# NONLINEAR COHERENT BEAM-BEAM OSCILLATIONS IN THE RIGID BUNCH MODEL

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Abstract. Within the framework of the rigid bunch model coherent oscillations of strong-strong colliding bunches are described by equations, which are specific for the weak-strong beam case. In this report some predictions of the model for nonlinear coherent beam-beam oscillations as well as for associated limitations of the ring luminosity are discussed.

1. The importance of collective phenomena for the interaction of colliding beams, especially in strong-strong case, is well known [1, 2]. The treatment of this problem in the linear approximation on amplitudes of coherent oscillations can be found anywhere [3]. Though, both experimental results and multiparticle tracking [4—6] definitely indicate the interest to the study of nonlinear beam-beam coherent effects. Since the general solution of this many-fold problem is still too hard for analytical methods, in this report we shall briefly discuss the description of the beam-beam coherent oscillations within the framework of the rigid bunch model. More detailed calculations can be found in [7]. Recently [8] this model was used to calculate nonlinear correction to the beam-beam coherent tune shift for the beam parameters diagnostic, based on the measurement of the beam response spectra. The excitation of nonlinear coherent beam-beam resonances generally can disturb results of such measurements [7, 9].

2. In the rigid bunch model coherent oscillations of a bunch are described by the displacement of its distribution function as a whole

$$f(\vec{r},\vec{p},t)=f(\vec{r}-\langle\vec{r}(t)\rangle,\quad \vec{p}-\langle\vec{p}(t)\rangle) \qquad (1)$$

by means of the following equations

$$\frac{d}{dt}\langle\vec{r}\rangle=\langle\vec{v}\rangle,\quad \frac{d}{dt}\langle\vec{p}\rangle=\langle\vec{F}\rangle, \qquad (2)$$

where $\vec{F}$ is the total force acting on a particle. Since $\langle\vec{F}\rangle$ generally depends on the higher order momenta of $f(r,p,t)$ eqs (2) are not closed. For instance, we have

$$\frac{d}{dt}\sigma_z^2=2(\langle zv_z\rangle-\langle z\rangle\langle v_z\rangle),$$

$$\frac{d}{dt}\sigma_{p_z}^2=2(\langle p_zF_z\rangle-\langle p_z\rangle\langle F_z\rangle), \qquad (3)$$

$$\sigma_z^2=\langle z^2\rangle-\langle z\rangle^2,\quad \sigma_{p_z}^2=\langle p_z^2\rangle-\langle p_z\rangle^2.$$

Nevertheless, if the distribution function has the form of eq. (1), both $\sigma_z^2$ and $\sigma_{p_z}^2$ (as well as the higher order spreads) will be conserved, whereas higher order momenta of $f$ can be calculated via $\langle z\rangle$, $\langle p_z\rangle$ and those constant spreads. Due to the dilution of the distribution (1) in phases of oscillations and subsequent enlargement of the effective beam emittance the model gives the adequate description of the bunch coherent oscillations only during time intervals $\Delta t\ll\Delta\omega^{-1}$, which are limited by the frequency spread in the beam $\Delta\omega$ or, probably, the rise time of the instability.

For the sake of simplicity we shall consider coherent oscillations of two relativistic ($\gamma=E/Mc^2\gg1$) bunches, which have densities $N_1\rho^{(1)}$ and $N_2\rho^{(2)}$ move in the same ring and collide at one interaction point (IP). In the smooth focussing approximation and without the beam cooling eqs (2) read

$$\langle\ddot{x}^{(1)}\rangle+\omega_x^2\langle x^{(1)}\rangle=\langle\delta F_x^{(1,2)}\rangle/\gamma M,$$

$$\langle\ddot{x}^{(2)}\rangle+\omega_x^2\langle x^{(2)}\rangle=\langle\delta F_x^{(2,1)}\rangle/\gamma M, \qquad (4)$$

where $\omega_x=\omega_0\nu_x$ is the frequency of radial betatron oscillations, while the average force $\langle\delta\vec{F}\rangle$ distorting the motion of particles from the counter-moving beam for the distribution (1) is [7, 8]

$$\langle\delta\vec{F}^{(1,2)}\rangle=\frac{N_2e^2\delta_T(t)}{c}\frac{\partial}{\partial\vec{b}}U(\vec{b}),$$

