing fast and slow variations separately. For example, it is used to analyze the motion of a particle in a confining magnetic field where fast gyration and slow guiding-center motions occur. Although the method has mostly been ignored in AG applications, we show that it is quite useful in treating continuous AG focusing forces and space charge.

In the following sections we treat three topics: **(I)** We consider the envelope equation for a K-V beam in a general quadrupole lattice, getting explicit results for beam radius, ripple, and phase advances in terms of the quadrupole parameters and the beam perveance and emittance. **(II)** We then specialize to a hard-edge FODO lattice, with results that agree closely with exact results over a wide range of phase advances σ_0 and σ. Our formulas are generally simpler and easier to apply than those obtained from the matrix formulation [6], and are nearly as accurate. In some cases (discussed elsewhere) our results improve on those obtained from the averaging scheme used in Refs. [7] and [8]. **(III)** We examine ESQ-focused acceleration, which is currently of interest in fusion applications [9 – 11]. Additional topics are discussed in [12].

Balanced Beam in Symmetric Quadrupoles

Envelope Equations for K-V Distribution

A non-relativistic beam with a K-V distribution transported by a series of linear symmetric quadrupoles is described by the paraxial envelope equations

$$a'' = -K(z)\,a + \frac{\epsilon^2}{a^3} + \frac{2Q}{a+b} \tag{1}$$

$$b'' = +K(z)\,b + \frac{\epsilon^2}{b^3} + \frac{2Q}{a+b} \tag{2}$$

where a and b are the beam radii, $K(z)$ represents the quadrupole gradient, ϵ is the emittance (we assume that $\epsilon_x = \epsilon_y$), and Q is the (nonrelativisitic) perveance, $Q = (4\pi\varepsilon_0)^{-1}(m/2q)^{1/2} I\, V^{-3/2}$, with I the beam current and qV the beam energy. The form of K(z) is arbitrary except for the assumption of symmetry, implying $\langle K(z)\rangle = 0$, averaged over a quadrupole cell.

Splitting and Expansion of Terms for Balanced Beam

We split a and b into slow and fast parts:

$$a = A(z) + a_1(z), \qquad b = B(z) + b_1(z), \tag{3}$$

where

$$A = \langle a\rangle, \qquad a_1 = a - A, \tag{4}$$

and similarly for B and b_1. In this paper we simplify by assuming that $A(z) = B(z)$: the beam is balanced in the x and y directions, although not necessarily matched. (The unbalanced case $A(z) \neq B(z)$ is treated in [12].) To first order, $a_1(z) = -b_1(z)$, so that $a + b = 2A$ and the coupling between Eqs. (1) and (2) is eliminated:

$$a'' = -K(z)\,a + \frac{\epsilon^2}{a^3} + \frac{Q}{A}. \tag{5}$$

Using Eq. (3) and expanding the ϵ^2/a^3 term, we have

$$A'' + a_1'' = -K(z)A - K(z)a_1 + \frac{\epsilon^2}{A^3}(1 - 3\rho + 6\rho^2 - \cdots) + \frac{Q}{A}, \tag{6}$$

where we define the ripple ratio

$$\rho(z) = \frac{a_1(z)}{A}. \tag{7}$$

The peak ripple ratio is not necessarily very small—for example, the case $a^{max} = 2a^{min}$ implies $\rho^{max} = 1/3$. However, the series in Eq. (6) converges fairly rapidly for $\rho^{max} < 1/3$; we shall see that it usually suffices to keep terms up to ρ^2 in Eq. (6).

Fast Differential Equation and its Solution

The terms with nonzero average in Eq. (6) will be treated in the next subsection; the fluctuating terms that will drop out when averaged are

$$a_1'' = -AK\,h(z) - 3a_1\frac{\epsilon^2}{A^4} + \cdots. \tag{8}$$

We assume that the constant part of the force is of higher order of smallness than the fluctuating force and neglect the ϵ^2 term to simplify our equations; higher order effects are considered in a consistent way in [13]. In Eq. (8) we have written

$$K(z) = K\,h(z), \tag{9}$$

where K is the maximum value of K(z), and h(z) ranges between +1 and −1 in a symmetric system. We start integrating with z = 0 at the midpoint of the region where h(z) is positive so that $a_1'(0) = 0$ and $a_1(0) = a_1^{max}$. In a symmetric quadrupole cell of length 2L, a_1 will pass through 0 at z = L/2, so that

$$a_1^{max} = AK\int_0^{L/2} dz \int_0^{z} h(z')\,dz'. \tag{10}$$

(We assume that A(z) changes slowly and take A outside the integral sign.) The ripple ratio can be written as

$$\frac{a_1}{A} = \rho(z) = K\int_z^{L/2} dz' \int_0^{z'} h(z'')\,dz''. \tag{11}$$

