EVOLUTION OF LONGITUDINAL EQUILIBRIUM DISTRIBUTION IN THE ADIABATIC REGIME*

J. Wei, S.Y. Lee, and A.G. Ruggiero

Brookhaven National Laboratory
Upton, New York 11973 USA

Abstract

Evolution of longitudinal equilibrium distribution of a hadron bunch under the beam-environment interaction is investigated based on a self-consistent solution of the Vlasov equation. The effect of this interaction on the distribution can be characterized by a dimensionless quantity in analogy to the one describing the microwave-instability criterion. In the case that the coupling impedance (Z/n) is reactive and frequency independent, the change in the distribution results in a stabilization that keeps the bunch below the instability threshold; microwave instability is thus eliminated. Monte Carlo simulation for the microwave instability agrees with analytic solution of the Vlasov equation, provided that bunch shape distortion due to the coupling is taken into account.

1. Introduction

It is well known that electromagnetic fields (self fields) of wavelengths much shorter than the bunch length of a hadron beam, produced by the interaction of the beam with its environment, may induce microwave instability. On the other hand, self fields of wavelengths comparable to the bunch length mainly distort bunch shape [1-3]. According to the frequency dependence of the beam-environment coupling impedance, the bunch-shape distortion may enhance the Landau damping[4], which then stabilizes the microwave instability.

This paper is devoted to a detailed examination of a simple model pertaining to the evolution of the longitudinal particle distribution in the presence of beam self fields. The discussion is restricted to the adiabatic regime, where the fractional change in the distribution is small during the period of synchrotron oscillation. It is demonstrated that the effect of the self fields on the equilibrium distribution is characterized by a scaling parameter D similar to the one describing the microwave-instability criterion. When the coupling changes adiabatically, the distribution changes in a way to damp coherent instability. This damping mechanism prevents instability if the coupling is reactive and frequency independent.

2. Evolution under adiabatic potential variation

Longitudinal motion of hadron beam in synchrotrons and storage rings can be described by two variables, the phase $\phi_s+\phi$ of the rf field at the moment the particle passes the cavity, and $W = \Delta E/h\omega_s$, with the energy deviation ΔE from the synchronous value. Consider an Hamiltonian system[3] with reactive coupling (Z/n independent of frequency $n\omega_s$),

$$\mathcal{H} = \frac{h^2\omega_s^2\eta}{2E\beta^2}W^2 - \frac{qe\hat{V}\cos\phi_s}{2\pi h}\left\{\frac{\phi^2}{2} - \frac{\epsilon_Z}{2}\left[\lambda(\phi) - \lambda(0)\right]\right\}, \quad (1)$$

where the coupling factor

$$\epsilon_Z = -\frac{2qeh^2\omega_s|Z/n|\,\mathrm{Sgn}(Z)}{\hat{V}\cos\phi_s},$$

Z is the longitudinal coupling impedance at frequency $n\omega_s$, and $\mathrm{Sgn}(Z)$ is equal to $+1$ if the impedance is capacitive and -1 if it is inductive. The particle density can be expressed in terms of the normalized distribution function Ψ in phase space as

$$\lambda(\phi) = N\int \Psi(\phi, W)\,dW.$$

In these expressions, $\hat{V}$, h, and ϕ_s are the peak voltage, harmonic number, and synchronous phase of the rf system, $\eta = 1/\gamma_T^2 - 1/\gamma^2$, γ_T is the transition energy, ω_s, $E = Am_0c^2\gamma$, and βc are the synchronous revolution frequency, energy, and velocity, and q and A are the charge and the atomic number of the particles, respectively.