$$U(\vec{b})=\int_{-\infty}^{\infty}\frac{d^2k}{\pi k^2}\exp(i\vec{k}\vec{b})\,\rho^{(1)}(-\vec{k})\,\rho^{(2)}(\vec{k}), \qquad (5)$$

$$\delta_T(t)=\sum_{l=-\infty}^{\infty}\delta\left[t-\frac{2\pi l}{\omega_0}\right],$$

$$\vec{b}=\langle\vec{r}^{(1)}\rangle-\langle\vec{r}^{(2)}\rangle.$$

Equations (4), (5) indicate that for beams moving in the same ring the beam-beam interaction affects only the relative motion of colliding bunches ($\pi$-mode). Such oscillations are described by the impact parameter $\vec{b}$, which satisfies the following equation

$$\ddot{b}_x+\omega_x^2b_x=\frac{2Ne^2}{\gamma Mc}\delta_T(t)\frac{\partial}{\partial b_x}U(\vec{b}), \qquad (6)$$

$$U(\vec{b})=(N_2U_{(1,2)}+N_1U_{(2,1)})/2N,\quad N=(N_1+N_2)/2.$$

It is remarkable that eq. (6) describing the interaction of two generally strong-strong colliding bunches has exactly the same form as the equation of motion of a particle in the weak-strong beam approximation but with the special distribution of particles in the strong beam

$$\rho_{eff}(k)=(N_2\rho^{(1)}(-\vec{k})\,\rho^{(2)}(\vec{k})+(1\to2))/2N. \qquad (7)$$

Hence, all the results of the weak-strong theory can be applied for the description of coherent oscillations of strong-strong beams within the framework of the rigid bunch model. In particular, if $\rho^{(1,2)}$ are Gaussian distributions, $\rho_{eff}$ is also Gaussian and one can get predictions for the strong-strong case using simple scale transformation of results of weak-strong calculations. Though, as the model assumes the oscillations statistically well off-equilibrium, it mainly describes their dynamic properties.

3. For the illustration let us discuss some properties of horizontal coherent oscillations of flat colliding beams

$$\rho^{(1,2)}=\frac{N\delta(z)}{(2\pi)^{1/2}\sigma}\exp(-x^2/2\sigma^2). \qquad (8)$$

The unperturbed coherent oscillations are described by formulae

$$b_x=(2J\beta)^{1/2}\cos\psi,$$

$$b_x'=db_x/d\theta=-\nu_x(2J\beta)^{1/2}\sin\psi,$$

$$\psi'=\nu_x,\quad \theta=\omega_0t,\quad \beta=R_0/\nu_x. \qquad (9)$$

which generate the canonical transformation from variables $b_x$, $b'_x$ to action-phase variables $(J, \psi)$. Since $U$ in eq. (6) is a periodic function of $\psi$, the interaction of bunches excites nonlinear resonances, if $\nu_x \approx n/m$, which without the beam cooling are described by the following Hamiltonian

$$h = \begin{cases} \frac{\Delta}{\xi} y + U_0(y) + (-1)^{m/2} 2U_m(y)\cos(m\varphi), & m=2l, \\ \frac{\Delta}{\xi} y + U_0(y), & m=2l+1. \end{cases} \tag{10}$$

Here $\Delta = \nu_x - n/m$, $\varphi = \psi - (n/m)\theta$, $y = J/4\varepsilon$, $\sigma^2 = \varepsilon\beta$,

$$\xi = \frac{Nr_0}{2\pi\gamma\varepsilon} \tag{11}$$

is the beam-beam parameter for horizontal coherent oscillations,

$$U_0(y) = \int_0^y dt(1 - e^{-t})/t,$$

$$U_m(y) = \frac{y^{m/2}\,r(m/2)}{m!} F(m/2, m+1, -y), \tag{12}$$

$F(\alpha, \beta, y)$ is the confluent hypergeometric function. As $h$ is conserved, specific features of the motion can yield the enspection of curves $h^{\pm}(J) = h(J, \cos m\varphi = \pm 1)$ (see Figs 1, 2). It indicates the instability of oscillations with small amplitudes inside the stop bands