This is the explicit solution for the fast part of Eq. (5). The ratio of maximum to mean envelope radius is

$$\frac{a^{max}}{A} = 1 + \frac{a_1^{max}}{A} = 1 + \rho^{max}. \tag{12}$$

Slow Differential Equation

We now average Eq. (6), where, by definition, $\langle a_1\rangle = 0$ and $\langle K(z)\rangle = 0$:

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$$A'' = \frac{\epsilon_*^2}{A^3} + \frac{Q}{A} - \langle a_1 K(z) \rangle$$
$$= \frac{\epsilon_*^2}{A^3} + \frac{Q}{A} + AK^2 \left\langle h \int_0^z dz' \int_0^{z'} h\,dz'' \right\rangle$$
$$= \frac{\epsilon_*^2}{A^3} + \frac{Q}{A} - AK^2 \left\langle \left[\int_0^z h\,dz' \right]^2 \right\rangle , \qquad (13)$$

where we changed the order of integration and used $\langle h \rangle = 0$. We also introduced

$$\epsilon_*^2 = \epsilon^2 \left(1 + 6\langle \rho^2 \rangle\right) \qquad (14)$$

with ρ given by Eq. (11). We now define

$$K_{eff} = K^2 \left\langle \left[\int_0^z h\,dz' \right]^2 \right\rangle , \qquad (15)$$

so that K_{eff} is a constant, dependent only on the lattice parameters, and write

$$A'' = -K_{eff} A + \frac{\epsilon_*^2}{A^3} + \frac{Q}{A} . \qquad (16)$$

This is similar to Eq. (5), with A replacing a, K_{eff} replacing K(z), and ϵ_* replacing ϵ. We correct a popular misconception by noting that K_{eff} is proportional to the mean square *integral* of the force, not the mean square force.

Equation (16) resembles the result from Reiser's method [7], [8], except that: (1) K_{eff} is calculated from integrals rather than particle advance matrices, so that it is unnecessary to divide the force K(z) into a series of constant steps, and (2) the ϵ term is corrected for ripple according to Eq. (14), greatly improving the accuracy when the phase advance is large. (Accuracy can be further improved by including all second order corrections, but with loss of simplicity [13].)

In cases where the fast envelope ripple is of no particular interest, the quadrupole problem is replaced by a much simpler problem: a round beam in a constant focusing channel. In cases where the ripple *is* of interest, we have the explicit solution, Eq. (11).

Matched Beam Radius and Mismatched Balanced Beam

Setting A'' equal to zero gives the condition for a matched beam,

$$Q = A_0^2 K_{eff} - \frac{\epsilon_*^2}{A_0^2} . \qquad (17)$$

If Q is a given quantity and Q > 0, Eq. (17) is easily solved for A_0^2.

The balanced oscillation mode, when linearized, obeys

$$\delta A'' = -\left(2K_{eff} + 2\frac{\epsilon_*^2}{A_0^4}\right)\delta A; \qquad (18)$$

thus, the wave number for linearized envelope oscillations is

$$k_A^2 = 2K_{eff} + 2\frac{\epsilon_*^2}{A_0^4} . \qquad (19)$$

Reference [12] discusses the unbalanced oscillation mode.