In order to obtain an explicit expression describing the bunch configuration under the coupling, a process is studied, in which the strength ϵ_Z of the coupling is increased adiabatically; transient effects are thus disregarded. In the absence of the coupling, a parabolic particle distribution is described by

$$\Psi_0(\phi, W) = \begin{cases} \dfrac{3}{4\pi J_0}\sqrt{1 - \dfrac{W^2}{\hat{W}_0^2} - \dfrac{\phi^2}{\hat{\phi}_0^2}}, & \text{when } \dfrac{W^2}{\hat{W}_0^2} + \dfrac{\phi^2}{\hat{\phi}_0^2} \le 1; \\ 0, & \text{otherwise,} \end{cases} \quad (2)$$

where $2\pi J_0 = \pi\hat{W}_0\hat{\phi}_0$ is the bunch area, $\hat{\phi}_0$ and $\hat{W}_0$ are respectively the phase and energy spread,

$$\hat{W}_0^2 = \frac{qe\hat{V}|\cos\phi_s|E\beta^2}{2\pi h^3|\eta|\omega_s^2}\hat{\phi}_0^2,$$

and the subscript 0 labels the absence of the coupling. Each of the N particles in the bunch performs synchrotron oscillation on the elliptic contour

$$\xi_i = \frac{W^2}{\hat{W}_0^2} + \frac{\phi^2}{\hat{\phi}_0^2}, \quad 0 \le \xi_i \le 1, \quad (3)$$

$i = 1,\cdots,N$. When the coupling is adiabatically turned on, these contours are deformed such that the phase-space area

$$\oint_C W\,d\phi = \xi_i\, 2\pi J_0, \quad i = 1,\cdots,N, \quad (4)$$

with the integral being taken over the path of one synchrotron oscillation, is an adiabatic invariant. Define a quantity ζ such that these deformed contours are described by

$$\xi_i = \frac{1}{\zeta^2}\left\{\frac{W^2}{\hat{W}_0^2} + \frac{\phi^2}{\hat{\phi}_0^2} - \frac{\epsilon_Z}{\hat{\phi}_0^2}\left[\lambda(\phi) - \lambda(0)\right]\right\}, \; 0 \le \xi_i \le 1, \quad (5)$$

$i = 1,\cdots,N$. Liouville's theorem implies that the density in phase space, at the location that each particle occupies, remains

* Work performed under the auspices of the U.S. Department of Energy

constant under the deformation. The equilibrium distribution thus becomes

$$\Psi^{(0)}(\phi, W) = \begin{cases} \dfrac{3}{4\pi J_0}\sqrt{1 - \dfrac{1}{\zeta^2}\left(\dfrac{W^2}{\hat{W}_0^2} + \dfrac{\phi^2}{\hat{\phi}_0^2} + \dfrac{\epsilon_Z \tilde{\lambda}\phi^2}{\hat{\phi}_0^2}\right)}, \\ \qquad \text{when } \dfrac{W^2}{\hat{W}_0^2} + \dfrac{\phi^2}{\hat{\phi}_0^2} + \dfrac{\epsilon_Z \tilde{\lambda}\phi^2}{\hat{\phi}_0^2} \le \zeta^2; \\ 0, \qquad \text{otherwise,} \end{cases} \tag{6}$$

where $\tilde{\lambda}\phi^2 = \lambda(0) - \lambda(\phi)$, and the superscript (0) indicates the equilibrium state. Integration of Eq. (6) over W yields the density distribution $\lambda(\phi)$ in phase, as

$$\lambda(0) = \zeta\hat{\lambda}_0, \quad \text{and} \quad \tilde{\lambda} = \frac{\hat{\lambda}_0}{\hat{\phi}_0^2\zeta}\left(1 - \frac{\epsilon_Z \hat{\lambda}_0}{\hat{\phi}_0^2\zeta}\right)^{-1}, \tag{7}$$

with $\hat{\lambda}_0 = 3N/4\hat{\phi}_0$ the peak density. With Eqs. (4) and (5), ζ satisfies

$$\zeta^2\sqrt{1 - \frac{\epsilon_Z \hat{\lambda}_0}{\hat{\phi}_0^2\zeta}} = 1, \quad \text{or} \quad 1 - \epsilon_Z\tilde{\lambda} = \zeta^4. \tag{8}$$

It is shown in figure 1 that with this simple model, ζ depends on the dimensionless quantity, $D_0 \equiv \frac{\epsilon_Z \hat{\lambda}_0}{\hat{\phi}_0^2}$, only. If $D_0 = 0$, then $\zeta = 1$.