$$-2\xi < \nu_x - n/2 < 0, \quad n = 1, 2. \tag{13}$$

Figs 1, 2 show that the nonlinearity of the interaction generally saturates this instability only in the region $J \gg 4\varepsilon$. Though, slightly above the stop band (13) the modulation of amplitudes still remain strong enough (Fig. 3). In the region $h \geqslant 0$ oscillations can be captured into the bucket. If the beam is cooled and the cooling decrement $\lambda$ is less then the frequency of small coherent phase oscillations in the bucket $\lambda \ll \Omega_m = 2|\Delta|\ m^{1/2}$, the oscillations will be dumped towards the bottom of the bucket

$$y \to y_s = \xi/|\Delta| \gg 1.$$

For resonances with $m \geqslant 4$ (see in Fig. 4) this can take place if initial amplitudes are large enough (say, after injection or strong kick). This can limit the luminosity of a collider by the values

$$L = L_0(|\Delta|/\pi\xi_z)^{1/2}$$

for the flat beam $\xi_z \gg \xi_x$, and

$$L = L_0\,|\Delta|/\pi\xi$$

for round beam and two-dimensional oscillations. In both cases the saturation of the luminosity will not be accompanied by the increase of beam sizes.

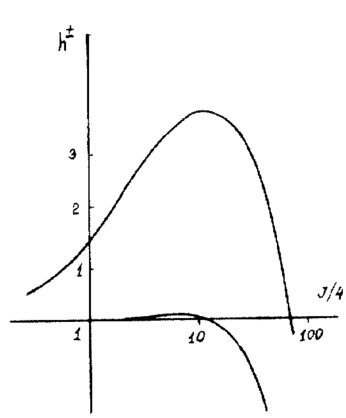


*Fig. 1.* Hamiltonians $h^\pm$ for resonances $\nu \simeq n/2$, $\Delta/\xi = -.1$.

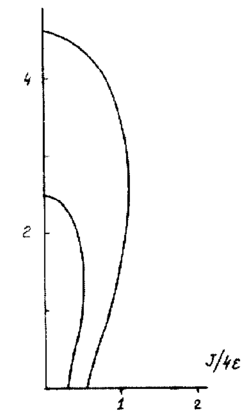


*Fig. 3.* The same as in Fig. 2, but $\Delta/\xi = .1$ and from the top $h = 1,\ 5$.

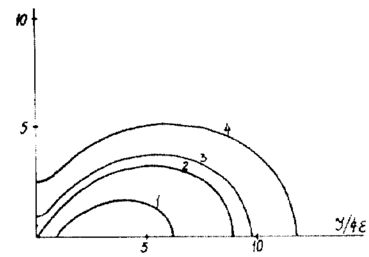


*Fig. 2.* Upper right quarter of phase trajectories of coherent oscillations for resonances $\nu \simeq n/2$, $\Delta/\xi = -.5$.

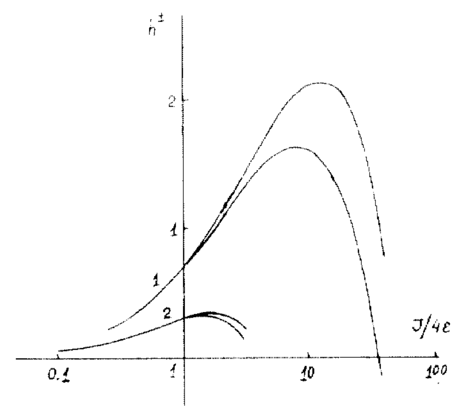


*Fig. 4.* Hamiltonians $h^\pm$ for resonances $\nu \simeq n/6$, 1: $\Delta/\xi = -.1$, 2: $\Delta/\xi = -.5$.

## References


[1] *A. Piwinski.* Proc. of the 8th Intern. Conf. on High Energy Accel. CERN Geneva, 1971, p.357.
[2] *A.Chao A, E. Keil.* CERN-ISR-TH/79-30.
[3] *N. Dikansky, D. Pestrikov.* Physics of Intense Beams in Storage Rings. Novosibirsk, Nauka, 1989.
[4] *T. Ieri et al.* NIM, v.A265, 1988, p.346.
[5] *E. Keil.* NIM, v.188, 1981, p.9.
[6] *S. Myers.* NIM, v.211, 1983, p.265.
[7] *N. Dikansky, D. Pestrikov.* Preprint INP 90-14, Novosibirsk, 1990.
[8] *K. Hirata.* NIM, v.A269, 1989, p.7.