Phase Advance and Transportable Current

The approximate phase advances σ and σ_0, with and without space charge, are derived in [12]:

$$\sigma = 2L\frac{\epsilon_*}{A_0^2}\left(1 + 3\langle \rho^2 \rangle\right), \qquad (20)$$

$$\sigma_0 = 2LK_{eff}^{1/2}\left(1 + 3\langle \rho^2 \rangle\right). \qquad (21)$$

For the transportable current at a chosen σ_0 limit and given beam radius, we use Eqs. (17) and (21) to get

$$Q_{transp} = \sigma_0^2 \left(\frac{A_0}{2L}\right)^2 \left(1 + 3\langle \rho^2 \rangle\right)^{-2} - \frac{\epsilon_*^2}{A_0^2} . \qquad (22)$$

Often, the maximum beam radius is specified; if so, one may use Eq. (12) to replace A_0 by a^{max} in Eq. (22).

Although the basic results, Eqs. (12), (17), (20), (21) and (22) for $a^{max}/\langle a \rangle$, A_0, σ_0, σ and Q_{transp}, are simpler in form than equivalent results obtained by matrix methods, we find that they are reasonably accurate for phase advances up to about 100 degrees.

Example: Hard Edge FODO Lattice

Although the above quantities are easily evaluated for realistic models without discontinuities in K(z) [12], we illustrate our results with a hard edge FODO lattice in order to compare with published results [6] – [8] which use this lattice model. The function h(z) for this lattice is shown in Fig. 1.

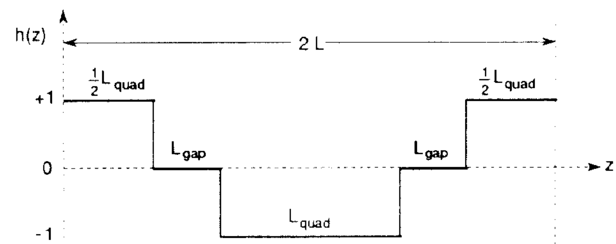

Fig. 1. Normalized electric field gradient h(z) for hard edge FODO model. The cell length is 2L. The occupancy factor η is the ratio L_{quad}/L.

Solution of Fast Equation

For the FODO cell, we have from Eqs. (11) and (12)

$$\frac{a^{max}}{A} = 1 + \int_0^{L/2} dz \int_0^z h(z')dz' = 1 + \frac{1}{8}\eta(2 - \eta)L^2, \qquad (23)$$

where η is the occupancy factor (see Fig. 1). Table 1 shows that this gives good results for cases with reasonably large beam ripple.

Slow Equation, Phase Advance, and Transportable Current

For the FODO model we obtain from Eq. (15)

$$K_{eff} = \frac{1}{12}\eta^2(3 - 2\eta)K^2L^2. \qquad (24)$$

To calculate ϵ_* we use Eq. (14) and find

$$\langle \rho^2 \rangle = \frac{1}{120}\eta^2\left(\frac{5}{2} - \frac{5}{2}\eta^2 + \eta^3\right)K^2L^4 . \qquad (25)$$

Because of the assumed FODO symmetry, it is sufficient to calculate the above averages over the range 0 to L/2.

The phase advance σ_0, from Eq. (21), is

$$\sigma_0^2 = \frac{1}{3}\eta^2(3 - 2\eta)K^2L^4\left(1 + 3\langle \rho^2 \rangle\right), \qquad (26)$$

with $\langle \rho^2 \rangle$ given by Eq. (25). Table 1 shows that Eq. (26) is accurate enough for most design purposes. In [12] this equation is rewritten, after dropping higher terms, as

$$1 - \cos\sigma_0 = \frac{1}{6}\eta^2(3 - 2\eta)K^2L^4 , \qquad (27)$$

the same as the result quoted in [6]; this is good for σ_0 up to 140 degrees. A similar approach could be used, for large σ_0, with lattice models that are more realistic than FODO.

TABLE 1
Comparison of our results with exact solutions of Eqs. (1)–(2) for FODO (Fig. 1) with $\eta = 0.65$.

$\sqrt{K}L$ given	Q given	ϵ given	a_{beam} exact	a_{max} exact	a_{max} approx	σ_0 exact	σ_0 Eq.(26)	σ exact	σ Eq.(20)
1.3997	.00000	.03100	0.815	1.015	1.009	57.2	58.8	57.2	58.8
1.3997	.00073	.03100	0.914	1.132	1.131	57.2	58.8	45.3	46.8
1.3997	.00073	.00775	0.617	0.761	0.761	57.2	58.8	24.7	25.9
1.3997	.00073	.00039	0.557	0.684	0.683	57.2	58.8	1.5	1.6
1.6162	.00000	.01096	0.430	0.582	0.563	79.1	82.3	79.1	82.3
1.6162	.00205	.01096	0.738	0.973	0.969	79.1	82.3	25.6	27.8
1.6162	.00581	.01550	1.190	1.567	1.561	79.1	82.3	13.9	15.1

Note: The approximate a_{max} is from Eq. (23), using A_0 from Eq. (17). The exact solutions for a_{beam}, a_{max}, and σ were obtained numerically.