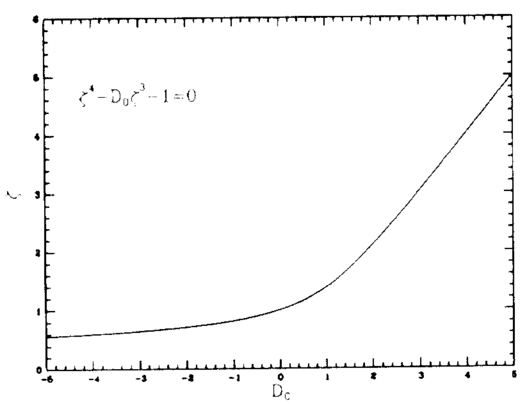

Figure 1: ζ as a function of $D_0 \equiv \frac{\epsilon_Z \hat{\lambda}_0}{\hat{\phi}_0^2}$.

It is seen from Eqs. (6), (7), and (8) that the deformed equilibrium distribution

$$\Psi^{(0)}(\phi, W) = \begin{cases} \dfrac{3}{4\pi J_0}\sqrt{1 - \dfrac{W^2}{\hat{W}^2} - \dfrac{\phi^2}{\hat{\phi}^2}}, & \text{when } \dfrac{W^2}{\hat{W}^2} + \dfrac{\phi^2}{\hat{\phi}^2} \le 1; \\ 0, & \text{otherwise,} \end{cases}$$

remains parabolic, except that its phase and energy spread follow the adiabatic change of the coupling, as

$$\hat{\phi} = \frac{1}{\zeta}\hat{\phi}_0, \quad \text{and} \quad \hat{W} = \zeta\hat{W}_0. \tag{9}$$

The dimensionless quantity D_0 given by Eq. (8) essentially describes the strength of the coupling. It can be rewritten in a more familiar form,

$$D_0 = \frac{qe\beta^2\hat{I}_0\,|Z/n|\ \mathrm{Sgn}(Z)}{2\pi E\eta\left(\hat{\epsilon}_0/\sqrt{2}\right)^2}, \tag{10}$$

where $\hat{\epsilon}_0$ is the fractional energy spread, and $\hat{I}_0 = 3\pi h\bar{I}/2\hat{\phi}_0$ and $\bar{I} = Nqe\omega_s/2\pi$ are the peak and the average current, respectively. Define D as a dimensionless quantity which describes the instantaneous bunch configuration under the coupling,

$$D = \frac{qe\beta^2\hat{I}\,|Z/n|\ \mathrm{Sgn}(Z)}{2\pi E\eta\left(\hat{\epsilon}/\sqrt{2}\right)^2}. \tag{11}$$

It is a monotonic function of D_0 and is similar to D_0 in its form. However, $\hat{I} = 3\pi h\bar{I}/2\hat{\phi}$ and $\hat{\epsilon}$ are instantaneous values instead of the un-perturbed values. The evolution of the bunch configuration under the adiabatic change of the coupling may be described with Eqs. (7), (8), and (10), as

$$D = 1 - \frac{1}{\zeta^4}\begin{cases} = 0, & \text{when } D_0 = 0; \\ \to 1, & \text{when } D_0 \to +\infty; \\ \to -\infty, & \text{when } D_0 \to -\infty. \end{cases} \tag{12}$$

3. Condition for microwave instability

The condition for microwave instability of a bunched beam with parabolic distribution (Eq. (2)) can be investigated[5] using the Vlasov equation and the perturbation method. Assuming that collision and diffusion processes can be neglected, Ψ satisfies

$$\frac{\partial\Psi}{\partial t} + \dot{\phi}\frac{\partial\Psi}{\partial\phi} + \dot{W}\frac{\partial\Psi}{\partial W} = 0. \tag{13}$$

Ψ may be considered as a superposition of the equilibrium distribution $\Psi^{(0)}$ and a density fluctuation $\Psi^{(1)}$, as

$$\Psi = \Psi^{(0)} + \Psi^{(1)}, \quad \Psi^{(1)} = e^{-i\Omega t}f_1(\phi, W). \tag{14}$$

Correspondingly, the Hamiltonian $\mathcal{H}$ may be written as

$$\mathcal{H}(\phi, W; t) = \mathcal{H}^{(0)}\left(\phi, W; \Psi^{(0)}\right) + \mathcal{H}^{(1)}\left(\phi, W; t; \Psi^{(1)}\right).$$

It has been shown above that $\mathcal{H}^{(0)}$ causes deformation of the equilibrium bunch shape. $\mathcal{H}^{(1)}$, on the other hand, may induce coherent instability.