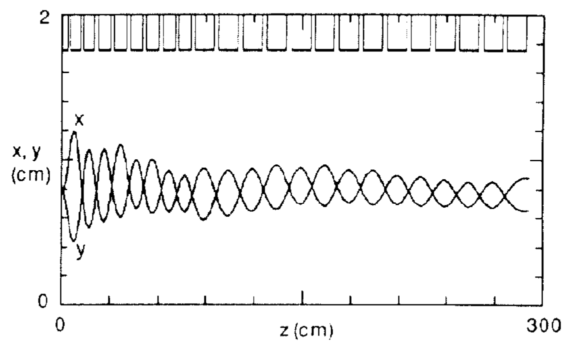

Fig. 2. Beam envelopes and quad locations for a low-gradient 1-MV ESQ accelerator, including matching section; 200 mA of H^- (or H^+) per channel is accelerated from 100 keV to 1 MeV. Transverse temperature was 4 eV. ESQ focus voltages are found using Eq. (33). (Taken from Ref. [11].)

The transportable current may be found from Eq. (17) or (22) using the above expressions for σ_0 and ϵ_*.

The depressed tune σ is given by Eq. (20) using Eq. (25). Table 1 shows that the results agree with the exact values within one or two degrees. If still greater accuracy were needed, an equation analogous to Eq. (27) might be used for σ. Or, one could include higher order corrections [12], but the expressions would not be as easy to use.

ESQ–Focused Acceleration

For a nonrelativistic beam with electrostatic focusing, K is

$$K = \frac{U}{V} a_Q^{-2}, \tag{28}$$

where U is the quadrupole voltage, a_Q is the quadrupole radius, and qV is the beam energy in electron volts. Therefore

$$K_{eff} = g(\eta) L^2 \frac{U^2}{V^2} a_Q^{-4}, \tag{29}$$

where, for the hard edge model,

$$g(\eta) = \frac{1}{12}\eta^2(3-2\eta). \tag{30}$$

If the beam is gradually accelerated, the matching condition, Eq. (17), still holds but all the terms are functions of the beam energy. Nonrelativistically, we replace the variable emittance ϵ by the constant normalized emittance ϵ_N:

$$\epsilon^2 = \frac{1}{2}\epsilon_N^2 \frac{M}{q}\frac{W}{V}; \tag{31}$$

M is the mass in atomic units, qV is the beam energy in eV, and W is the proton rest energy in eV. Using also Eq. (29), the matched beam condition [Eq. (17)] can be written in the form

$$g(\eta)\frac{L^2}{a_Q^4}U^2 = C_P \frac{I}{A_0^2} V^{1/2} + \frac{\epsilon_{*N}^2}{A_0^4}\frac{M}{q} W V. \tag{32}$$

where C_P is the perveance coefficient $(4\pi\varepsilon_0)^{-1}(m/2q)^{1/2}$, and where ϵ_{*N} is assumed to scale with energy according to Eq. (31). A slight correction would be needed for a large ripple fraction that varies much with energy. In the low current limit, $U \propto V^{1/2}$ approximately. For a bright beam with small emittance (strongly depressed tune), we may neglect the last term and find

$$U \propto V^{1/4}. \tag{33}$$

This simple quarter-power rule (or the complete result, Eq. (32)) has been useful in designing ESQ-focused dc accelerators. Figure 2, from [11], shows a typical design based on these results.

Note Added

T.P. Wangler has pointed out an earlier reference [14] where Hill's equation was treated by a simple two time scale method [2] and space charge was considered. However, space charge was unrealistically modeled as a round cylinder with constant radius equal to the quadrupole radius. Also, there was no attempt to include higher order terms such as our ripple corrections [Eqs. (14), (25), (26), etc.].

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