By using Eq. (14), the Vlasov equation (Eq. (13)) can be rewritten as a linear equation in $\Psi^{(1)}$, which may be expressed in a matrix form by defining

$$\rho_m = \int\int f_1(\phi, W)\, e^{i\frac{m}{h}\phi}\, d\phi\, dW,$$

as

$$\rho_n = \sum_{m=-\infty}^{+\infty} T_{nm}\rho_m, \qquad n = \text{all integers}. \tag{15}$$

Solving Eq. (13) is equivalent to obtaining the eigenvalues of a matrix $\mathbf{T}$ of infinite dimensions, i.e. $\det(\delta_{nm} - T_{nm}) = 0$. The so called fast blow-up regime[6] is defined as the one in which the rise time of the instability is short compared with the period of synchrotron oscillation and long compared with the period of the disturbing fields. In this regime, T_{nm} can be simplified as

$$T_{nm} = \frac{3iNq^2e^2\omega_s Z k_1^2\beta_L^{1/2}}{8\pi^2(2J_0)^{3/2}}\int \frac{\mathrm{J}_0\left[\frac{(m-n)}{hB(W)}\right] W\,dW}{h\beta_L^2\Omega - mk_1W},$$

where $Z = Z(m\omega_s)$, J_0 is the Bessel function of 0th order,

$$B(W) = \frac{\beta_L}{\sqrt{2J_0\beta_L - W^2}}, \quad k_1 = -\frac{qe\hat{V}\cos\phi_s}{2\pi h}\left(1 - \epsilon_Z\tilde{\lambda}\right),$$

and

$$\beta_L = \sqrt{\frac{E\beta^2 k_1}{h^2\omega_s^2\eta}}$$

is the longitudinal amplitude function[3]. Instability that most likely occurs has the largest eigen-frequency. Using the identity[7]

$$\sum_{m-n=-\infty}^{\infty} J_0\left[\frac{(m-n)}{hB(W)}\right] \approx 2hB(W),$$

this eigen-frequency Ω is shown to satisfy the dispersion relation

$$1 = (U - iV)\int_{-1}^{1} \frac{1}{x - x_1}\frac{dg(x)}{dx}dx, \tag{16}$$

where U and V are both real,

$$U - iV = \frac{3iNq^2e^2h\omega_s Zk_1\beta_L^{3/2}}{8\pi(2J_0)^{3/2}}, \quad x_1 = \frac{h\beta_L^{3/2}\Omega}{nk_1(2J_0)^{1/2}},$$

and, for the parabolic distribution,

$$g(x) = \frac{2}{\pi}\sqrt{1-x^2}, \quad |x| \le 1.$$

In the case of the reactive coupling, $V = 0$, and the condition for instability to occur is

$$\frac{qe\beta^2\hat{I}_0|Z/n|\,\mathrm{Sgn}(Z)}{2\pi E\eta\left(\epsilon_0/\sqrt{2}\right)^2}\left(1 - \epsilon_Z\tilde{\lambda}\right)^{-1/4} \ge 1. \tag{17}$$

The quantity $\left(1 - \epsilon_Z\tilde{\lambda}\right)^{-1/4}$ represents the contribution from the bunch-shape deformation caused by the coupling. Using Eqs. (8) and (9), this condition becomes

$$D \ge 1. \tag{18}$$

On the other hand, it follows from the discussion of the last section that the bunch adjusts itself under the coupling in such a way that the Landau damping is enhanced. It follows from Eq. (12) that

$$D < 1,$$

which implies that Eq. (18) can never be satisfied due to this damping enhancement. The bunch keeps changing its shape without encountering microwave instability.

4. Discussion

The conclusion of the last section is based on the assumption that the coupling impedance is reactive and frequency independent, and that the particle motion is adiabatic. It is not necessarily true, for example, for a general broad-band coupling impedance. Such an impedance does not significantly affect the equilibrium bunch shape. When the coupling is strong enough to satisfy the instability condition, the particle motion may indeed be coherently unstable. Furthermore, in the case that the particle motion is non-adiabatic (e.g. when an intense beam is injected into the machine), both of the microwave instability and the adjustment of the bunch towards an equilibrium shape may occur, at the same time, in a period comparable to the synchrotron-oscillation period.

Under general circumstances, it is not easy to obtain an explicit and complete description of the evolution of the equilibrium distribution. An analytic solution is not always obvious even if the coupling is assumed to be purely reactive and frequency independent. For example, when the coupling is turned on adiabatically, an originally un-perturbed Gaussian bunch

$$\Psi_0(\phi, W) = \frac{1}{2\pi\sigma_\phi\sigma_W}\exp\left(-\frac{W^2}{2\sigma_W^2} - \frac{\phi^2}{2\sigma_\phi^2}\right), \tag{19}$$

will change into a new form, $\Psi(\phi, W)$. Following the discussion above, in contrast to the case of a parabolic distribution, ζ depends not only on the dimensionless quantity D_0, but also on the relative location of the contour ξ_i (Eq. (4)). Therefore, Ψ will not even retain the following Gaussian-like form,

$$\Psi(\phi, W) = C_1\exp\left[-C_2\mathcal{H}(\phi, W)\right],$$

where C_1 and C_2 are both constant. The problem may have to be solved numerically. This fact cautions that when dealing with the problem of coherent instability using perturbative methods, the effect of the coupling on the nominal particle distribution has to be properly considered; an ideal Gaussian distribution (Eq. (19)) exists only if the coupling is disregarded.

The findings in the last two sections were confirmed by revolution-by-revolution computer simulation[3] of a single bunch of particles in space-charge fields, with the bin length chosen as the cutoff wavelength and with a total of 7200 macro-particles. When the coupling is adiabatically turned on, the equilibrium distribution evolves according to Eqs. (9) and (12), either below or above the transition energy, without encountering microwave instability. In contrast to the behavior in the adiabatic regime, the bunch may easily suffer from microwave instability when it crosses the transition energy, or if the coupling is non-adiabatically turned on. The threshold and the growth rate evaluated from the simulation results agree with the analytic solution (Eqs. (17) and (16)) within the statistical accuracy of the numerical method.

Acknowledgement

We are grateful to Dr. J. Claus and Dr. R. Baartman for helpful discussion. We would also like to thank Dr. E.D. Courant for comments.

References

1. A.N. Lebedev, "Longitudinal Instability in the Presence of an rf Field", Proceedings of the 6th International Conference on High-Energy Accelerators, Cambridge, Mass., 1967, p.284.
2. A. Hofmann and F. Petersen, "Bunches with Local Elliptic Energy Distributions", IEEE Transactions on Nuclear Science **NS-26** (1979) 3526.
3. J. Wei, Longitudinal Dynamics of the Non-Adiabatic Regime on Alternating-Gradient Synchrotrons, Ph.D thesis, State University of New York at Stony Brook, 1990.
4. L.D. Landau, "On the Vibrations of the Electronic Plasma", J. Physics USSR **10**, 25 (1946).
5. J. Wei and S.Y. Lee, "Microwave Instability near Transition Energy", Particle Accelerators **28**, 77 (1990).
6. J.M. Wang and C. Pellegrini, "On the Condition for a Single Bunch High Frequency Fast Blow-Up", Proceedings of the 11th International Conference on High-Energy Accelerators, Geneva, 1980, p.554.
7. I.S. Gradshteyn and I.M. Ryshik, Table of Integrals, Series, and Products, Academic Press, New York, 1965, Eqs. 6.511 and 8.521.